Oscillations in a half-empty bottle

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(Received 26 June 2016; accepted 6 October 2017)

When a half-empty bottle of water is pushed to roll on a flat surface, the oscillations of the fluid inside the bottle induce an overall jerky motion. These velocity fluctuations of the bottle are studied through simple laboratory experiments accessible to undergraduate students and can help them to grasp fundamental concepts in mechanics and hydrodynamics. We first demonstrate through an astute experiment that the rotation of the fluid and the bottle is decoupled. The equations of motion are then derived using a mechanical approach, while the hydrodynamics of the fluid motion is explained. Finally, the theory is tested against two benchmark experiments.
https://doi.org/10.1119/1.5009664

I. INTRODUCTION

When a half-empty bottle initially resting horizontally on a flat table is pushed, the fluid it contains is set into motion, which leads to an oscillatory rolling motion of the bottle. Figure 1 shows an example of the trajectory of a 1-l glass bottle half-filled with water, manually pushed at the time indicated by the dashed line. The space-time diagram showing the position of the cap as a function of time clearly illustrates the intermittent overall motion of the bottle. In the particular case shown in Fig. 1, the bottle undergoes a stop-and-go motion with a typical frequency of a few Hertz. The goal of the present paper is to understand and model this oscillatory motion.

The phenomenon of surface gravity waves on water and oscillations of fluid has been studied for a wide variety of container shapes,[1,2] and the dependence of frequency on the curvature in a vertical cylinder is known.[3] However, when the container is subjected to an external force and free to move, the problem becomes more complex and the free surface of the fluid adopts different shapes, depending on the excitation and the geometry of the container. This issue is of great importance in a wide range of applications involving liquid transport, from the problems encountered by space agencies in aerospace vehicles[4,5] to tank carriages on highway or rail roads.[6,7]

The motion of a soft drink can on an incline was studied by Jackson *et al.*[8] The authors used water (and varied the filling fraction of the can) as well as granular matter (lead shots and glass marbles) and proposed a model that describes the limiting cases of non-viscous and infinitely viscous fluids. Later, Lin *et al.*[9] compared the rolling dynamics of cans fully filled with liquid water and solid ice, Ireson and Twidle[10] showed that shaking a can of soda noticeably affects its rolling speed, and Micklavzina[11] investigated the influence of the fluid viscosity. In this article, the bottles are modeled by simpler cylindrical tubes. The main goal of this article is to study the oscillating speed of a half-empty bottle. The ratio of filling is therefore fixed to one half, and the fluid used in all of our experiments is tap water.

The oscillatory motion of a bottle is a simple hands-on experiment which can illustrate important concepts in solid and fluid mechanics. The Euler-Lagrange equations[12] are used to investigate the motion of a coupled system, and the conservation of energy during an elastic shock allows one to predict the bounce of a bottle on a wall. Moreover, the phenomenon exposes hydrodynamics concepts to explain a fluid motion: characteristic times relevant to the phenomenon are identified, the validity of a potential flow approach[13] is discussed, and the eigenmodes of fluid motion in a container are studied.

In order to model the overall oscillating motion of a half-empty cylinder, we begin in Sec. II by focusing on the coupling between the water and the solid and show that their rotation is decoupled for short times. The motion of the fluid within a half-empty still cylinder, known as sloshing, is presented in Sec. III and allows one to define the moment of

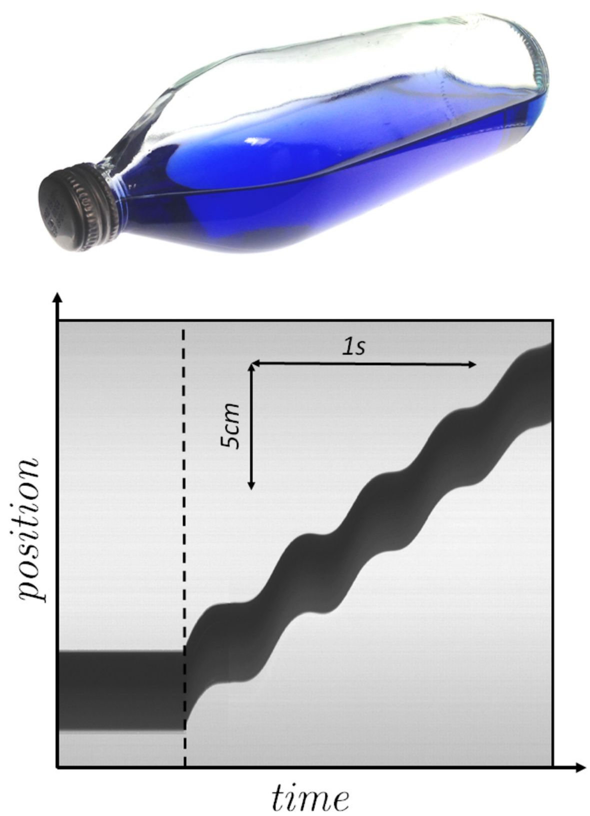

Fig. 1. Picture of a half-empty glass bottle and a space-time plot showing the motion of the cap after the bottle was given a push. The oscillations in speed, adding to the overall forward motion at constant velocity, are clearly visible. The diagram is obtained from a 1500-pixel line video at 500 fps.

inertia of the water, which can be used in the equations of motion derived from the purely mechanical model developed in Sec. IV. Finally, in Sec. V, we show experimental results of oscillations obtained when a bottle bounces on a wall or rolls down a gentle slope.

II. ROTATIONAL COUPLING

A. Race down a slope

The study of a bottle rolling down an incline[8–11] is extremely informative regarding the motion of the fluid inside the rolling container. In particular, Jackson *et al.*[8] showed that the velocity of an empty can is noticeably less than that of a can filled with water. This indicates that the relative moment of inertia of an empty can (i.e., normalized by mass × radius squared) must be larger than that of a full or half-empty can. Indeed, although the fluid clearly contributes to the overall mass of the system (and therefore its weight), it may not contribute significantly to the total moment of inertia since, in general, it is not in solid-body rotation within the container.

We have reproduced these experiments by releasing four "bottles" of identical radius (3 cm) and length (12 cm) on a 2° slope: a hollow tube (or empty bottle) with 5 mm-thick walls, a solid Plexiglass cylinder, a half-empty bottle, and a full bottle. Note that on this gentle slope, all objects roll with no slip. A picture taken 2 s after the start of the race is shown in Fig. 2, while Fig. 3 displays the position and speed of the various bottles. These data were obtained from a video using the "Analyze Particle" tool of ImageJ, a free software program developed by NIH.

Figure 3 shows that the velocities increase roughly linearly with time, which indicates that friction plays a negligible role. As expected, the hollow tube accelerates more slowly than the solid cylinder. The behavior of the half-empty and full bottles is surprising: both roll down the slope with a larger acceleration than the hollow tube and the empty bottle, which indicates a smaller moment of inertia. Their motion is close to that of a frictionless sliding block (indicated by a dashed line in Fig. 3). Note that the velocity is computed from the position (obtained through particle tracking) using a simple first-order differentiation scheme (finite differences). This explains why the velocity data are "noisy," while the position data seem to be surprisingly smooth. Again, these results indicate that the water inside the bottle (whether half-empty or full) undergoes little rotation but is instead simply translated. In conclusion, this simple experiment shows that the rotation of the bottle is decoupled from that of the fluid (over the short duration of the race).

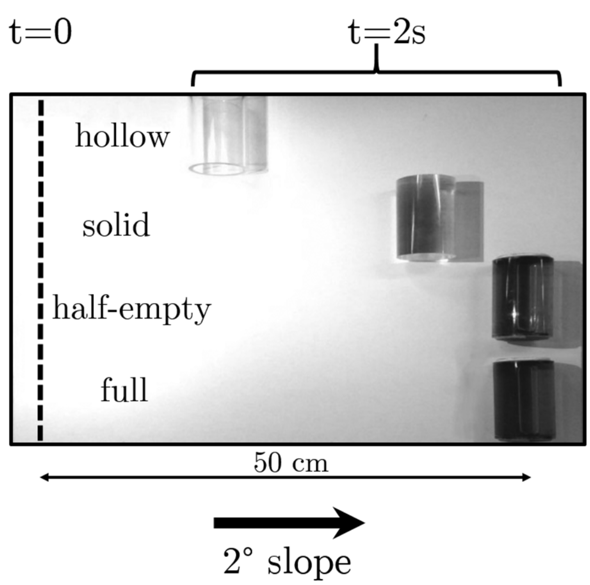

Fig. 2. Picture showing four bottles racing down a 2° slope. The initial positions are shown by the dashed line, and the picture is taken after 2 s of rolling. The race shows that the moments of inertia of half-empty and full bottles are lower than those of the solid cylinder, which indicates that the water undergoes little rotation and only a simple translation.

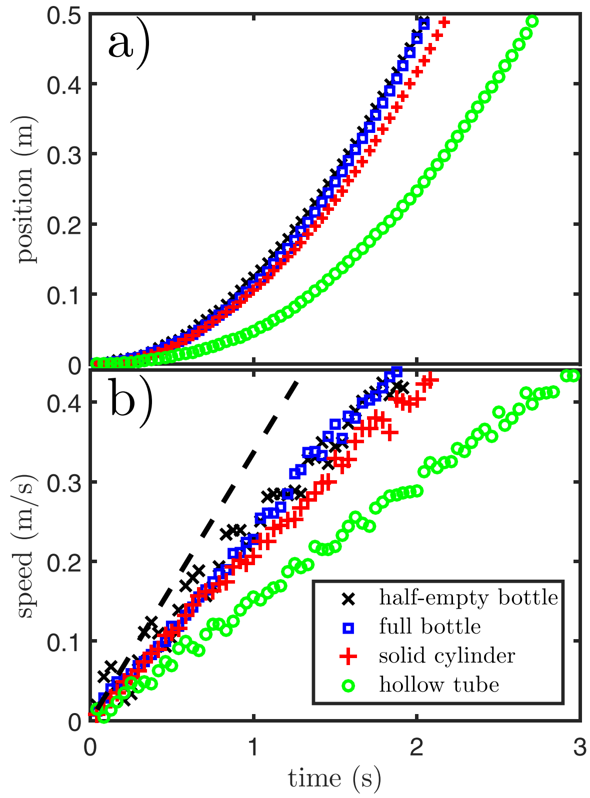

Fig. 3. Experimental measurements of the positions (top) and velocities (bottom) along a 2° slope for empty, half-empty, and full bottles, as well as a solid cylinder. The full and half-empty bottles are the fastest, whereas the empty bottle is the slowest. The greater acceleration of the half-empty and full bottles is due to their reduced moment of inertia. The dashed line indicates a constant acceleration of $g\sin(2°)$.

B. Diffusion time

The discussion in the previous paragraph indicates that a bottle can spin without inducing significant rotation within the water. This behavior only holds for short times, less than the typical diffusion time of momentum. Indeed, over longer times, the viscosity of the fluid should induce motion of the fluid. In this paragraph, we discuss the momentum diffusion in a vertical rotating bottle. We have performed a simple experiment in which a tall cylinder (height 400 mm and radius $R = 50$ mm) is placed on a rotating table (an old record-player). The motion of the water is followed using neutrally buoyant tracers (700 μm polystyrene beads) from a 1000×1000 pixel video recorded at 30 fps. At time $t = 0$, the rotation is started at 33 rpm.

Figure 4 shows the time evolution of the local rotation speed of the water (normalized by that of the rotating support) measured at various distances from the outer wall ($R/10 = 5$ mm, $R/5 = 10$ mm, and $R/2 = 25$ mm). It takes roughly 5 s for the fluid near the bottle wall (at $R/10$) to be set into motion. Therefore, one can conclude that over short times (typically less than 5 s), the vast majority of fluid is not affected by the rotation of the bottle. Therefore, as a first-order approximation, we will consider in the rest of this paper that there is no noticeable coupling between the rotation of the container and that of the liquid.

As a side-note, it is worth mentioning that the moment of inertia of a half-empty bottle set into motion on an incline should increase in time. One might design and perform experiments to show that the resulting acceleration is not constant but instead decreases over time due to the increasing inertia of the system.

An estimate of the diffusion length δ of the velocity within the fluid (or its momentum) over the time T of a typical experiment can be found knowing the kinematic viscosity ν of the fluid:[14,15] $\delta = \sqrt{\nu T} \simeq 3$ mm for $T = 10$ s, which is compatible with the experimental data of Fig. 4 and negligible compared to the radius of the bottle. For an oscillatory motion of typical frequency $1/T = 3$ Hz (see Fig. 1), the boundary layer is given by the penetration length[14,15] (analogous to the electromagnetic skin-depth[16–18]) $\delta = \sqrt{\nu T/\pi} \simeq 0.3$ mm, which is an order of magnitude smaller.

In conclusion, we have shown that the rotation of the fluid and of the bottle is decoupled as long as the experiment lasts for less than about 10 s.

III. SLOSH DYNAMICS IN A STILL BOTTLE

In this section the motion of the fluid in a still bottle is discussed. This "slosh dynamics" was studied as early as the XIXth century by Rayleigh,[19] and a comprehensive review can be found in the study by Lamb[20] and in the study by Ibrahim.[21]

A. Potential flow

In a horizontal cylinder, the first eigenmode of sloshing displays a flat (although not constantly horizontal) free surface that oscillates up and down. Rayleigh[19] showed that under the assumption of potential flow (where in particular the viscosity of the fluid can be neglected), the moment of inertia of the fluid around the center of the bottle is given by

$$I_s = \pi \rho R^4 L\left(\frac{4}{\pi^2} - \frac{1}{4}\right) \simeq 0.31 m R^2, \tag{1}$$

where ρ is the density of the fluid and L the length of the bottle.

The potential from which the velocity is derived is given in terms of a series,[20,21] and the corresponding velocity field is plotted in Fig. 5(b). A few comments can highlight the differences with a rigid-body rotation. First, the velocity clearly decays with the increasing distance from the free surface, whereas it increases linearly in the case of a rigid-body rotation. Second, one can notice that the streamlines are not half-circles centered on the center of the bottle. Instead, the streamlines are flatter near the free surface. Finally, the velocity at the center of the bottle is not zero. It is purely horizontal and oscillates back and forth. The theoretical velocity field is remarkably similar to the experimental observations. The picture in Fig. 5(a) shows neutrally buoyant markers, indicating the motion of the fluid in a bottle pushed and held against a wall. This excellent agreement supports the assumption that the viscosity of the fluid plays no major role in the slosh dynamics, aside from the boundary layer discussed earlier.

B. Experimental validation

Experiments were performed by pushing a half-empty cylinder against a wall in order to induce fluid motion. The cylinder is then firmly held still against the wall, and the slosh dynamics of the water is studied. The altitude of the free surface against the curved side of the cylinder was recorded at 500 fps and measured through particle tracking (see the inset in Fig. 6).

A simple way to check whether the potential flow is an accurate description of the actual flow is to measure the frequency of the oscillations. The Fourier transform of the signal is shown in Fig. 6. The frequencies of the first three modes of sloshing can be analytically predicted[21] ($\omega_1 \simeq 1.17\sqrt{g/R}$, $\omega_2 \simeq 2.17\sqrt{g/R}$, and $\omega_3 \simeq 2.82\sqrt{g/R}$) and are indicated by vertical lines.

A decay of the amplitude of oscillations is visible in the inset and is a long-term effect of the viscosity of the fluid. However, the agreement between the observed frequency and the prediction under the assumption of an ideal fluid is excellent, again supporting the hypothesis that the viscosity plays no major part in the flow (although it causes a slow damping). Moreover, even if the force initially applied to the bottle induces a sloshing motion which is more complex than an oscillating flat surface, the first sloshing mode is always dominant. The second or third modes could be observed if the cylinder were shaken at the corresponding resonance frequency, but in the case of an inherently asymmetrical initial push, the first mode always dominates. Therefore, in the following, only the first mode—in which the free surface remains flat, although obviously not horizontal—will be considered.

IV. A SOLID BODY TOY MODEL

Having understood that the rotation of the fluid and of the bottle is decoupled over short times (less than 10 s), one can

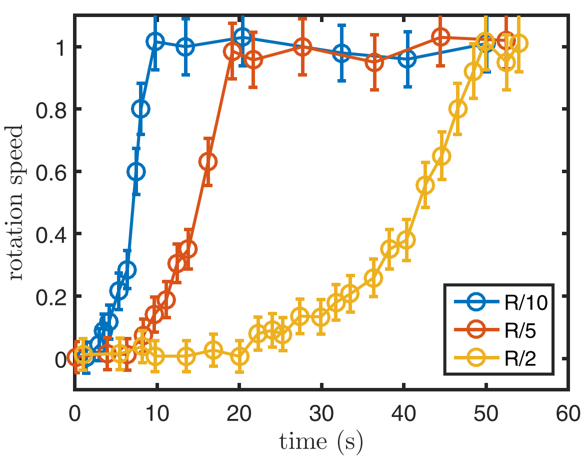

Fig. 4. Rotation speed of water (measured at $R/10$, $R/5$, and $R/2$ from the wall) in a tall vertical cylinder initially at rest and set into rotation (33 rpm) at $t = 0$. The lines are guides to the eye.

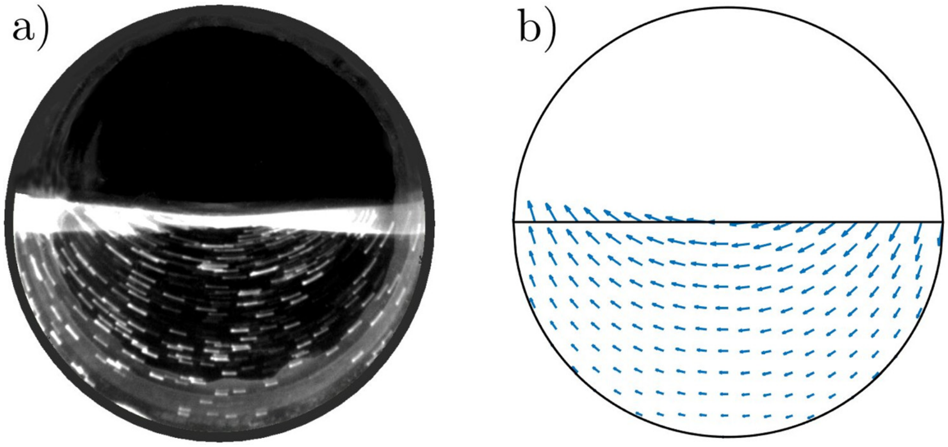

Fig. 5. (a) Picture of the sloshing motion of water within a still cylinder (radius $R = 5$ cm and exposure time = 1/10 s). The picture shows neutrally buoyant tracers (polystyrene spheres of density 1.06 g/cm^3 and diameter 700 μm). (b) Velocity field computed from the potential flow given by Ibrahim (Ref. 21). Note that the velocity decreases with the increasing distance from the free surface.

propose a mechanical model for the motion of a half-empty bottle in which the fluid is seen as a solid half-cylinder. This simplification is obviously inappropriate (as discussed in Sec. III) but allows for a derivation and analysis of the equations of motion of a half-cylinder, representing the water, mounted on wheels, representing the bottle [see Fig. 7(a)]. The mass and moment of inertia of the water are denoted m and I_G (computed around its center of mass G), while those of the bottle alone are denoted M and J. The horizontal position of the bottle is described by x and the inclination of the half-cylinder θ. The distance between the center of the bottle and the center of mass of the water is given by $l = OG = 4R/3\pi$. The model presented here is very similar to that presented by Jackson *et al.*,[8] in which further technical details can be found.

Assuming a slip-free rotation of the bottle allows one to express its rotation speed as $\dot{x}/R$, with the overdot representing differentiation with respect to time. If all sources of dissipation are neglected (air drag, solid and rolling friction, viscosity of the fluid, etc.), the equations of motion can be derived from the Euler-Lagrange equations with kinetic T and potential V energies

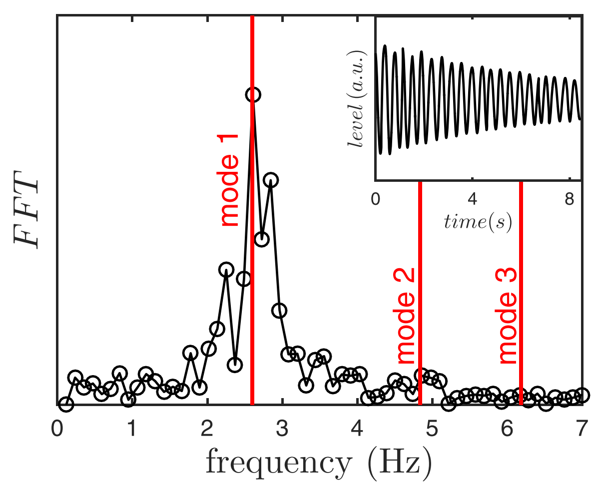

Fig. 6. Fourier power spectrum of the motion of the free surface in a bottle first pushed and then held still. The experimental maximum coincides with the theoretical frequency of the first sloshing mode. The inset shows the damping of the amplitude over long times.

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{R}\right)^2 + \frac{1}{2}m\left(\vec{\dot{x}} + l\vec{\dot{\theta}}\right)^2 + \frac{1}{2}I_G\dot{\theta}^2, \qquad (2)$$

$$V = mgl(1 - \cos\theta). \qquad (3)$$

In the small-angle approximation, the equations of motion then read

$$\left(M + \frac{J}{R^2}\right)\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\right) = 0, \qquad (4)$$

$$(I_G + ml^2)\,\ddot{\theta} = -mgl\theta - ml\ddot{x}. \qquad (5)$$

Interestingly, these equations are equivalent to a system consisting of a pendulum with an effective moment of inertia $I_{\text{eff}} = I_G + ml^2$ attached to a block sliding with no friction [see Fig. 7(b)] with an effective mass $M_{\text{eff}} = M + J/R^2$. Note that the effective moment of inertia I_{eff} is simply equal to the moment of inertia I_O of the half-cylinder about the symmetry axis O of the bottle.

The equations of motion (4) and (5) can be combined and solved (see Ref. 8 for details) to give

$$\ddot{\theta} = A\sin(\Omega t + \varphi), \qquad (6)$$

$$\ddot{x} = -A\left(\frac{m}{m + M_{\text{eff}}}\right)l\sin(\Omega t + \varphi), \qquad (7)$$

where A (in rad/s^{-2}) and φ (in rad) are constants that depend on the initial conditions and

$$\Omega^{-2} = \frac{l}{g}\left(\frac{I_O}{ml^2} - \frac{m}{M_{\text{eff}} + m}\right). \qquad (8)$$

V. SLOSH DYNAMICS IN A ROLLING BOTTLE

A. Adaptation of the theory to a rolling bottle

The results presented above can be combined to study the motion of a half-empty bottle. The moment of inertia I_O in Eq. (8) simply needs to be replaced by that of the first mode of sloshing given in Sec. III: $I_s = \pi\rho R^4 L((4/\pi^2) - (1/4))$. Moreover, for simplicity, the moment of inertia of the bottle

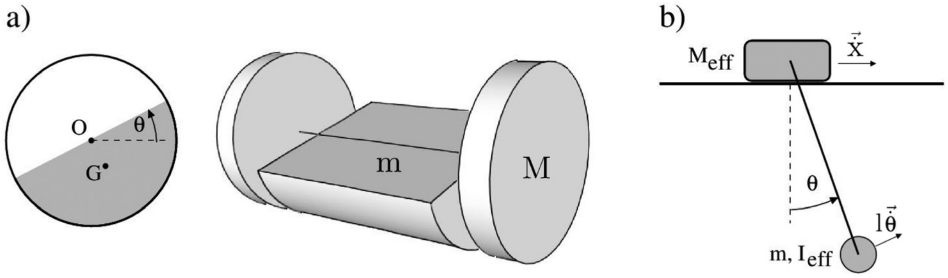

Fig. 7. Schematic diagrams of (a) a half-cylinder mounted on wheels and (b) an equivalent system consisting of a pendulum attached to a sliding block with the effective moment of inertia and masses.

itself is assumed to be $J = MR^2$ and the effective mass defined in Sec. IV is then simply $M_{\text{eff}} = 2M$. The angular frequency of oscillations is therefore given by

$$\Omega^{-2} = \frac{l}{g}\left(\frac{I_s}{ml^2} - \frac{m}{2M + m}\right). \quad (9)$$

Note that the ratio I_s/ml^2 is only a geometrical constant ($\simeq 1.73$) since $l = 4R/3\pi$. Moreover, the right-hand side is ensured to be positive since the term in parentheses can be rewritten as $I_G/ml^2 + 2M/(2M + m)$.

Knowing the initial conditions therefore allows one to determine the constants A and φ in Eqs. (6) and (7) and to integrate them forward in time. In what follows we will discuss the motion in two different situations: a half-empty bottle bouncing off a wall and a half-empty bottle rolling down a gentle slope.

B. Soft collision of a half-empty bottle

1. Theoretical prediction

As explained above, the motion strongly depends on initial conditions. When a bottle is manually pushed, it is difficult to accurately determine what force (i.e., acceleration) or velocity is imposed and the corresponding initial conditions remain unclear. In order to study a well-defined set of initial conditions, a half-empty bottle rolling on a flat surface with speed v_0 and without any oscillations is sent to bounce off a wall (Fig. 8).

When an empty bottle hits a wall, it rapidly bounces back and much of its energy is dissipated in the collision. However, in the case of a half-empty bottle, the water it contains rises (as the free surface tilts) and its initial kinetic energy is converted into gravitational potential energy. If the mass of the liquid is noticeably larger than that of the bottle itself, the contact is not instantaneous and the energy of the water is essentially conserved (as long as viscous effects are neglected as discussed in Sec. II B). This procedure is a reproducible and controlled way to set initial conditions at the time when the bottle leaves the wall (see Fig. 8): $x_0 = 0$, $\dot{x}_0 = 0$, and $\theta_0 = 0$, where the initial rotation speed $\dot{\theta}_0$ is given by

$$\frac{1}{2} I_s \dot{\theta}_0^2 = \frac{1}{2} m v_0^2. \quad (10)$$

Note that Eq. (10) must be seen as a first-order approximation, only valid if the mass of the water dominates over that of the empty bottle. Let us emphasize again that viscous dissipation is neglected, that the water is assumed to be at rest prior to the collision (i.e., the thin boundary layer set in motion is neglected), that the inelasticity of the collision of the bottle itself is neglected, that only the first mode of sloshing is considered, and that we focus only on the case of small

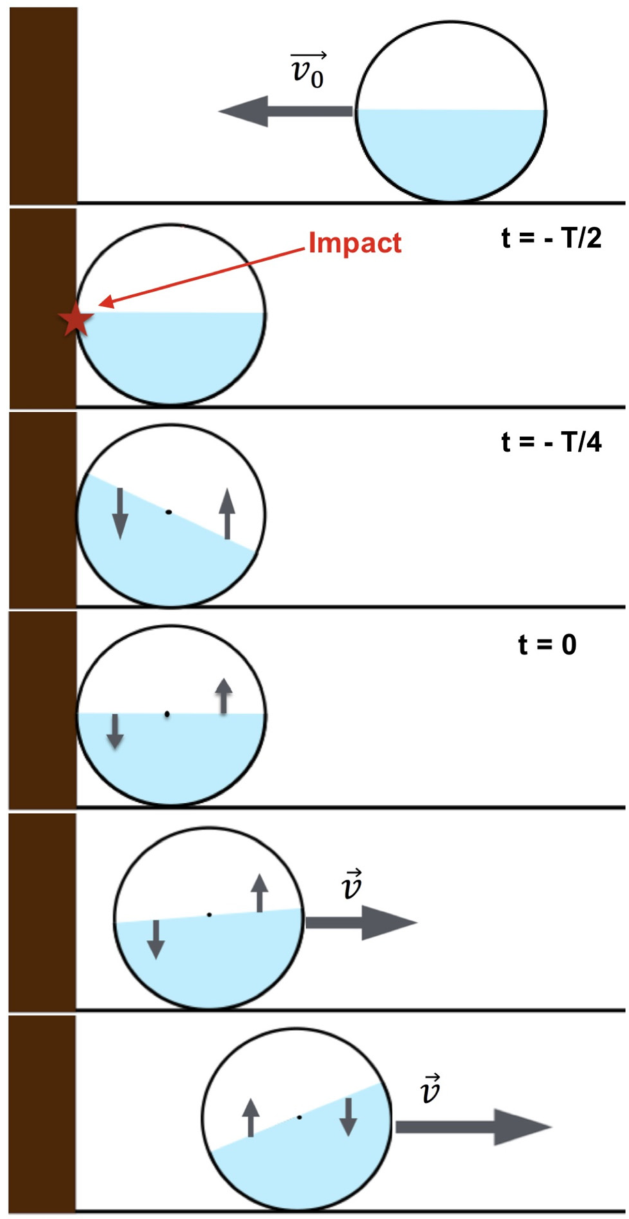

Fig. 8. A half-empty bottle is sent to bounce off a wall. During the impact, the fluid is set into rotation, which triggers oscillations in the velocity when the bottle leaves the wall.

amplitude oscillations. In a more thorough and comprehensive analysis, these effects should be taken into account although this would likely preclude a simple physical understanding. Moreover, as we will show, the agreement with experimental measurements is excellent, which in retrospect supports the simplifying assumptions made.

Under the conditions described above, the motion of the fluid and the bottle given in Eqs. (6) and (7) read

$$\dot{\theta} = \dot{\theta}_0 \cos(\Omega t), \tag{11}$$

$$\dot{x} = \beta v_0 \left(\frac{m}{2M+m}\right)[1 - \cos(\Omega t)], \tag{12}$$

where $\beta = \sqrt{ml^2/I_s} \simeq 1.32$.

2. Experimental validation

A cylindrical Plexiglas bottle with a radius (R) of 50 mm, a thickness of 4 mm, and a length (L) of 50 cm was used to test the theoretical predictions. The mass of the empty bottle (including the end caps) was $M = 0.910$ kg, and the mass of water used (corresponding to a half-empty bottle) was $m = 1.74$ kg. The bottle's position was determined using video analysis (at 30 fps), and the velocity was computed as its numerical derivative. The results are presented in Fig. 9. The velocity prior to the impact, v_0, is seen to be approximately constant (with small fluctuations, in part due to the image processing), while after the impact (indicated by the vertical dashed lines), the velocity is oscillatory. One can see that the bottle undergoes a stop-and-go motion since the velocity periodically reaches zero. This feature is visible in the space-time diagram in Fig. 1.

The prediction given by Eq. (12) is shown in Fig. 9 by the solid curve (red online). It is worth noting that there are no fitting parameters to this prediction. The initial velocity v_0 is measured as the average before impact, while the oscillation frequency Ω is determined from the duration of the impact, which corresponds to one half-period. The agreement between the experimental data and the theory is excellent, supporting the validity of the numerous assumptions made.

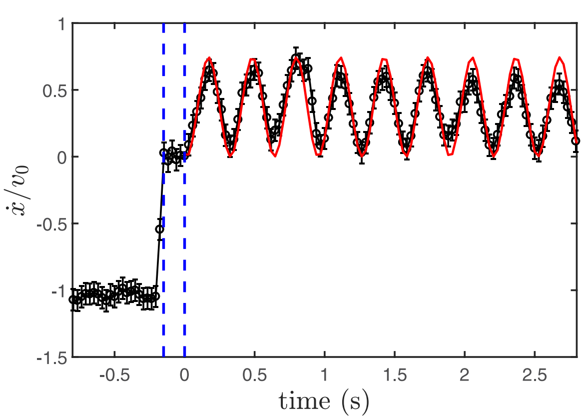

Fig. 9. Velocity of the half-empty bottle experiencing a soft collision with a wall. The velocity is constant before the impact (first dashed line), and oscillations are triggered after the bottle leaves the wall (second dashed line). The solid curve (red online) shows the velocity predicted using Eq. (12); there are no fitting parameters, and the agreement is excellent.

C. Half-empty bottle on a gentle slope

1. Theoretical prediction

Another protocol used to produce reproducible and controlled initial conditions consists of letting a half-empty bottle roll down an inclined plane of slope α (which must remain small enough to ensure no-slip motion of the bottle). Here, the inclination of the free surface, θ, is measured with respect to the plane angle α. The gravitational potential energy of the bottle and the fluid needs to be included in the Euler-Lagrange equations, which yield equations of motion (in the small-angle approximation)

$$2M\ddot{x} + m(\ddot{x} + l\ddot{\theta}) = (M+m)g\alpha, \tag{13}$$

$$I_s\ddot{\theta} = -mgl(\theta - \alpha) - ml\ddot{x}. \tag{14}$$

For a bottle initially at rest (with $\theta_0 = \alpha$), the solution is found to be

$$\dot{x} = \frac{M+m}{2M+m}\alpha g t + \frac{m(M+m)}{(2M+m)^2}\alpha l \Omega \sin(\Omega t). \tag{15}$$

The velocity is composed of two separate terms: the first is an average (constant) acceleration (smaller than αg as discussed in Sec. II), while the second is oscillatory.

2. Experimental validation

A cylindrical Plexiglas bottle of radius $R = 44$ mm and length $L = 170$ mm was used to test the theoretical predictions. The mass of the empty bottle was $M = 0.140$ kg, and the mass of water used (corresponding to a half-empty bottle) was $m = 0.510$ kg. The bottle was placed on a table inclined at an angle of $\alpha = 15°$, and its position was determined using video analysis (at 500 fps) with the velocity computed as its numerical derivative. The results are presented in Fig. 10.

The experimental data confirm the basic theoretical predictions: the bottle rolls down the slope with an average constant acceleration and oscillations in the speed are clearly visible. The average experimental acceleration is in excellent agreement with the predicted value. The velocity fluctuations $\Delta\dot{x}$ defined as the difference in velocity from the linear behavior are shown in the bottom of Fig. 10, where the solid curve (red online) shows the theoretical predictions. The agreement for the oscillations is fairly good initially but gets worse as time goes on. The discrepancy might be due to the failure of the decoupling assumption. It should be emphasized once again that there are no fitting parameters: all values are deduced from the simple measurement of the masses and sizes. The amplitude, frequency ($\Omega/(2\pi)$), and phase are those given by Eqs. (8) and (15).

VI. CONCLUSION

In conclusion, we have investigated and modeled the motion of a half-empty bottle rolling on a flat surface. We showed that over the duration of a typical experiment, the rotation of the water and of the bottle is decoupled, which can also be predicted from the Navier-Stokes equation. A simple mechanical toy-model allowed for the derivation of the equation of motion, while the exact motion of the fluid is well described by the first mode of sloshing under the

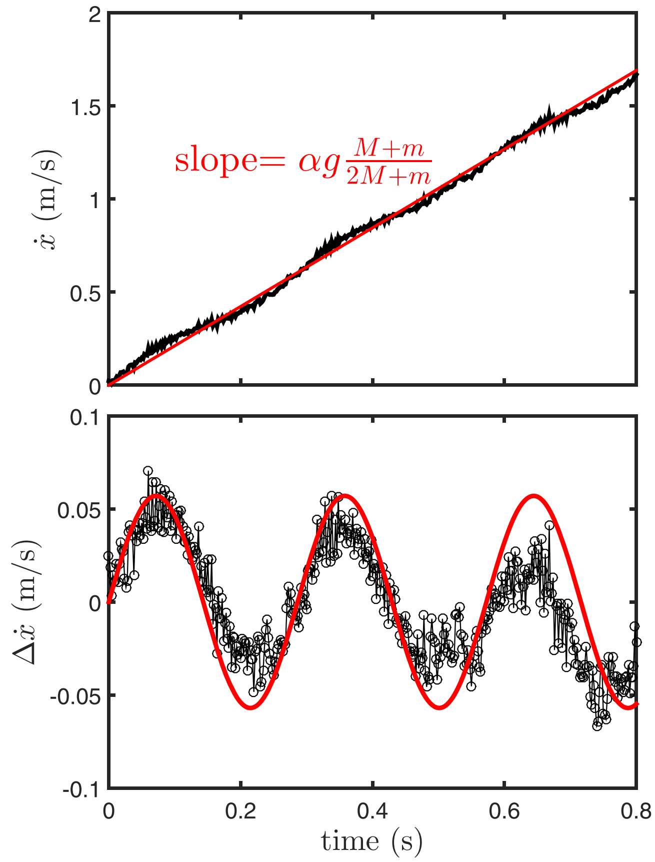

Fig. 10. Top: velocity of a bottle released on a 15° incline. On average, the bottle undergoes a constant acceleration, but oscillations in the velocity are clearly visible. The solid line (red online) shows the predicted average acceleration. Bottom: deviation from the constant acceleration highlighting the oscillations. The solid curve (red online) shows the theoretical prediction.

assumption of an inviscid fluid. Experiments performed on a bottle bouncing off a wall and rolling down a gentle slope provided experimental support for the assumptions used in the models.

The limitations of our work deserve further attention. For example, the decoupling of the rotation of the fluid and the bottle will not hold for smaller radii. Thus, if a test-tube of radius $R = 1$ cm is used, the diffusion time is of the order of a few seconds and the entire fluid can be set into a rigid-body rotation. The same conclusion can be drawn if a more viscous fluid is used; vegetable oils are typically 50 times more viscous than water, leading to a diffusion time of the order of one second. Finally, we should recall that the equations were derived in the small-angle approximation and that an exact study of larger amplitudes could reveal surprising results.

The potential flow gives an accurate description of the motion of the water in the bottle, but the effect of the viscosity is visible in Fig. 6. A comprehensive study of the viscous dissipation in the boundary layer might deserve further attention. Similarly, the effect of static and rolling friction of a bottle rolling on a surface, as well as that of drag caused by the surrounding air, could be studied to provide one with a more realistic description of the motion over long times.

It would be interesting to vary the filling fraction of the bottle. Obviously, there can be no oscillations in a bottle either completely full or empty. Therefore, there must exist an optimal filling fraction for the oscillatory motion studied in this paper. It is unclear whether the optimal value should correspond to a filling fraction of one half.

Finally, whether the bottle should be seen as half-empty or half-full remains an open question which goes far beyond the scope of this paper and certainly deserves further investigation.

ACKNOWLEDGMENTS

The authors acknowledge support from the PALSE program of the University of Lyon Saint-Etienne, the University Lyon Claude Bernard, the Société Française de Physique, and the École Normale Supérieure de Lyon and its Physics Department and Laboratoire de Physique. The authors are thankful to B. Capitaine for technical help and A. Chemin, A. Guittonneau, C. Gouiller, C. Malciu, and J. Sautel for fruitful discussions. The work presented here was done in preparation for the International Physicists Tournament,[22] a world-wide competition for undergraduate students. The authors are grateful to both local and international organizing committees of the International Physicists Tournament for having put together an exciting event.

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[22]More information on the International Physicists Tournament can be determined at <http://iptnet.info/>.