Chapter 1

Polynomial approximations

In this chapter, we present various theoretical and algorithmic results regarding polynomial approximations of functions. We will mainly deal with real-valued continuous functions over a compact interval \([a, b]\), \(a, b \in \mathbb{R}, a \leq b\). We will denote \(C([a, b])\) the real vector space of continuous functions over \([a, b]\). In the framework of function evaluation one usually works with the following two norms over this vector space:

- the least-square norm \(L^2\): given a weight\(^1\) function \(w\), if \(dx\) denotes the Lebesgue measure, we write\(^2\)
  
  \[ g \in L^2([a, b], w, dx) \]

  if

  \[ \int_a^b w(x)|g(x)|^2dx < \infty, \]

  and then we define

  \[ \|g\|_{2,w} = \sqrt{\int_a^b w(x)|g(x)|^2dx}; \]

- the supremum norm (aka Chebyshev norm, infinity norm, \(L^\infty\) norm): if \(g\) is bounded on \([a, b]\), we set

  \[ \|g\|_{\infty} = \sup_{x \in [a,b]} |g(x)|, \]

  (observe that for a continuous function \(g\), we have \(\|g\|_{\infty} = \max_{x \in [a,b]} |g(x)|\)).

For both norms, one of the main questions we are interested in here is the following.

**Problem 1.1.** (Best approximation) Given \(f \in C([a, b])\) and \(n \in \mathbb{N}\), minimize \(\|f - p\|\) where \(p\) describes the space \(\mathbb{R}_n[x]\) of polynomials with real number coefficients and degree at most \(n\).

In the \(L^2\) case, the answer to this question is easy. The space \(C([a, b])\) is a subset of \(L^2([a, b], w, dx)\) which is a Hilbert space, i.e. a vector space equipped with an inner product

\[ \langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx, \]

and \(\| \cdot \|_2\) is the associated norm, for which \(L^2([a, b], w, dx)\) is complete. The best polynomial approximation of degree at most \(n\) is the projection \(p = \text{pr}_n(f)\) of \(f\) onto \(\mathbb{R}_n[x]\). We will give more details on the \(L^2\) case in Chapter 2. The situation in the \(L^\infty\) case is more intricate and we will focus on it in the sequel of this chapter.

\(^1\)Here, we will assume that it means that \(w \in C((a, b))\) and \(w > 0\) almost everywhere.
1.1 Density of the polynomials in \( C([a, b]), \| \cdot \|_\infty) \)

For all \( f \in C([a, b]) \) and \( n \in \mathbb{N} \), let

\[
E_n(f) = \inf_{p \in \mathbb{R}_n[x]} \| f - p \|_\infty.
\]

We first recall that \( E_n(f) \to 0 \) as \( n \to \infty \), a result due to Weierstraß:

**Theorem 1.2.** [Weierstraß, 1885] For all \( f \in C([a, b]) \) and for all \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \), \( p \in \mathbb{R}_n[x] \) such that \( \| p - f \|_\infty < \varepsilon \).

Various proofs of this result have been published, in particular, those by Runge (1885), Picard (1891), Lerch (1892 and 1903), Volterra (1897) Lebesgue (1898), Mittag-Leffler (1900), Fejér (1900 and 1916), Landau (1908), la Vallée Poussin (1908), Jackson (1911), Sierpinski (1911), Bernstein (1912), Montel (1918). The text [Pinkus, 2000] is an interesting account on Weierstraß’ contribution to Approximation Theory and, in particular, his fundamental result on the density of polynomials in \( C([a, b]) \) stated in Theorem 1.2.

We give now one proof inspired by Bernstein’s one.

**Proof of Theorem 1.2.** Up to a change of variable, we can assume \([a, b] = [0, 1]\). Define the Bernstein polynomials as

\[
B_n(g, x) = \sum_{k=0}^{n} \binom{n}{k} g(k/n) x^k (1 - x)^{n-k}
\]

for \( g \in C([0, 1]) \).

We have

\[
B_n(1, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1,
\]

\[
B_n(x, x) = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1 - x)^{n-k} = x \sum_{k=0}^{n} \frac{n-1}{k-1} x^{k-1} (1 - x)^{n-k}
\]

\[= x \sum_{k=0}^{n} \frac{n-1}{k} x^k (1 - x)^{n-1-k} = x,
\]

\[
B_n(x^2, x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{k}{n} \right)^2 x^k (1 - x)^{n-k} = x \sum_{k=1}^{n} \frac{k}{n} \binom{n-1}{k-1} x^{k-1} (1 - x)^{n-k}
\]

\[= x \sum_{k=1}^{n} \frac{k-1}{n} \frac{n-1}{k-1} x^{k-1} (1 - x)^{n-k} + \frac{x}{n}
\]

\[= \frac{n-1}{n} x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1 - x)^{n-k} + \frac{x}{n} = \frac{x}{n} + x^2 \frac{n-1}{n}.
\]

Now consider the sequence

\[
f(x) - B_n(f, x) = \sum_{k=0}^{n} (f(x) - f(k/n)) b_{n,k}(x) \text{ where } b_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \forall x \in [0, 1].
\]

Fix \( \varepsilon > 0 \). The function \( f \) is continuous and hence uniformly continuous over \([0, 1]\), hence there exists \( \delta > 0 \) such that

\[
\forall x_1, x_2 \in [0, 1], \quad |x_2 - x_1| < \delta \quad \Rightarrow \quad |f(x_2) - f(x_1)| < \varepsilon.
\]

Let \( M = \max_{x \in [0, 1]} |f(x)| \). Since \( b_{n,k}(x) \geq 0 \) for all \( x \in [0, 1] \), we can write
\[ |f(x) - B_n(f, x)| \leq \left| \sum_{k=0}^{n} (f(x) - f(k/n))b_{n,k}(x) \right| + \left| \sum_{k=0}^{n} (f(x) - f(k/n))b_{n,k}(x) \right| + \left| \sum_{k=0}^{n} (f(x) - f(k/n))b_{n,k}(x) \right|
\]

\[ \leq \varepsilon \sum_{k=0}^{n} b_{n,k}(x) + 2M \sum_{k=0}^{n} b_{n,k}(x) = \varepsilon + 2M \sum_{k=0}^{n} b_{n,k}(x). \]

Note that we actually have

\[ \sum_{k=0}^{n} b_{n,k}(x) = \sum_{k=0}^{n} b_{n,k}(x) \text{ since } b_{n,k}(x) \geq 0 \]

\[ \leq \sum_{k=0}^{n} \left( \frac{x - k/n}{\delta} \right)^2 \frac{b_{n,k}(x)}{\delta} \]

\[ \leq \sum_{k=0}^{n} \left( \frac{x - k/n}{\delta} \right)^2 \frac{b_{n,k}(x)}{\delta} \]

\[ = \frac{1}{\delta^2} \left( x^2 - 2x^2 + x^2 \frac{n - 1}{n} + \frac{x}{n} \right) = \frac{x(1 - x)}{n^3}. \]

Therefore, we obtained \( |f(x) - B_n(f, x)| \leq \varepsilon + \frac{M}{2n^2}. \) The upper bound does not depend on \( x \) and can be made as small as desired. \( \square \)

**Remark 1.3.** One of the very nice features of this proof is that it provides an explicit sequence of polynomials which converges to the function \( f \). It is worth mentioning that Bernstein polynomials prove useful in various other domains (computer graphics, global optimization, ...). See [Farouki, 2012] for instance.

Note that, in the proof, we only used the values of the \( B_n(f, x) \) for \( 0 \leq n \leq 2 \). In fact, we have the following result.

**Theorem 1.4 (Bohman and Korovkin).** Let \( L_n \) be a sequence of monotone linear operators on \( C([a, b]) \), that is to say: for all \( f, g \in C([a, b]) \)

- \( L_n(\mu f + \lambda g) = \lambda L_n(f) + \mu L_n(g) \) for all \( \lambda, \mu \in \mathbb{R} \),
- \( f(x) \geq g(x) \) for all \( x \in [a, b] \) then \( L_n f(x) \geq L_n g(x) \) for all \( x \in [a, b] \).

the following conditions are equivalent:

i. \( L_n f \to f \) uniformly for all \( f \in C([a, b]) \);

ii. \( L_n f \to f \) uniformly for the three functions \( x \to 1, x, x^2 \);

iii. \( L_n \to 1 \) and \( (L_n \phi_t)(t) \to 0 \) uniformly in \( t \in [a, b] \) where \( \phi_t : x \in [a, b] \to (t - x)^2 \).

**Proof.** See [Cheney, 1998]. \( \square \)

Actually, a refinement of Weierstrass’ theorem yields a statement about the speed of convergence of the \( B_n(f, x) \) to \( f \). It is obtained in terms of the modulus of continuity.
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**Definition 1.5.** The modulus of continuity of \( f \) is the function \( \omega \) defined as

\[
\forall \delta > 0, \quad \omega(\delta) = \sup_{|x - y| < \delta, \; x, y \in [a, b]} |f(x) - f(y)|.
\]

**Proposition 1.6.** If \( f \) is a continuous function over \([0, 1]\), \( \omega \) its modulus of continuity, then

\[
\|f - B_n(f, x)\|_\infty \leq \frac{\omega}{2} \left( n^{-\frac{1}{2}} \right).
\]

**Proof.** Let \( \delta > 0 \) and \( x \in [0, 1] \). Let \( k \in \{0, \ldots, n\} \) such that \( |x - k/n| \leq \delta \), then \( |f(x) - f(k/n)| \leq \omega(\delta) \).

Since \( b_{n,k}(y) \geq 0 \) for all \( y \in [0, 1] \), we have

\[
\sum_{k=0}^{n} (f(x) - f(k/n))b_{n,k}(x) \leq \omega(\delta) \sum_{k=0}^{n} b_{n,k}(x) = \omega(\delta).
\]

Now, let \( k \in \{0, \ldots, n\} \) such that \( |x - k/n| \geq \delta \). Let \( M = \left\lfloor \frac{|x-k/n|}{\delta} \right\rfloor \), let \( y_j = x + \frac{j}{M+1}(k/n - x) \) for \( j = 0, \ldots, M + 1 \). Note that, for all \( j = 0, \ldots, M \), we have \( |y_{j+1} - y_j| < \delta \), from which follows

\[
|f(x) - f(k/n)| \leq \sum_{j=0}^{M} |f(y_{j+1}) - f(y_j)| \leq (M + 1)\omega(\delta)
\]

\[
\leq \omega(\delta) \left( 1 + \frac{1}{\delta} |x - k/n| \right) \leq \omega(\delta) \left( 1 + \frac{1}{\delta} \left( x - \frac{k}{n} \right)^2 \right).
\]

For all \( x \in [0, 1] \), we can write

\[
|f(x) - B_n(f, x)| \leq \sum_{k=0}^{n} (f(x) - f(k/n))b_{n,k}(x) \leq \omega(\delta) + \omega(\delta) \left( 1 + \frac{1}{\delta^2} |x - k/n|^2 \right) b_{n,k}(x)
\]

\[
\leq \omega(\delta) \left( 2 + \frac{1}{\delta^2} \sum_{k=0}^{n} \left( x - \frac{k}{n} \right)^2 b_{n,k}(x) \right)
\]

\[
\leq \omega(\delta) \left( 2 + \frac{x(1-x)}{n\delta^2} \right) \leq \omega(\delta) \left( 2 + \frac{1}{4n\delta^2} \right).
\]

Finally, replace \( \delta \) with \( 1/\sqrt{n} \).

**Remark 1.7.** This result is not optimal. For improvements and refinements, see Section 4.6 of [Cheney, 1998] or Chapter 16 of [Powell, 1981] for a presentation of Jackson theorems. See also Theorem 2.23.

**Corollary 1.8.** When \( f \) is Lipschitz continuous, \( E_n(f) = O(n^{-1/2}) \).
1.2 Best $L^\infty$ (or minimax) approximation

The infimum $E_n(f)$ is reached, thanks to the following proposition.

**Proposition 1.9.** Let $(E, \| \cdot \|)$ be a normed $\mathbb{R}$-vector space, let $F$ be a finite dimensional subspace of $(E, \| \cdot \|)$. For all $f \in E$, there exists $p \in F$ such that $\| p - f \| = \min_{q \in F} \| q - f \|$. Moreover, the set of best approximations to a given $f \in E$ is convex.

**Proof.** Let $f \in E$. Consider $F_0 = \{ p \in F : \| p \| \leq 2\| f \| \}$. Then $F_0$ is nonempty (it contains 0), closed, bounded, and we assumed $\dim F < \infty$. Hence $F_0$ is compact. Let $\varphi(p) = \| f - p \|$. The function $\varphi$ is 1-Lipschitz and hence continuous. It follows that $\varphi(F_0)$ is compact, which implies the existence of $p^* \in F_0$ s.t. $\varphi(p^*) = \min_{p \in F_0} \| f - p \|$. Moreover, if $p \in F \setminus F_0$, $\| f - p \| \geq \| p \| - \| f \| > \| f \| \geq \varphi(p^*)$ since $0 \in F_0$. Thus, $\| f - p^* \| = \min_{p \in F} \| f - p \|$. Now, let $p$ and $q \in F$ be two best approximations to $f$. For all $\lambda \in [0, 1]$, the vector $\lambda p + (1 - \lambda)q$ is an element of the vector space $F$ and we have, from the triangle inequality, $\| \lambda p + (1 - \lambda)q - f \| \leq \lambda \| p - f \| + (1 - \lambda)\| q - f \| = \min_{q \in F} \| q - f \|$: the vector $\lambda p + (1 - \lambda)q$ is also a best approximation to $f$. \qed

The best $L^2$ approximation is unique, which is not always the case in the $L^\infty$ setting.

**Exercise 1.2.1.** Consider the following simple situation: the interval is $[-1, 1]$, $f$ is the constant function 1 and $F = \mathbb{R}g$ where $g : x \to x^2$. Determine the set of best $L^\infty$ approximations to $f$.

In the case of $L^\infty$, it is necessary to introduce an additional condition known as the Haar condition.

**Definition 1.10.** Consider $n + 1$ functions $\varphi_0, \ldots, \varphi_n$ defined over $[a, b]$. We say that $\varphi_0, \ldots, \varphi_n$ satisfy the Haar condition if

a) the $\varphi_i$ are continuous;

b) and the following equivalent statements hold:

- for all $x_0, x_1, \ldots, x_n \in [a, b]$,

\[
\begin{vmatrix}
\varphi_0(x_0) & \cdots & \varphi_n(x_0) \\
\vdots & & \vdots \\
\varphi_0(x_n) & \cdots & \varphi_n(x_n)
\end{vmatrix} = 0 \iff \exists i \neq j, x_i = x_j;
\]

- given pairwise distinct $x_0, \ldots, x_n \in [a, b]$ and values $y_0, \ldots, y_n$, there exists a unique interpolant

\[
p = \sum_{k=0}^{n} \alpha_k \varphi_k, \text{ with } \alpha_k \in \mathbb{R}, \forall k = 0, \ldots, n,
\]

such that $p(x_i) = y_i, \forall i = 0, \ldots, n$;

- the $\varphi_k, k = 0, \ldots, n$, are $\mathbb{R}$-linearly independent and any $p = \sum_{k=0}^{n} \alpha_k \varphi_k \neq 0$ has at most $n$ distinct zeros in $[a, b]$.

**Exercise 1.2.2.** Prove that the conditions above are equivalent.

A set of functions that satisfy the Haar condition is called a Chebyshev system. The prototype example is $\varphi_i(x) = x^i$, for which we have

\[
\begin{vmatrix}
\varphi_0(x_0) & \cdots & \varphi_n(x_0) \\
\vdots & & \vdots \\
\varphi_0(x_n) & \cdots & \varphi_n(x_n)
\end{vmatrix} = \begin{vmatrix} 1 & \cdots & x_0^n \\
\vdots & & \vdots \\
1 & \cdots & x_n^n \end{vmatrix} = V_n = \prod_{0 \leq i < j \leq n} (x_j - x_i). \quad (1.1)
\]

(Sketch of a proof: considering $x_n = z$ as an indeterminate and looking at the roots of the polynomial $V_n$, we see that $V_n = V_{n-1}(z - x_0) \cdots (z - x_{n-1})$.)
Exercise 1.2.3. Show that the following families of functions are Chebyshev systems as well:

- \( \{ e^{\lambda_0 x}, \ldots, e^{\lambda_n x} \} \) for \( \lambda_0 < \lambda_1 < \cdots < \lambda_n \);
- \( \{ 1, \cos x, \sin x, \ldots, \cos(nx), \sin(nx) \} \) over \([a, b]\) where \( 0 \leq a < b < 2\pi \);
- \( \{ x^{\alpha_0}, \ldots, x^{\alpha_n} \} \), \( \alpha_0 < \cdots < \alpha_n \), over \([a, b]\) with \( a > 0 \).

Let \( E \) be a real vector space, \( e_1, e_2, \ldots, e_m \in E \), we will denote \( \text{Span}_R \{ e_1, \ldots, e_m \} \) the set

\[
\text{Span}_R \{ e_1, \ldots, e_m \} = \left\{ \sum_{k=1}^{m} \alpha_k e_k; \alpha_1, \ldots, \alpha_m \in \mathbb{R} \right\}.
\]

If \( \{ \varphi_0, \ldots, \varphi_n \} \) is a Chebyshev system over \([a, b]\), any element of \( \text{Span}_R \{ \varphi_0, \ldots, \varphi_n \} \) will be called a generalized polynomial.

Beside its beauty, the following characterization of the minimax approximation proved crucial to the design of an algorithm for computing it.

Theorem 1.11. [Alternation Theorem. Kirchberger (1902)] Let \( \{ \varphi_0, \ldots, \varphi_n \} \) be a Chebyshev system over \([a, b]\). Let \( f \in C([a, b]) \). A generalized polynomial \( p = \sum_{k=0}^{n} \alpha_k \varphi_k \) is the best approximation (or minimax approximation) to \( f \) iff there exist \( n+2 \) points \( x_0, \ldots, x_{n+1}, a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b \) such that, for all \( k \),

\[
f(x_k) - p(x_k) = (-1)^k (f(x_0) - p(x_0)) = \pm \| f - p \|_\infty.
\]

In other words, \( p \) is the best approximation if and only if the error function \( f - p \) has (at least) \( n+2 \) extrema, all global (of the same absolute value) and with alternating signs.

Example 1.12. Let \( f : x \in [0, 1] \mapsto \frac{1}{\cos(x)} \), \( p = \sum_{k=0}^{10} c_k x^k \) its minimax approximation. The graph of the error function \( \varepsilon = f - p \) is:

Example 1.13. Let \( f : x \in [-0.9, 0.9] \mapsto \arctan(x) \), \( p = \sum_{k=0}^{15} c_k x^k \) its minimax approximation. The graph of the error function \( \varepsilon = f - p \) is:
Example 1.14. The best approximation to $\cos$ over $[0,10\pi]$ on the Chebyshev system $\{1, x, x^2\}$ is the constant function $0!$ Moreover, the same is true for $\{1, x, \ldots, x^5\}$ up to and including $h = 9$.

Proof. We can assume that $f \notin \text{Span}_\mathbb{R} \{\varphi_0, \ldots, \varphi_n\}$.

We already proved the existence of a best approximation.

We now show that the equioscillation property implies optimality of the approximation. Let $p$ be an approximation with equioscillating error function, and suppose that there exists $q = \sum \beta_j \varphi_j$ with $\|f - q\| < \|f - p\|$. Writing $p - q = (p - f) - (q - f)$, we see that $p - q$ changes sign between each pair of consecutive $x_i$. It follows from the intermediate value theorem that there exist $(n + 1)$ points $y_0, \ldots, y_n$ such that $x_0 < y_0 < x_1 < \cdots < x_{n} < y_{n} < x_{n+1}$ and $p(y_i) = q(y_i)$. By definition of a Chebyshev system, this implies that $p = q$.

Conversely, optimality implies equioscillation.

For simplicity, we assume that $\{\varphi_0, \ldots, \varphi_n\} = \{1, x, \ldots, x^n\}$ (see [Cheney, 1998, Powell, 1981] for a proof of the general case). Let $p$ be a best approximation. First, note that the global minimum and the global maximum of $f - p$ must have the same absolute value: otherwise, we can improve the approximation by shifting $p$ by a constant. Now suppose that $f - p$ equioscillates at $\ell$ points $x_0 < x_1 < \cdots < x_{\ell}$ at most, with $1 \leq \ell < n + 1$. We can choose the $\{x_i\}_{0 \leq i \leq \ell}$ as follows.

The point $x_0$ is the smallest number in $[a, b]$ at which $|p - f|$ reaches its maximum: the set $A = \{x \in [a, b] : |p(x) - f(x)| = \|p - f\|\}$ is nonempty, bounded and closed since $|p - f|$ is continuous and $A = (\|p - f\|)^{-1}(\{\|p - f\|\}) \cap [a, b]$: $A$ is compact and let $x_0$ be the minimum of $A$. Likewise, the point $x_1$ is defined as the smallest number in $[x_0, b]$ at which $p - f$ is equal to $-(p - f)(x_0)$ and so on and so forth.

Assume wlog that $p(x_0) - f(x_0) = -\|p - f\|$. For $j = 0, \ldots, \ell - 1$, let $B_j = \{x_j \leq x \leq x_{j+1} : p(x) = f(x)\}$, the set $B_j$ is nonempty since $(p(x_j) - f(x_j))(p(x_{j+1}) - f(x_{j+1})) < 0$, closed (p - f is continuous and $B_j = (p - f)^{-1}(\{0\}) \cap [x_j, x_{j+1}]$) and bounded: it is a compact set, which has a maximum $y_{j+1}$ distinct from $x_{j+1}$. We remark that $y_1 < y_2 < \cdots < y_{\ell}$, in particular.

We now define $Q(x) = (y_1 - x) \cdots (y_{\ell} - x)$. Note that $Q(a) \geq 0$. If we set $y_0 = a$ and $y_{\ell+1} = b$, let

$$K_1 = [a, y_1] \cup [y_2, y_3] \cup \cdots = \bigcup_{k=0}^{\lfloor \ell/2 \rfloor} [y_{2k}, y_{2k+1}],$$

$$K_2 = [y_1, y_2] \cup [y_3, y_4] \cup \cdots = \bigcup_{k=0}^{\lfloor (\ell-1)/2 \rfloor} [y_{2k+1}, y_{2k+2}].$$

The sets $K_1$ and $K_2$ are finite unions of compact sets, and hence compact. We have:
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- \([a, b] = K_1 \cup K_2\) and \(K_1 \cap K_2 = \{y_k\}_{1 \leq k \leq \ell}\),
- \(-\|p - f\|_\infty \leq p - f < \|p - f\|_\infty\), \(Q \geq 0\) on the compact \(K_1\) and \(Q > 0\) on \(K_1 \setminus \{y_k\}_{1 \leq k \leq \ell}\),
- \(-\|p - f\|_\infty < p - f \leq \|p - f\|_\infty\), \(Q \leq 0\) on the compact \(K_2\) and \(Q < 0\) sur \(K_2 \setminus \{y_k\}_{1 \leq k \leq \ell}\).

Hence there exists \(\lambda \in (0, +\infty)\) such that

\[ -\|p - f\|_\infty < p + \lambda Q - f < \|p - f\|_\infty, \]

which contradicts the optimality of \(p\).

Finally, let us prove the uniqueness. Let \(p, q\) be two best approximations, and let

\[ \mu = \|f - p\|_\infty = \|f - q\|_\infty. \]

It follows from Proposition 1.9 that \(\frac{1}{2}(p + q)\) is a best approximation too. Thus there exist \(t_0 < t_1 < \cdots < t_{n+1}\) such that

\[ \left(\frac{p + q}{2}\right)(t_i) - f(t_i) = \pm (-1)^i \mu. \]

Thus, we have \(p(t_i) - f(t_i) = q(t_i) - f(t_i) = \pm (-1)^i \mu\) for all \(i = 0, \ldots, n + 1\), and hence \(p = q\) by the Haar condition.

Next result is also a key element for the design of an algorithm for computing the minimax approximation.

**Theorem 1.15.** (La Vallée Poussin) Let \(f \in \mathcal{C}([a, b])\). Let \(\{\varphi_0, \ldots, \varphi_n\}\) be a Chebyshev system over \([a, b]\), and let \(p \in \text{Span}_\mathbb{R}\{\varphi_0, \ldots, \varphi_n\}\). If there exist \(x_0 < x_1 < \cdots < x_{n+1}\) such that \(p - f\) alternates at the \(x_i\), then

\[ \min_i |f(x_i) - p(x_i)| \leq E_n(f) \leq \|f - p\|_\infty, \]

where \(E_n(f) = \inf_{q \in \text{Span}_\mathbb{R}\{\varphi_i\}} \|f - q\|_\infty\).

**Proof.** The second inequality is obvious. If the first one does not hold, assume wlog that \(f(x_0) > p(x_0)\). Then, if \(p\) is the best approximation of \(f\), we have, for all \(k = 0, \ldots, n + 1\), \((-1)^k(f(x_k) - p(x_k)) > (-1)^k(f(x_k) - p^*(x_k))\): the generalized polynomial \(p - p^*\) changes sign \(n + 1\) times over \([a, b]\), which is not possible.

**Remark 1.16.** Let \(\{\varphi_0, \ldots, \varphi_n\}\) be a Chebyshev set over \([a, b]\). The statements from Theorems 1.11 and 1.15 remain valid if \([a, b]\) is replaced with any closed subset of \([a, b]\) containing at least \(n + 2\) points (see [Cheney, 1998]).

Now let’s see whether the Haar condition is necessary or not to guarantee uniqueness. Before stating the result, we introduce the function \(\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}\):

\[ \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{otherwise}. \end{cases} \]

**Theorem 1.17 (Haar’s Uniqueness Theorem).** Let \(\{\varphi_0, \ldots, \varphi_n\}\) be a set of continuous functions over \([a, b]\). The minimax approximation to a continuous function \(f\) by a generalized polynomial \(p = \sum_{k=0}^n \alpha_k \varphi_k\) is unique for all choices of \(f\) iff \(\{\varphi_0, \ldots, \varphi_n\}\) satisfies the Haar condition.

**Proof.** We already proved the “only if” direction in Theorem 1.11.

We assume that \(\{\varphi_0, \ldots, \varphi_n\}\) does not satisfy the Haar condition. Then, there exist \(a \leq x_0 < \cdots < x_n \leq b\) such that \(\varphi_j(x_j) |_{0 \leq j \leq n} = 0\). Hence, there are \(\{a_0, \ldots, a_n\}\) and \(\{b_0, \ldots, b_n\} \in \mathbb{R}^{n+1} \setminus \{0\}\) such that \(\sum_{i=0}^n a_i g_i(x_j) = 0\) for all \(j = 0, \ldots, n\) and \(\sum_{j=0}^n b_j g_i(x_j) = 0\) for all \(i = 0, \ldots, n\). The latter implies

\[ \sum_{j=0}^n b_j P(x_j) = 0 \]

for all generalized polynomial \(P\). (1.2)
Let \( Q(x) = \sum_{j=0}^{n} a_j g_j(x) \). We have \( Q(x_j) = 0 \) for all \( j = 0, \ldots, n \) and we may assume that \( \|Q\|_\infty < 1 \).

Now, we consider \( f \in C([a, b]) \) such that \( \|f\|_\infty = 1 \) and \( f(x_j) = \text{sgn} b_j \) for all \( j = 0, \ldots, n \). Then, we introduce \( F = f(1 - |Q|) \). We have \( F(x_j) = f(x_j) = \text{sgn} b_j \) for all \( j = 0, \ldots, n \).

Now, we prove that for any generalized polynomial \( P \), we have \( \|F - P\|_\infty \geq 1 \). Suppose that there exists \( P_0 \) satisfying \( \|F - P_0\|_\infty < 1 \), then \( \text{sgn} P_0(x_j) = \text{sgn} F(x_j) = \text{sgn} b_j \) for all \( j = 0, \ldots, n \). And yet, we have \( \sum_{j=0}^{n} b_j P_0(x_j) = 0 \): contradiction.

Finally, we notice that for all \( \lambda \in [0, 1] \), the generalized polynomial \( \lambda Q \) is a best approximation to \( F \) since, for all \( x \in [a, b] \),

\[
|F(x) - \lambda Q(x)| \leq |f(x)||1 - |Q(x)|| + \lambda|Q(x)| \leq 1 - |Q(x)| + \lambda|Q(x)| \leq 1.
\]

\[\Box\]

Remez [Remez, 1934] published in 1934 Algorithm 1 which allows one to approximate, as close as desired, the minimax polynomial. This algorithm is used for the design of mathematical functions but it is its variant, due to Parks and McClellan [Parks and McClellan, 1972], in the framework of the design of filters for signal processing which has been extremely successful.

**Algorithm 1** Remez second algorithm

**Input:** An interval \([a, b]\), a function \( f \in C([a, b]) \), a natural integer \( n \), a Chebyshev system \( \{\varphi_k\}_{0 \leq k \leq n} \), a tolerance \( \Delta \).

**Output:** An approximation of the degree \( n \)-minimax polynomial of \( f \) on the system \( \{\varphi_k\}_{0 \leq k \leq n} \).

1. Choose \( n + 2 \) points \( x_0 < x_1 < \cdots < x_{n+1} \) in \([a, b]\), \( \delta \leftarrow 1, \varepsilon \leftarrow 0 \).
2. while \( \delta \geq \Delta |\varepsilon| \) do
3. Determine the solutions \( a_0, \ldots, a_n \) and \( \varepsilon \) of the linear system

\[
\sum_{k=0}^{n} a_k \varphi_k(x_j) - f(x_j) = (-1)^j \varepsilon, \quad j = 0, \ldots, n + 1.
\]

4. Choose \( x_{\text{new}} \in [a, b] \) such that

\[
\|p - f\|_\infty = |p(x_{\text{new}}) - f(x_{\text{new}})|, \quad \text{with} \quad p = \sum_{k=0}^{n} a_k \varphi_k.
\]

5. Replace one of the \( x_i \) with \( x_{\text{new}} \), in such a way that the sign of \( p - f \) alternates at the points of the resulting discretization \( x_{0,\text{new}}, \ldots, x_{n+1,\text{new}} \).

6. \( \delta \leftarrow |p(x_{\text{new}}) - f(x_{\text{new}})| - |\varepsilon| \).
7. end while
8. Return \( p \).

We will not give more details concerning this algorithm. See [Cheney, 1998, Powell, 1981, Filip, 2016a, Filip, 2016b]. Regarding its speed of convergence, one can find in [Cheney, 1998] the following statement.

**Theorem 1.18.** Let \( p_k \) denote the value of \( p \) after \( k(n+2) \) loop turns, and let \( p^* \) be such that \( E_n(f) = \|f - p^*\|_\infty \). There exists \( \theta \in (0, 1) \) such that \( \|p_k - p^*\|_\infty = O(\theta^k) \).

Under mild regularity assumptions, the bound \( O(\theta^k) \) can in fact be improved to \( O(\theta^{2k}) \) [Veidinger, 1960].

### 1.3 Polynomial interpolation

Now we restrict our study to polynomials in \( \mathbb{R}_n[x] \).

At this stage, it seems natural to focus on techniques for computing polynomials that interpolate functions at a given finite family of points:
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- sometimes a finite number of values is the only information we have on the function,
- Step 2.a of Remez’ algorithm requires an efficient interpolation process,
- Theorem 1.11 shows that, for all \( n \), there exists \( a \leq z_0 < z_1 < \ldots < z_n \leq b \) such that \( f(z_i) = p^*(z_i) \) for \( i = 0, \ldots, n \), where \( p^* \) is the minimax approximation of \( f \): the polynomial \( p^* \) is an interpolation polynomial of \( f \).

Let \( A \) be a commutative ring (with unity). Given pairwise distinct \( x_0, \ldots, x_n \in A \) and corresponding \( y_0, \ldots, y_n \in A \), the interpolation problem is to find \( p \in A_n[x] \) such that \( p(x_i) = y_i \) for all \( i \). Write \( p = \sum a_k x^k \). The problem can be restated as

\[
V \cdot a = y
\]

where \( V \) is a Vandermonde matrix. If \( \det V \) is invertible, there is a unique solution.

From now on we assume \( A = \mathbb{R} \). The expression (1.1) of the Vandermonde determinant shows that as soon as the \( x_i \) are pairwise distinct, there is a unique solution. We now discuss several ways to compute the interpolation polynomial.

Linear algebra. We could invert the system (1.3) using standard linear algebra algorithms. This takes \( O(n^3) \) operations using Gaussian elimination. In theory, the best known complexity bound is currently \( O(n^\theta) \) where \( \theta \approx 2.3727 \) (Williams). In practice, Strassen’s algorithm yields a cost of \( O(n \log^2 n) \). There are issues with this approach, though:

- the problem is ill-conditioned: a small perturbation on the \( y_i \) leads to a significant perturbation of the solution,
- we can do better from the complexity point of view: \( O(n^2) \) or even \( O(n \log O(1) n) \) in general, \( O(n \log n) \) if the \( x_i \) are so-called Chebyshev nodes.

The divided-difference method. Newton’s divided-difference method allows us to compute interpolation polynomials incrementally. The idea is as follows. Let \( p_k \in \mathbb{R}[x] \) be such that \( p_k(x_i) = y_i \) for \( 0 \leq i \leq k \leq n \), and write

\[
p_{n+1}(x) = p_n(x) + a_{n+1}(x - x_0)(x - x_1) \cdots (x - x_n).
\]

Then we have

\[
p_{n+1}(x_j) = y_j, \quad 0 \leq j \leq n, \tag{1.4}
\]

\[
p_{n+1}(x_{n+1}) = p_n(x_{n+1}) + a_{n+1}(x_{n+1} - x_0)(x_{n+1} - x_1) \cdots (x_{n+1} - x_n). \tag{1.5}
\]

Given \( y_0, \ldots, y_k \), we denote by \( [y_0, \ldots, y_k] \) the corresponding \( a_k \): Then, we can compute \( a_k \) using the relation

\[
[y_0, \ldots, y_k+1] = [y_1, \ldots, y_{k+1}] - [y_0, \ldots, y_k] \quad x_{k+1} - x_0.
\]

This leads to a tree of the following shape.

Hence, the cost for computing the coefficients in \( O(n^2) \) operations.

The evaluation cost at a given point \( z \) is in \( O(n) \) operations in \( \mathbb{R} \).
Lagrange's Formula. For all \( j \), let
\[
\ell_j(x) = \prod_{0 \leq k \leq n, k \neq j} \frac{x - x_k}{x_j - x_k}.
\]
Then we have \( \deg \ell_j = n \) and \( \ell_j(x_i) = \delta_{i,j} \) for all \( 0 \leq i, j \leq n \). The polynomials \( \ell_j, 0 \leq j \leq n \), form a basis of \( \mathbb{R}_n[x] \), and the interpolation polynomial \( p \) can be written
\[
p(x) = \sum_{i=0}^{n} y_i \ell_i(x).
\]
Thus, writing the interpolation polynomial on the Lagrange basis is straightforward.

What about the cost of evaluating the resulting polynomial at a given point \( z \)? If we do it naively, computing \( \ell_j(z) \) costs (say) \( 2n \) subtractions, \( 2n + 1 \) multiplications and 1 division. The total cost is \( O(n^2) \) operations in \( \mathbb{R} \).

But we can also write
\[
p(x) = W(x) \sum_{i=0}^{n} \frac{y_i}{(x - x_i) W'(x_i)} W(x) = \prod_{i=0}^{n} (x - x_i).
\]
Assuming the \( W'(x_i) \) are precomputed, the cost of evaluating \( p(z) \) using this formula is only \( O(n) \) arithmetical operations.

The notion of “barycentric Lagrange interpolation” is particularly relevant regarding these stability issues [Trefethen, 2013].

1.4 Interpolation and approximation, Chebyshev polynomials

How useful is interpolation for our initial \( L_{\infty} \) approximation problem? It turns out that the choice of the points is critical. The more points, the better? Actually, with equidistant points, the error can grow with the number of points (Runge’s phenomenon).

Exercise 1.4.1. Using your computer algebra system of choice, interpolate the function
\[
f : x \mapsto \frac{1}{1 + 5x^2}
\]
at the points \(-1 + \frac{2k}{n}, 0 \leq k \leq n\), for \( n = 10, 15, \ldots, 30 \). Compare with \( f \) on \([-1, 1]\).

In short, we should never use equidistant points when approximating a function by interpolation. Are there better choices?

Theorem 1.19. [Faber] For each \( n \), let a system of \( n+1 \) distinct nodes \( \xi^{(n)}_0, \ldots, \xi^{(n)}_n \in [a, b] \). Then there exists \( f \in C([a,b]) \) such that the sequence of errors \( \{ \| f - p_n \|_{\infty} \}_{n \in \mathbb{N}} \) is unbounded, where \( p_n \in \mathbb{R}_n[x] \) denote the polynomial which interpolates \( f \) at the \( \xi^{(n)}_0, \ldots, \xi^{(n)}_n \).

Proof. See Remark 2.17. \( \square \)

We discuss better choices below. We start with the following analogue of the Taylor-Lagrange formula.

Theorem 1.20. Let \( a \leq x_0 < \cdots < x_n \leq b \), and let \( f \in C^{n+1}([a, b]) \). Let \( p \in \mathbb{R}_n[x] \) be such that \( f(x_i) = p(x_i) \) for all \( i \). Then, for all \( x \in [a, b] \), there exists \( \xi_x \in (a, b) \) such that
\[
f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} W(x), \quad W(x) = \prod_{i=0}^{n} (x - x_i).
\]
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Theorem 1.21. Let $n \in \mathbb{N}, n \neq 0$. The minimum value of the set

$$ \left\{ \max_{x \in [-1,1]} |p(x)| : p \in \mathbb{R}_n[x], \text{lc}(p) = 1 \right\} $$

is uniquely attained for $T_n/2^{n-1}$ and is therefore equal to $2^{-n+1}$.

Proof. We have

$$ A_n = \left\{ \max_{x \in [-1,1]} |p(x)| : p \in \mathbb{R}_n[x], \text{lc}(p) = 1 \right\} = \left\{ \max_{x \in [-1,1]} |x^n - q(x)| : q \in \mathbb{R}_{n-1}[x] \right\}. $$

Hence, minimizing $A_n$ is equivalent to determining $E_{n-1}(x^n)$. We deduce from (1.6) that the leading coefficient of $T_n$ is $2^{n-1}$. Moreover, $\|T_n\|_\infty \leq 1$ and $T_n \cos \left( \frac{(n-k)n}{n+1} \right) = (-1)^{n-k}$ for $k = 0, \ldots, n$. We can now apply Theorem 1.11 and conclude that $T_n/2^{n-1} - x^n$ is the minimax approximation of degree at most $n - 1$ to $x^n$ and so $A_n = E_{n-1}(x^n) = \|T_n/2^{n-1}\|_\infty = 2^{1-n}$. 

Forcing $W(x) = 2^{-n}T_{n+1}(x)$ leads to the interpolation points

$$ \mu_k = \cos \left( \frac{(2k+1)\pi}{2n+1} \right), \quad k = 0, \ldots, n, $$
called the Chebyshev nodes of the first kind.

Another important family is that of Chebyshev polynomials of the second kind $U_n(x)$, defined by

$$ U_n(\cos x) = \frac{\sin((n + 1)x)}{\sin x}. $$

They can also be defined by

$$ U_0(x) = 1, U_1(x) = 2x, U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), \forall n \in \mathbb{N}. $$

For all $n \geq 0$, we have $\frac{d}{dx} U_n = n U_{n-1}$. So the extrema of $T_{n+1}$ are $nU_{n-1}$. The zeros of $U_n$, that is,

$$ \nu_k = \cos \left( \frac{k\pi}{n} \right), \quad k = 0, \ldots, n, $$
called the Chebyshev nodes of the second kind. With $W(x) = 2^{-n+1}(1 - x^2)U_{n-1}(x)$, we have $\|W\|_\infty \leq 2^{-n+1}$.

It is obvious that $\deg T_n = \deg U_n = n$ for all $n \in \mathbb{N}$. Therefore, in particular, the family $(T_k)_{0 \leq k \leq n}$ is a basis of $\mathbb{R}_n[x]$. In the sequel of the chapter, we give results that allow for the (fast) computation of the coefficients of interpolation polynomials, at the Chebyshev nodes, expressed in the basis $(T_k)_{0 \leq k \leq n}$.

Let $\sum \gamma$ denote a sum such that the first and the last terms of the sum have to be halved.
1.5. Clenshaw’s method for Chebyshev sums

**Proposition 1.22.** (Discrete orthogonality.) Let \( j, \ell \in \{0, \ldots, n\} \).

i. We have
\[
\sum_{k=0}^{n} T_j(\mu_k)T_\ell(\mu_k) = \begin{cases} 
0, & j \neq \ell, \\
1, & j = \ell = 0, \\
\frac{n+1}{2}, & j = \ell \neq 0.
\end{cases}
\]

ii. We have
\[
\sum_{k=0}^{n} T_j(\nu_k)T_\ell(\nu_k) = \begin{cases} 
0, & j \neq \ell, \\
n, & j = \ell \in \{0, n\}, \\
\frac{n}{2}, & j = \ell \notin \{0, n\}.
\end{cases}
\]

**Exercise 1.4.2.** Prove the previous proposition.

The discrete orthogonality property implies the following (\( \sum' \) denotes that the first term of the sum has to be halved).

**Proposition 1.23.** i. If \( p_{1,n} = \sum_{0 \leq j \leq n} c_{1,j} T_j(x) \in \mathbb{R}_n[x] \) interpolates \( f \) on the set \( \{ \mu_k : 0 \leq k \leq n \} \), then
\[
c_{1,j} = \frac{2}{n+1} \sum_{k=0}^{n} f(\mu_k) T_j(\mu_k) \text{ for } j = 0, \ldots, n.
\]

ii. Likewise, if \( p_{2,n} = \sum_{0 \leq j \leq n} c_{2,j} T_j(x) \) interpolates \( f \) at \( \{ \nu_k : 0 \leq k \leq n \} \), then
\[
c_{2,j} = \frac{2}{n} \sum_{k=0}^{n} f(\nu_k) T_j(\nu_k) \text{ for } j = 0, \ldots, n.
\]

**Proof.** Exercise. \( \square \)

### 1.5 Clenshaw’s method for Chebyshev sums

Given coefficients \( c_0, \ldots, c_N \) and a point \( t \), we would like to compute the sum
\[
\sum_{k=0}^{N} c_k T_k(t).
\]

Recall that the polynomials \( T_k \) satisfy \( T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x) \). A first idea would be to use this relation to compute the \( T_k(t) \) that appear in the sum. Unfortunately, this method is numerically unstable. This is related to the fact that the \( U_k(x) \) satisfy the same recurrence but grow faster: we have
\[
\|T_k\|_\infty = 1, \quad \|U_k\|_\infty = k + 1.
\]

Clenshaw’s algorithm below does better.

**Algorithm 2** Clenshaw’s evaluation scheme

**Input:** Chebyshev coefficients \( c_0, \ldots, c_n \), a point \( t \in [-1, 1] \)

**Output:** \( \sum_{k=0}^{n} c_k T_k(t) \)

1: \( b_{n+1} \leftarrow 0, b_n \leftarrow c_n \)
2: \( \text{for } k = n - 1, n - 2, \ldots, 1 \text{ do} \)
3: \( b_k \leftarrow 2tb_{k+1} - b_{k+2} + c_k \)
4: \( \text{end for} \)
5: return \( c_0 + tb_1 - b_2 \)
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The sum simplifies to

$$\sum_{k=0}^{N} c_k T_k(t) = c_0 + (b_1 - 2tb_2 + b_3) T_1(t) + \cdots + (b_{N-1} - 2tb_N + b_{N+1}) T_{N-1}(t) + c_N T_N(t).$$

Proof. By definition of the \(b_k\), we have

$$\sum_{k=0}^{N} c_k T_k(t) = c_0 + (b_1 - 2tb_2 + b_3) T_1(t) + \cdots + (b_{N-1} - 2tb_N + b_{N+1}) T_{N-1}(t) + c_N T_N(t).$$

The sum simplifies to \(c_0 + b_1 t + b_2 (T_2 - 2tT_1)\) using the recurrence relation and the values of \(b_N, b_{N+1}\). □

This algorithm runs in \(O(N)\) arithmetic operations.

1.6 Computation of the coefficients of the interpolants at Chebyshev nodes

Now, how do we compute the \(c_{1,k}\) and \(c_{2,k}\)? First, we introduce three functions, fundamental in the field of Signal Processing [Oppenheim and Schafer, 2010]: let \(M \in \mathbb{N}\),

- let \(\omega = e^{-2\pi/M}\) a \(M\)-th primitive root of unity, the Discrete Fourier Transform (DFT) is the map \((x_0 \cdots x_{M-1}) \in \mathbb{C}^M \mapsto (X_0 \cdots X_{M-1}) \in \mathbb{C}^M\) defined by:

$$X_j = \sum_{k=0}^{M-1} x_k \omega^{jk} = \sum_{k=0}^{M-1} x_k e^{-2\pi ijk/M} \text{ for } j = 0, \ldots, M - 1.$$

The DFT sends the coefficient vector of a polynomial \(P(Y) = \sum_{n=0}^{M-1} x_n Y^n\) to its values \(P(1), P(\omega), \ldots, P(\omega^{M-1})\).

- the type I Discrete Cosine Transform (DCT-I) is the map \((x_0 \cdots x_{M-1}) \in \mathbb{R}^M \mapsto (X_0 \cdots X_{M-1}) \in \mathbb{R}^M\) defined by:

$$X_j = \sum_{k=0}^{M-1} x_k \cos \left( jk \frac{\pi}{M-1} \right) \text{ for } j = 0, \ldots, M - 1.$$

- the type II Discrete Cosine Transform (DCT-II) is the map \((x_0 \cdots x_{M-1}) \in \mathbb{R}^M \mapsto (X_0 \cdots X_{M-1}) \in \mathbb{R}^M\) defined by:

$$X_j = \sum_{k=0}^{M-1} x_k \cos \left( (j(k + 1/2)) \frac{\pi}{M} \right) \text{ for } j = 0, \ldots, M - 1.$$

Note that DCT-I and DCT-II can be expressed in function of the DFT. For instance, a DCT-I of length \(M\) can be computed thanks to a DFT of length \(2M - 2\): for \(j = 0, \ldots, M - 1\),

$$X_j = \sum_{k=0}^{M-1} x_k \cos \left( jk \frac{\pi}{M-1} \right) = \sum_{k=0}^{M-1} x_k \frac{e^{jk \pi/(M-1)} + e^{-jk \pi/(M-1)}}{2} = \sum_{k=0}^{M-1} x_k e^{jk \pi/(2M-1)} = \frac{1}{2} \sum_{k=0}^{2M-3} x_{\min(2(M-1) - k, k)} e^{-jk \pi/(2M-1)} = \frac{1}{2} \sum_{k=0}^{2M-3} x_{\min(2(M-1) - k, k)} e^{2jk \pi/(2M-1)},$$

$$= \frac{1}{2} \text{DFT} \left( (x_{\min(2(M-1) - k, k)})_{k=0,\ldots,2M-3} \right).$$
1.6. Computation of the coefficients of the interpolants at Chebyshev nodes

Likewise, a DCT-II of length $M$ can be computed thanks to a DFT of length $4M$.

The Fast Fourier Transform was introduced in 1965 by Cooley and Tukey [Cooley and Tukey, 1965, Duhamel and Vetterli, 1990, Loan, 1992], but can be traced back to Gauss [Heidemann et al., 1984]. We now briefly recall its operation [von zur Gathen and Gerhard, 2013]. Assume that $M = 2m$ is even, then $\omega^m = -1$. Rewrite $P(Y) = \sum_{k=0}^{M-1} x_n Y^n$ as

$$P(Y) = Q_0(Y)(Y^m - 1) + R_0(Y) = Q_1(Y)(Y^m + 1) + R_1(Y)$$

with $\deg R_0, \deg R_1 < m$. More precisely,

$$R_0(Y) = \sum_{j=0}^{m-1} (x_j + x_{j+m}) Y^j$$
$$R_1(Y) = \sum_{j=0}^{m-1} (x_j - x_{j+m}) Y^j.$$

Then $P(\omega^\ell) = R_0(\omega^\ell)$ if $\ell$ is even and $P(\omega^\ell) = R_1(\omega^\ell)$ if $\ell$ is odd. Therefore, if $R_1^*(Y)$ denotes the polynomial $R_1(\omega Y) = \sum_{j=0}^{m-1} (x_j - x_{j+m}) Y^j$, evaluating $P$ at $1, \omega, \ldots, \omega^{M-1}$ reduces to evaluating $R_0$ and $R_1^*$ at $1, \omega^2, (\omega^2)^2, \ldots, (\omega^2)^{m-1}$. If we apply this recursively, it leads to a number of operations in $O(M \log M)$, which yields a similar estimate for the computations of DCT-I and DCT-II.

Now, let’s rewrite Proposition 1.23 the following way:

i. If $p_1,n = \sum_{0 \leq j \leq n} c_{1,j} T_j \in \mathbb{R}_n[x]$ interpolates $f$ on the set $\{\mu_k : 0 \leq k \leq n\}$, then, for $j = 0, \ldots, n$,

$$c_{1,j} = \frac{2}{n+1} \sum_{k=0}^{n} f(\mu_k) T_j \left( \cos \left( \frac{(k+1/2)\pi}{n + 1} \right) \right)$$
$$= \frac{2}{n+1} \sum_{k=0}^{n} f(\mu_k) \cos \left( \frac{j(k+1/2)\pi}{n + 1} \right)$$
$$= \frac{2}{n+1} \text{DCT-II}(f(\mu_k)_{k=0,\ldots,n}).$$

ii. Likewise, if $p_2,n = \sum_{0 \leq j \leq n} c_{2,j} T_j \in \mathbb{R}_n[x]$ interpolates $f$ at $\{\nu_k : 0 \leq k \leq n\}$, then, for $j = 0, \ldots, n$,

$$c_{2,j} = \frac{2}{n} \sum_{k=0}^{n} f(\nu_k) T_j \left( \cos \left( \frac{k\pi}{n} \right) \right)$$
$$= \frac{2}{n} \sum_{k=0}^{n} f(\nu_k) \cos \left( \frac{jk\pi}{n} \right)$$
$$= \frac{2}{n} \text{DCT-I}(f(\nu_k)_{k=0,\ldots,n}).$$

Thus, we conclude that, if we already have the $f(\mu_k)$s, resp. the $f(\nu_k)$s, we can compute these coefficients in $O(n \log n)$ operations.
Chapter 2

Orthogonal polynomials - Chebyshev series

2.1 Orthogonal polynomials

Let \((a, b) \subset \mathbb{R}\) be an open interval (note that, in this section, it does not need to be bounded), and let \(w\) be a weight function, that is to say \(w : (a, b) \to (0, \infty)\) is a continuous function (this last hypothesis is not strictly necessary, we use it for ease of presentation). We assume

\[
\forall n \in \mathbb{N}, \quad \int_a^b |x|^n w(x) dx < \infty.
\]

This is the case, for instance, if \((a, b)\) is bounded and

\[
\int_a^b w(x) dx < \infty.
\]

Let

\[
E(w) = \left\{ f \in \mathcal{C}((a, b)) : \|f\|_2 := \left( \int_a^b f(x)^2 w(x) dx \right)^{1/2} < \infty \right\}.
\]

Observe that \(\mathbb{R}[x] \subset E(w)\). The space \(E(w)\) is equipped with an inner product

\[
\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx;
\]

and \(\| \cdot \|_2\) is the norm associated to this inner product.

Definition 2.1. A family of orthogonal polynomials associated with \(w\) is a sequence \((p_n) \in \mathbb{R}[x]^\mathbb{N}\) where \(\deg p_k = k\) for all \(k \in \mathbb{N}\), and

\[
i \neq j \quad \Rightarrow \quad \langle p_i, p_j \rangle = 0.
\]

Theorem 2.2. For any weight \(w\), there exists a family of orthogonal polynomials associated with \(w\). If additionally we request that the \(p_k\) are all monic, this family is unique.

Proof. We use Gram-Schmidt orthogonalization. Starting with \(p_0 = 1\), we iteratively construct polynomials \(p_k\) obeying the three conditions:

- \(p_k\) is monic;
- \(\text{Span}_{\mathbb{R}} \{p_0, \ldots, p_k\} = \mathbb{R}_k[x]\);
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- \( p_k \in \text{Span}_\mathbb{R} \{ p_0, \ldots, p_{k-1} \} \).

In view of the first two conditions, the polynomial \( p_k \) is necessarily of the form

\[
p_k = x^k + \sum_{j=0}^{k-1} \lambda_{k,j} p_j, \quad \lambda_{k,j} \in \mathbb{R}.
\]

The third condition above is equivalent to the system

\[
0 = \langle p_k, p_j \rangle = \langle x^k, p_j \rangle + \lambda_{k,j} \| p_j \|_2^2, \quad j = 0, \ldots, k - 1.
\]

The unique solution to this system is

\[
\lambda_{k,j} = -\frac{\langle x^k, p_j \rangle}{\| p_j \|_2^2}.
\]

Thus uniqueness is established and the polynomial \( p_k \) thus constructed has the required properties. \( \square \)

The following statement gives us a way to recursively compute a sequence of orthogonal polynomials. Note also that if you adapt Clenshaw’s method (cf. 1.5) to this recurrence, it also yields an evaluation scheme in linear time for polynomials expressed in the corresponding basis of orthogonal polynomials.

**Theorem 2.3.** The polynomials \( (p_n)_{n \in \mathbb{N}} \) satisfy the recurrence relation

\[
p_n(x) = (x - \alpha_n)p_{n-1}(x) - \beta_n p_{n-2}(x) \quad (n \geq 2)
\]

with

\[
\alpha_n = \frac{\langle xp_{n-1}, p_{n-1} \rangle}{\| p_{n-1} \|_2^2}, \quad \beta_n = \frac{\| p_{n-1} \|_2^2}{\| p_{n-2} \|_2^2}.
\]

**Proof.** Let \( n \geq 2 \). The polynomial \( xp_{n-1} \) is monic and has degree \( n \), hence

\[
xp_{n-1} = p_n + \sum_{k=0}^{n-1} a_k p_k.
\]

The orthogonality of the \( p_n \)’s gives \( a_k = \frac{\langle xp_{n-1}, p_k \rangle}{\| p_k \|_2^2} \) for \( k = 0, \ldots, n - 1 \). If we notice that \( \langle xp_{n-1}, p_k \rangle = \langle p_{n-1}, xp_k \rangle \) for all \( k = 0, \ldots, n - 1 \), we obtain \( a_k = 0 \) if \( k \leq n - 3 \) since \( xp_k \in \mathbb{R}_{n-2}[x] \) and \( p_{n-1} \in (\mathbb{R}_{n-2}[x])^\perp \). Hence, there are at most two nonzero coefficients:

\[
a_{n-1} = \frac{\langle p_{n-1}, p_{n-1} \rangle}{\| p_{n-1} \|_2^2} = \alpha_n, \tag{2.1}
\]

\[
a_{n-2} = \frac{\langle p_{n-1}, p_{n-2} \rangle}{\| p_{n-2} \|_2^2} = \frac{\langle p_{n-1}, p_{n-1} + q \rangle}{\| p_{n-2} \|_2^2} \quad \text{with } q \in \mathbb{R}_{n-2}[x] \tag{2.2}
\]

\[
a_{n-2} = \frac{\langle p_{n-1}, p_{n-1} \rangle}{\| p_{n-2} \|_2^2} = \beta_n. \tag{2.3}
\]

\( \square \)

**Example 2.4.**

\[
\begin{align*}
(1, 0) & \quad w(x) = (1 - x^2)^{-1/2} & \text{Chebyshev polynomials of the first kind (up to normalization)} \\
(-1, 1) & \quad w(x) = 1 & \text{Legendre polynomials} \\
(0, +\infty) & \quad w(x) = e^{-x} & \text{Laguerre polynomials} \\
(-\infty, \infty) & \quad w(x) = e^{-x^2} & \text{Hermite polynomials}
\end{align*}
\]

**Exercise 2.1.1.** Prove that the first statement of Example 2.4 is correct.

**Theorem 2.5.** For any weight \( w \) and for all \( n \), the polynomial \( p_n \) has \( n \) distinct zeros in \( (a, b) \).
2.1. Orthogonal polynomials

Proof. Fix \( n \). Let \( x_1, \ldots, x_k \) be the distinct zeros of \( p_n \) in \((a, b)\), with respective multiplicities \( m_1, \ldots, m_k \). We introduce the polynomial

\[
q(x) = \prod_{j=1}^{k} (x - x_j)^{m_j \mod 2}.
\]

If \( k < n \), we have \( \deg q \leq k < n \), and hence

\[
\langle q, p_n \rangle = \int_a^b p_n(x)q(x)w(x)dx = 0,
\]

but the integrand is strictly positive over \((a, b)\setminus\{x_1, \ldots, x_k\} \): contradiction. \( \square \)

**Theorem 2.6.** Let \( f \in \mathcal{E}(w), n \in \mathbb{N} \). There exists a unique best \( L_2(w) \) polynomial approximation to \( f \) in \( \mathbb{R}_n[x] \), denoted \( p_{2,n} \):

\[
\|f - p_{2,n}\|_2 = \min_{p \in \mathbb{R}_n[x]} \|f - p\|_2.
\]

It is characterized by

\[
\forall p \in \mathbb{R}_n[x], \quad \langle f - p_{2,n}, p \rangle = 0.
\]

**Exercise 2.1.2.** Prove this theorem.

**Remark 2.7.** We have \( p_{2,n} = \sum_{k=0}^{n} \frac{\langle p_k, f \rangle}{\|p_k\|_2^2} p_k \).

**Theorem 2.8.** If \((a, b)\) is bounded, then for all \( f \in \mathcal{E}(w) \), we have

\[
p_{2,n} \xrightarrow{\|\cdot\|_2} f
\]

as \( n \to \infty \).

**Proof.** First assume that \( f \in \mathcal{C}([a, b]) \). Let \( p_n^* \) be the minimax degree-\( n \) approximation to \( f \): then

\[
\|f - p_{2,n}\|_2 \leq \|f - p_n^*\|_2 = \left( \int_a^b (f - p_n^*)^2 w(x)dx \right)^{1/2} \leq E_n(f) \left( \int_a^b w(x)dx \right)^{1/2}
\]

but we already know that \( E_n(f) \to 0 \) as \( n \to \infty \).

For general \( f \), for all \( \alpha > 0 \), let

\[
\varphi_\alpha : [a, b] \to [0, 1] = \begin{cases} \frac{2}{\alpha} (x - a - \frac{\alpha}{2}) & \text{if } x \in [a + \alpha/2, a + \alpha], \\ \frac{2}{\alpha} (b - x - \frac{\alpha}{2}) & \text{if } x \in [b - \alpha/2, b - \alpha], \\ 1 & \text{if } x \in [a + \alpha, b - \alpha]. \end{cases}
\]

defined more precisely by

\[
\varphi_\alpha(x) = \begin{cases} 0 & \text{if } x \in [a, a + \alpha/2] \cup [b - \alpha/2, b], \\ \frac{2}{\alpha} (x - a - \frac{\alpha}{2}) & \text{if } x \in [a + \alpha/2, a + \alpha], \\ \frac{2}{\alpha} (b - x - \frac{\alpha}{2}) & \text{if } x \in [b - \alpha/2, b - \alpha], \\ 1 & \text{if } x \in [a + \alpha, b - \alpha]. \end{cases}
\]

We have \( f \varphi_\alpha \in \mathcal{C}([a, b]) \) if we assume \( f \varphi_\alpha(a) = f \varphi_\alpha(b) = 0 \). For almost all \( x \in [a, b] \), we have

\[
|f(x) - (f \varphi_\alpha)(x)| \leq |f(x)| 1_{[a,a+\alpha] \cup [b-\alpha,b]}(x),
\]

where \( 1_{[a,a+\alpha] \cup [b-\alpha,b]} \) denotes the indicator function of the set \([a, a + \alpha] \cup [b - \alpha, b]\). Hence, for almost all \( x \in [a, b] \),
• \( \lim_{\alpha \to 0} |f(x) - (f \varphi_\alpha)(x)| = 0 \),
• \( |f(x) - (f \varphi_\alpha)(x)| \leq |f(x)| \), with \( f \in L^2([a,b], w) \).

It follows from Lebesgue’s dominated convergence theorem that
\[
\int_a^b |f(x) - (f \varphi_\alpha)(x)|^2 \, dx \xrightarrow{\alpha \to 0} 0.
\]

Denoting by \( p_{2,n}^{(\alpha)} \) the best \( L_2(w) \) degree-\( n \) approximation to \( f \varphi_\alpha \), we have
\[
\|f - p_{2,n}\|_2 \leq \|f - p_{2,n}^{(\alpha)}\|_2 \leq \|f - f \varphi_\alpha\|_2 + \|f \varphi_\alpha - p_{2,n}^{(\alpha)}\|_2
\]
for all \( n \) and \( \alpha \). Let \( \varepsilon > 0 \), there exists \( \alpha > 0 \) such that \( \|f - f \varphi_\alpha\|_2 < \varepsilon \). For this \( \alpha \), there exists \( n_0 \in \mathbb{N} \) such that \( \|f \varphi_\alpha - p_{2,n}^{(\alpha)}\|_2 < \varepsilon \) for all \( n \in \mathbb{N}, n \geq n_0 \).

**Remark 2.9.** The previous statement can be wrong if one does not assume that \((a,b)\) is bounded. Can you give a counter-example?

Note that, from Remark 2.7, the computation of the coefficients of the best approximations in the basis of orthogonal polynomials seems to require the evaluation of several integrals. Hence, this kind of polynomials approximation is often significantly more expensive than the approach via interpolation polynomials.

### 2.2 A little bit of quadrature: Gauss methods

Let \( w \) be a weight function over \((a,b)\), and let \( f \in C((a,b)) \). We briefly study methods which approximate the integral
\[
\int_a^b f(x)w(x) \, dx
\]
with a sum of the form
\[
\sum_{k=0}^n w_k f(x_k), \quad w_k \in \mathbb{R}, \quad x_k \in [a,b] \text{ pairwise distinct. (2.4)}
\]

First of all, if \( \ell_k(x) = \prod_{0 \leq j \leq n, j \neq k} \frac{x-x_j}{x_k-x_j} \), observe that if
\[
p(x) = \sum_{k=0}^n f(x_k)\ell_k(x) \in \mathbb{R}_n[x]
\]
interpolates \( f \) at the points \( x_0, \ldots, x_n \), then our approximation for the integral is equal to \( \int_a^b p(x)w(x) \, dx = \sum_{k=0}^n w_k f(x_k) \) with
\[
w_k = \int_a^b \ell_k(x)w(x) \, dx \text{ for } k = 0, \ldots, n.
\]

Thus we obtain an approximation of the integral that is exact at least for polynomials of degree up to \( n \). It is possible to obtain a much better result if one is allowed to choose the points \( x_0, \ldots, x_n \):

**Theorem 2.10.** There exists a unique choice of the points \( x_k \) and the weights \( w_k \) such that, whenever \( f \in \mathbb{R}_{2n+1}[x] \), the formula (2.4) is exact in the sense that
\[
\int_a^b f(x)w(x) \, dx = \sum_{k=0}^n w_k f(x_k).
\]

These points \( x_k \) belong to \((a,b)\) and are the roots of the \((n+1)\)-th orthogonal polynomial associated to \( w \).
2.3. Lebesgue constants

Proof. We start with the uniqueness. Assume that \( x_j, w_j \) are such that the method is exact for any \( f \in \mathbb{R}_n[x] \), \( m \leq 2n + 1 \). Set

\[
\pi_{n+1}(x) = \prod_{j=0}^{n}(x - x_j).
\]

For all \( p \in \mathbb{R}_n[x] \), we have \( \deg(p\pi_{n+1}) \leq 2n + 1 \). Hence

\[
\langle p, \pi_{n+1} \rangle = \int_a^b p(x)\pi_{n+1}(x)w(x)dx = \sum_{k=0}^{n} p(x_k)\pi_{n+1}(x_k)w_k = 0.
\]

The polynomial \( \pi_{n+1} \) is monic and belongs to \( (\mathbb{R}_n[x])^\perp \): it is the \( (n + 1) \)-th orthogonal polynomial associated to \( w \). The \( x_k \) are its roots and, as noted above, \( w_k = \sum_{k=0}^{n} w_k\ell_k(x_k) = \int_a^b \ell_k(x)w(x)dx \).

As for the existence, let \( x_0, \ldots, x_n \) be the distinct roots in \((a, b)\) of the \((n + 1)\)-th orthogonal polynomial (cf. Proposition 2.5), and let \( w_k = \int_a^b \ell_k(x)w(x)dx \) where \( \ell_k \) is the corresponding \( k \)-th Lagrange polynomial. Clearly the method is exact if \( f \in \mathbb{R}_n[x] \). If now \( f \in \mathbb{R}_{2n+1}[x] \), write

\[
f = q\pi_{n+1} + r, \quad \deg r \leq n.
\]

As \( \pi_{n+1} \in \mathbb{R}_n[x]^\perp \) et \( \deg q \leq n \), we have \( \int_a^b q(x)\pi_{n+1}(x)w(x)dx = 0 \). It follows that

\[
\int_a^b f(x)w(x)dx = \int_a^b r(x)w(x)dx = \sum_{k=0}^{n} w_k r(x_k) = \sum_{k=0}^{n} w_k f(x_k).
\]

\[\square\]

See Chapter 19 of [Trefethen, 2013] for an interesting and up-to-date account on Gauss methods. Note that recent works [Hale and Townsend, 2013, Bogaert, 2014, Johansson and Mezzarobba, 2018] showed that the weights and the nodes for Gauss-Legendre or Gauss-Chebyshev quadrature, for instance, can be computed in \( O(n) \) operations.

Remark 2.11. When \( w = 1 \), an alternative to Gauss quadrature with Legendre points is the so-called Clenshaw-Curtis quadrature, which uses Chebyshev points as interpolation nodes. The Chebyshev polynomials of the first kind satisfy

\[
\int_{-1}^{1} T_k(x)dx = \begin{cases} 
\frac{2}{2k^2}, & k \in 2N, \\
0, & k \not\in 2N.
\end{cases}
\]

Hence, if \( p = \sum_{k=0}^{n} c_k T_k \) is the interpolation polynomial of \( f \), we deduce that the integral with weight \( w = 1 \) of \( f \) is approximated by

\[
\int_{-1}^{1} p(x)dx = \sum_{0 \leq k \leq n, \ k \in 2N} \frac{2c_k}{1 - k^2}.
\]

Since the coefficients \( c_k \) can be computed in \( O(n \log n) \) arithmetic operations using the FFT, this yields a complexity in \( O(n \log n) \) for the computation of the quadrature approximant.

2.3 Lebesgue constants

For simplicity, we assume \([a, b] = [-1, 1]\).

Definition 2.12. We say that a linear mapping \( L : C([-1, 1]) \rightarrow \mathbb{R}_n[x] \) is a projection onto \( \mathbb{R}_n[x] \) if \( Lp = p \) for all \( p \in \mathbb{R}_n[x] \). The operator norm

\[
\Lambda = \sup_{f \in C([-1,1])} \frac{\|Lf\|_{\infty}}{\|f\|_{\infty}}
\]

is called the Lebesgue constant for the projection.
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Additionally, on the other hand, since the function \( f \in C([-1, 1]) \) and let \( p = Lf \). Let \( p^* \) denote the minimax approximation to \( f \). Then, we have

\[
\|f - p\|_\infty \leq (1 + \Lambda)\|f - p^*\|_\infty.
\]

Proof. We have \( L(f - p^*) = p - p^* \). It follows that

\[
\|p - f\|_\infty - \|f - p^*\|_\infty \leq \|p - p^*\|_\infty = \|L(f - p^*)\|_\infty \leq \Lambda \|f - p^*\|_\infty.
\]

\( \square \)

2.3.1 Lebesgue constants for polynomial interpolation

Let \( x_0, \ldots, x_n \) be pairwise distinct points in \([-1, 1]\). Consider the Lagrange interpolation operator

\[
L_n : C([-1, 1]) \to \mathbb{R}_n[x], \quad L_n f(x) = \sum_{k=0}^n f(x_k) \ell_k(x).
\]

Clearly, \( L_n \) is a linear projection of \( C([-1, 1]) \) onto \( \mathbb{R}_n[x] \). On the one hand, we have

\[
|L_n f(x)| \leq \|f\|_\infty \sum_{k=0}^n |\ell_k(x)|, \quad \text{for all } x \in [-1, 1],
\]

which implies that the corresponding Lebesgue constant \( \Lambda_n = \|L_n\| \) satisfies

\[
\Lambda_n \leq A := \max_{x \in [-1, 1]} \sum_{k=0}^n |\ell_k(x)|.
\]

On the other hand, since the function \( x \in [-1, 1] \mapsto \sum_{k=0}^n |\ell_k(x)| \) is continuous, there exists \( \xi \in [-1, 1] \) such that \( A = \sum_{k=0}^n |\ell_k(\xi)| \). Let \( g : [-1, 1] \to [-1, 1] \) be a continuous piecewise affine function such that \( g(x_i) = \text{sgn} \ell_i(\xi) \). Then, we have

\[
L_n g(\xi) = \sum_{k=0}^n |\ell_k(\xi)|
\]

and hence \( \|L_n g\|_\infty \geq A \|g\|_\infty \). We’ve just proved the following statement.

Theorem 2.14. The Lebesgue constant of degree-\( n \) Lagrange interpolation at \( x_0, \ldots, x_n \) is equal to

\[
\max_{x \in [-1, 1]} \sum_{k=0}^n |\ell_k(x)|.
\]

Theorem 2.15. The Lebesgue constant \( \Lambda_n \) satisfies

\[
\frac{2}{\pi} \left( \log(n + 1) + \gamma + \log \frac{4}{\pi} \right) \leq \Lambda_n, \quad \text{where } \frac{2}{\pi} \left( \gamma + \log \frac{4}{\pi} \right) = 0.52125 \ldots \quad (2.5)
\]

Additionally,

- for Chebyshev nodes (of the first and the second kinds), we have the bound

\[
\Lambda_n \leq \frac{2}{\pi} \log(n + 1) + 1 \text{ and } \Lambda_n \sim \frac{2}{\pi} \log n \text{ as } n \to +\infty \quad (2.6)
\]

- for equispaced points,

\[
\Lambda_n > \frac{2^{n-2}}{n^2} \text{ and } \Lambda_n \sim \frac{2^{n+1}}{en \log n} \text{ as } n \to +\infty. \quad (2.7)
\]
Proof. See Chapter 15 of [Trefethen, 2013] for a commented bibliography of the proofs of these results and [Brutman, 1997] for a detailed survey on this subject.

For a proof of (2.5), see [Brutman, 1978]. See [Ehlich and Zeller, 1966] for a proof of (2.6). A proof of the left estimate of (2.7) can be found in [Trefethen and Weideman, 1991]. Two proofs of the right estimate of (2.7) were independently published in [Turetskii, 1940] and [Schönhage, 1961].

Remark 2.16. We deduce from this theorem that Chebyshev interpolants (i.e. interpolation polynomials at Chebyshev nodes) are "near-best" approximations:

- \( \Lambda_{15} = 2.76 \ldots \): one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;
- \( \Lambda_{30} = 3.18 \ldots \): one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;
- \( \Lambda_{100} = 3.93 \ldots \): one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;
- \( \Lambda_{100000} = 8.32 \ldots \): one loses at most 4 bits if one uses a Chebyshev interpolant instead of the minimax polynomial.

Remark 2.17. The estimates (2.5) imply \( \sup_{n \in \mathbb{N}} \Lambda_n = +\infty \). We then deduce from Banach-Steinhaus theorem [Brezis, 2010] a proof of Theorem 1.19 (Faber’s Theorem).

2.3.2 Lebesgue constants for \( L_2 \) best approximation

Definition 2.18. When the space under consideration is \( E \left( \frac{1}{\sqrt{1-x^2}} \right) \), the best \( L_2 \) polynomial approximation to \( f \in \mathbb{R}_n[x] \) is called the truncated Chebyshev expansion of \( f \) of order \( n \) and is denoted \( f_n \). Its coefficients \( a_k \) are called the Chebyshev coefficients of \( f \). They are given by

\[
    a_k = \begin{cases} 
        \langle f, T_k \rangle / \langle T_k, T_k \rangle = \frac{2}{\pi} \int_{-1}^{1} f(x) T_k(x) \sqrt{1-x^2} dx, & k \neq 0, \\
        \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx, & k = 0, 
    \end{cases}
\]

and the formal series \( \sum_{k=0}^{\infty} a_k T_k(x) \) is called the Chebyshev expansion of \( f \).

Remark 2.19. The Chebyshev expansion of \( f \) is the Fourier expansion of \( f(\cos t) \), so that many results on the convergence of Chebyshev expansions can be deduced from corresponding results in the well-developed theory of Fourier series.

Theorem 2.20. The Lebesgue constant for the map \( f \in E \left( \frac{1}{\sqrt{1-x^2}} \right) \mapsto p_{2,n} \) is

\[
    \Lambda_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt. \tag{2.8}
\]

Its behaviour obeys

\[
    \frac{4}{\pi^2} \log(n+1) < \Lambda_n \begin{cases} 
        = 1 & \text{if } n = 0, \\
        < 2 & \text{if } n = 1, \\
        < \frac{4}{\pi^2} \log(n-1) + 3 & \text{otherwise.} 
    \end{cases} \tag{2.9}
\]

Proof. For \( f \in E \left( \frac{1}{\sqrt{1-x^2}} \right) \), \( n \in \mathbb{N} \), \( x \in [-1, 1] \), we have

\[
    S_n f(x) := \sum_{k=0}^{n} a_k T_k(x) = \frac{2}{\sqrt{1-y^2}} \int_{-1}^{1} \frac{f(y)}{\sqrt{1-y^2}} \left( \frac{T_0(x)T_0(y)}{2} + \sum_{k=1}^{n} T_k(x)T_k(y) \right) dy.
\]
If we put $x = \cos(\theta)$ and $y = \cos(u)$, it comes

$$S_n f(x) = \frac{2}{\pi} \int_0^\pi f(\cos u) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos(k \theta) \cos(k u) \right) du$$

$$= \frac{1}{\pi} \int_0^{\pi} f(\cos u) \left( 1 + \sum_{k=1}^{n} \cos(k(u + \theta)) + \cos(k(u - \theta)) \right) du$$

$$= \frac{1}{\pi} \int_{-\theta}^{\pi - \theta} f(\cos(v + \theta)) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos(kv) \right) dv \quad (\text{we put } v = u + \theta)$$

$$+ \frac{1}{\pi} \int_{\theta - \pi}^{\theta} f(\cos(v + \theta)) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos(kv) \right) dv \quad (\text{we put } v = -u + \theta)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos(v + \theta)) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos(kv) \right) dv \quad (\text{the integrand is even and } 2\pi\text{-periodic}).$$

Now, we use the fact that

$$\sin((n + 1/2)v) = \sin(v/2) + \sum_{k=1}^{n} \left( \sin((k + 1/2)v) - \sin((k - 1/2)v) \right)$$

$$= \sin(v/2) + \sum_{k=1}^{n} 2 \sin(v/2) \cos(kv) = 2 \sin(v/2) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos(kv) \right).$$

This yields, for all $n \in \mathbb{N}$, $x \in [-1,1]$,

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\cos(v + \theta)) \frac{\sin((n+1/2)v)}{\sin(v/2)} dv,$$

from which follows

$$\|S_n f\|_\infty \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sin((n+1/2)v) \right| \frac{dv}{\sin(v/2)} \right) \|f\|_\infty.$$
follows from Lebesgue’s dominated convergence theorem that

\[ S_n(\alpha_m \circ D_n \circ \arccos)(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\alpha_m \circ D_n \circ \arccos)(\cos(v)) \frac{\sin((n + 1/2)v)}{\sin(v/2)} dv \]

Then, for \( k \)

\[ S_n(\alpha_m \circ D_n \circ \arccos)(1) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varphi_n \circ \arccos)(\cos(v)) \frac{\sin((n + 1/2)v)}{\sin(v/2)} dv = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n + 1/2)v)}{\sin(v/2)} \right| dv. \]

We’ve just proved that, for all \( \varepsilon > 0 \), there exists \( g \in \mathcal{E} \left( \frac{1}{\sqrt{1-x^2}} \right) \) such that \( \|S_n g\|_\infty > |S_n g(1)| > \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+1/2)v)}{\sin(v/2)} \right| dv - \varepsilon \right) \|g\|_\infty \), which yields \( \Lambda_n \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+1/2)v)}{\sin(v/2)} \right| dv. \)

We now prove estimates (2.9). For all \( n \in \mathbb{N} \),

\[ \Lambda_n = \frac{2}{\pi} \int_{0}^{\pi/2} \frac{\left| \sin((2n+1)t) \right|}{\sin t} dt \quad \text{(we have put } t = v/2), \]

\[ \geq \frac{2}{\pi} \int_{0}^{\pi/2} \frac{\left| \sin((2n+1)t) \right|}{\sin t} dt \quad \text{since } \sin x \leq x \text{ for } x \in [0, \pi/2]. \]

Now, we use the change of variable \( v = \pi x/(2n + 1) \) which gives

\[ \Lambda_n \geq \frac{2}{\pi} \int_{0}^{n+1/2} \frac{\left| \sin(\pi v) \right|}{v} dv > \frac{2}{\pi} \int_{0}^{n} \frac{\left| \sin(\pi v) \right|}{v} dv. \]

Since, for \( k = 0, \ldots, n - 1 \) and \( v \in [k, k+1] \), we have \( \left| \frac{\sin(\pi v)}{v} \right| \geq \left| \frac{\sin(\pi v)}{k+1} \right| \), it comes

\[ \Lambda_n \geq \frac{2}{\pi} \int_{0}^{n} \frac{\left| \sin(\pi v) \right|}{v} dv = \frac{2}{\pi} \sum_{k=0}^{n-1} \int_{k}^{k+1} \frac{\left| \sin(\pi v) \right|}{v} dv \geq \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \int_{k}^{k+1} \left| \sin(\pi v) \right| dv = \frac{2}{\pi^2} \left[ \cos(\pi v) \right]_{0}^{1} \sum_{k=0}^{n-1} \frac{1}{k+1} = \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k}. \tag{2.10} \]

Now, for \( k = 1, \ldots, n \) and \( v \in [k, k+1] \), \( \frac{1}{k+1} \leq \frac{1}{v} \leq \frac{1}{k} \) implies \( \frac{1}{k+1} \leq \int_{k}^{k+1} \frac{1}{v} dv = \log(k+1) - \log k \leq \frac{1}{k} \), hence

\[ \sum_{k=1}^{n} \frac{1}{k+1} \leq \sum_{k=1}^{n} \left( \log(k+1) - \log k \right) = \log(n+1) \leq \sum_{k=1}^{n} \frac{1}{k}. \tag{2.11} \]

Estimates (2.10) and (2.11) yield \( \Lambda_n > \frac{4}{\pi^2} \log(n+1). \)

We now prove the second inequality of (2.9). We follow [Rivlin, 1981, Chap. 3]. The case \( n = 0 \) is straightforward. Then, we assume \( n \geq 1 \). First, note that

\[ \Lambda_n = \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{\sin(n v) \cos(v/2) + \cos(n v) \sin(v/2)}{\sin(v/2)} \right| dv \]

\[ = \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{\sin(n v)}{\tan(v/2)} \right| dv \leq \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{\sin(n v)}{\tan(v/2)} \right| dv + \frac{1}{\pi} \int_{0}^{\pi} \left| \cos(n v) \right| dv. \tag{2.12} \]
Now, recall that $\tan x \geq x$ for all $x \in [0, \pi/2]$. Therefore,
\[
\int_0^\pi \frac{|\sin(nv)|}{\tan(v/2)} \, dv \leq 2 \int_0^\pi \frac{|\sin(nv)|}{v} \, dv
\]
\[
= 2 \int_0^{n\pi} \frac{|\sin u|}{u} \, du \quad \text{(we have put } u = nv) \\
= 2 \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} \, du \\
= 2 \sum_{k=0}^{n-1} \int_0^{\pi} \frac{|\sin u|}{u + k\pi} \, du \quad \text{since } u \mapsto |\sin u| \text{ is } \pi\text{-periodic}
\]
\[
\leq 2 \int_0^\pi \frac{\sin u}{u} \, du + 2 \left( \int_0^\pi \frac{\sin u}{u} \, du \right) \sum_{k=1}^{n-1} \frac{1}{k\pi} \\
\leq 2 \int_0^\pi \frac{\sin u}{u} \, du + \frac{4}{\pi} \sum_{k=1}^{n-1} \frac{1}{k} \\
\leq 2 \int_0^\pi \frac{\sin u}{u} \, du + \frac{4}{\pi} (1 + \log(n - 1)) \text{ thanks to } (2.11). \quad (2.13)
\]

For all $x \geq 0$, $m \in \mathbb{N}$, $|\sin x - \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} x^{2k+1}| \leq \frac{1}{(2m+1)!} x^{2m+1}$, hence
\[
\left| \int_0^\pi \frac{\sin u}{u} \, du - \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} \int_0^\pi x^{2k} \, du \right| \leq \frac{1}{(2m+1)!} \int_0^\pi x^{2m} \, du,
\]
i.e.
\[
\left| \int_0^\pi \frac{\sin u}{u} \, du - \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} \frac{\pi^{2k+1}}{2k+1} \right| \leq \frac{\pi^{2m+1}}{(2m+1)! (2m+1)}.
\]

If we set $m = 4$, we obtain
\[
\int_0^\pi \frac{\sin u}{u} \, du \leq 1.86. \quad (2.14)
\]

Finally, we have
\[
\int_0^\pi |\cos(nv)| \, dv = \frac{1}{n} \int_0^{n\pi} |\cos(u)| \, du \quad \text{(we have put } u = nv) \\
= \frac{1}{n} \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} |\cos(u)| \, du \\
= \frac{1}{n} \sum_{k=0}^{n-1} \int_0^\pi |\cos(u)| \, du \quad \text{since } u \mapsto |\cos(u)| \text{ is } \pi\text{-periodic} \\
= \int_0^{\pi/2} \cos(u) \, du - \int_\pi^{\pi/2} \cos(u) \, du = 2. \quad (2.15)
\]

The estimates (2.12), (2.13), (2.14) and (2.15) yield
\[
\Lambda_n \leq \frac{1}{\pi} \left( 2 \cdot 1.86 + \frac{4}{\pi} + \frac{4}{\pi} \log(n - 1) \right) + \frac{2}{\pi} < 3 + \frac{4}{\pi^2} \log(n - 1).
\]

Note that $\Lambda_1 \leq \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} \, du + \frac{1}{\pi} \int_0^\pi \cos(v) \, dv = \frac{2}{\pi}(1.86 + 1) < 2$. \hfill \square

Remark 2.21. We deduce from this theorem that truncated Chebyshev series are "near-best" approximations:
2.4. Chebyshev expansions and interpolation polynomials at Chebyshev nodes

- $\Lambda_{15} = 4.12 \ldots$: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;
- $\Lambda_{30} = 4.39 \ldots$: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;
- $\Lambda_{100} = 4.87 \ldots$: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;
- $\Lambda_{100000} = 7.66 \ldots$: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial.

Remark 2.22. The estimates (2.9) imply $\sup_{n \in \mathbb{N}} \Lambda_n = +\infty$. We then deduce from Banach-Steinhaus theorem [Brezis, 2010] that there exists $f \in C \left( \frac{1}{\sqrt{1-x^2}} \right)$ such that its Chebyshev expansion that does not uniformly converge to $f$.

2.4 Chebyshev expansions and interpolation polynomials at Chebyshev nodes

2.4.1 Convergence results, certified estimates

Here is a summary of convergence results that we are going to rely on (see Theorems 3.1, 7.1, 7.2, 8.1, 8.2 in [Trefethen, 2013] for versions with weaker hypotheses).

Theorem 2.23. Let $f$ be continuous on $[-1, 1]$. Denote by $(a_k)_{k \in \mathbb{N}}$ its sequence of Chebyshev coefficients, by $(f_n)_{n \in \mathbb{N}}$ its sequence of truncated Chebyshev expansions and by $(p_n)_{n \in \mathbb{N}}$ the sequence of interpolation polynomials of $f$ at the Chebyshev nodes. Then

1. The coefficients $a_k$ tend to 0 when $k \to \infty$.
2. If $f$ is Lipschitz continuous on $[-1, 1]$, then $(f_n)$ converges uniformly to $f$ and $(p_n)$ converges uniformly to $f$.
3. If $f$ is Lipschitz continuous on $[-1, 1]$, then $(f_n)$ converges absolutely. Consequently, it converges normally, hence uniformly, to $f$.
4. If $f$ is $C^m$ and $f^{(m)}$ is Lipschitz continuous, then $a_k = O(1/k^{m+1})$. $\|f - f_n\|_\infty = O(n^{-m})$ and $\|f - p_n\|_\infty = O(n^{-m})$.
5. If $f$ is analytic inside the ellipse $\{z \in \mathbb{C} : |z + \sqrt{z^2 - 1}| \leq r \}$ with $r > 1$, then $a_k = O(r^{-k})$, $\|f - f_n\|_\infty = O(r^{-n})$ and $\|f - p_n\|_\infty = O(r^{-n})$.
6. Let $P_n^*$ denote the minimax polynomial of degree at most $n$ of $f$. If $f \in C^{n+1}([-1, 1])$, there exists $\xi_1, \xi_2, \xi_3 \in (-1, 1)$ such that

\[
\|f - P_n^*\|_\infty = \frac{|f^{(n+1)}(\xi_1)|}{2^n(n+1)!};
\]
\[
\|f - f_n\|_\infty = \frac{|f^{(n+1)}(\xi_2)|}{2^n(n+1)!};
\]
\[
\|f - p_n\|_\infty = \frac{|f^{(n+1)}(\xi_3)|}{2^n(n+1)!}.
\]

Proof. 1. This is Riemann-Lebesgue lemma [Zygmund, 2002, Chap. II].
2. This is obtained by combining Proposition 2.13, the bounds on $\Lambda_n$ from Theorems 2.15 and 2.20 and Corollary 1.8 that states $\tilde{E}_n(f) = O(n^{-1/2})$. Thus, if $(p_n)_{n \in \mathbb{N}}$ denotes the sequence of interpolation polynomials of $f$ at the Chebyshev nodes and $(f_n)_{n \in \mathbb{N}}$ denotes the sequence of truncated Chebyshev expansions of $f$, there exists $K$ such that for large enough $n$,

$$\|f - p_n\|_\infty \leq \left(2 + \frac{2}{\pi} \log(n + 1)\right) \frac{K}{\sqrt{n}} \to 0 \text{ as } n \to \infty,$$

$$\|f - f_n\|_\infty \leq \left(4 + \frac{4}{\pi^2} \log(n + 1)\right) \frac{K}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

3. See [Zygmund, 2002, Chap. VI] for a proof of absolute convergence. The normal convergence follows from $|a_n T_n(x)| \leq |a_n|$, for all $n \in \mathbb{N}$ and $x \in [-1, 1]$.

4. See Chapter 7 of [Trefethen, 2013].

5. See Chapter 8 of [Trefethen, 2013].

6. See [Bernstein, 1926] for a proof of (2.16) and [Elliott et al., 1987] for a proof of (2.17). The estimate (2.18) follows from Theorem 1.20 and Proposition 1.21.
Bibliography


