You are asked to complete Exercise 3 and to send me your solutions to:

nicolas.schabanel@ens-lyon.fr

as a PDF file named HW1-Lastname.pdf on Thu. 24/10 before 12:00.

Exercise 1 (Algorithmic Self-Assembly). Recall that the self-assembly process consists in, given a finite tileset (with infinitely many tiles of each type), starting from the seed tile (marked with a ⋆), gluing tiles with matching colors to the current aggregate so that each new tile is attached by at least two links to the aggregate (either on the same border or on two borders). Recall that a shape is final if no tiles can be attached to it anymore.

Question 1.1) What is the exact family of final shapes self-assembled by the following tileset? (No proof nor justification is asked.) Indicate the local order of assembly by drawing arrows over the tiles of a generic final shape. Which are the two competing tiles that decide the size of the resulting final shape?

Exercise 2 (Guess the shape). Recall that the self-assembly process consists in, given a finite tileset (with infinitely many tiles of each type), starting from the seed tile (marked with a ⋆), gluing tiles with matching colors to the current aggregate so that each new tile is attached by at least $T^\circ = 2$ links to the aggregate (either on the same border or on two borders). Recall that a shape is final if no tiles can be attached to it anymore.

Question 2.1) What is the exact family of final shapes self-assembled by the following tileset at temperature $T^\circ = 2$? (No proof nor justification is asked.)

Indicate the local order of assembly by drawing arrows over the tiles of a generic final shape. Which are the competing tiles that decide the size of the resulting final shape?
Exercise 3 (Counter at $T^o = 2$ ($\star$)). Given an integer $n$, and an seed configuration consisting of an isosceles rectangle triangle isocèle of side $\ell = \lceil \log_2 n \rceil$ where the bits of $n$ are encoded on the diagonal as shown in grey below.

Propose a well-ordered (finite) tileset which assembles the yellow at $T^o = 2$ to realise a rectangle of size $\ell \times (n + \ell)$. Carefully indicate the position of the glue of strength 1 and 2 on the diagonal of the seed configuration. Indicate the assembly order ($1$). What does the tiles encode?

Exercise 4. Assume a random Poisson model where the random time $X$ between two consecutive appearances of a tile of a given type $\tau$ at a given empty location follows an exponential law: $p(x) = c \cdot e^{-cx}$ where $c > 1$ is the concentration of the tiles of type $\tau$. We want to prove the following theorem:

**Theorem 1 (Adleman et al, 2001).** Consider an ordered tile system $T$ that assembles deterministically a single shape $S$. Let $\prec$ be the partial order of the assembly, i.e. such that $(i, j) \prec (k, l)$ if the tile at position $(i, j)$ is attached before the tile at $(k, l)$ by $T$. With very high probably, the assembly time of a shape $S$ by $T$ is:

$$O(\gamma \times \text{rank}(S))$$

where $\gamma$ only depends on the concentrations and $\text{rank}(S)$ is the highest rank in the shape $S$ (i.e. the length of the longest path in $\prec$).

**Question 4.1** Let $X$ be an exponential random variable such that $p(X = x) = ce^{-cx}$ for all real $x \geq 0$, for some $c > 0$. Show that $X$ is memoryless, i.e. for all $u, t \geq 0$,

$$p(X = t + u|X \geq u) = p(X = t)$$

Let $T$ be the assembly time of the shape $S$, i.e. the time at which the last tile of shape $S$ is attached. We denote by $w(P)$ the random variable for the weight of a $\prec$-path $P$, defined as: $w(P) \sum_{(i,j) \in P} X_{i,j}$.

**Question 4.2** Let $X_{i,j}$ be the independant exponential random variable for the time between two consecutive appearances of the tile to be attached at position $(i, j)$ in $S$. Show that:

$$T = \max_{\prec \text{-path } P} w(P)$$

> **Hint.** Proceed by recurrence on the rank of the tiles and show that for all tile $(i, j)$, its assembly time is the random variable $T_{ij} = \max_{\prec \text{-path } P \text{ from } (0,0) \text{ to } (i,j)} w(P)$. 

**Question 4.3** Let $X_1, \ldots, X_\ell$ be independent exponential variables s.t. $p(X_1 = x) = c_i e^{-c_i x}$ with $c_i > 1$. Show that there is $\gamma$ which depends only of $\min_i c_i$ such that: for all $n \geq \ell$, 

$$\Pr\{X_1 + \cdots + \ell \geq \gamma \cdot n\} \leq 1/4^{\ell} \cdot e^{-\gamma(n-\ell)}$$

> **Hint.** Note that $E[e^{-X_i}] < \infty$ and apply Markov inequality to $Z = e^{X_1 + \cdots + X_\ell}$. 

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Question 4.4) Conclude.

Exercise 5. Propose a staged assembly scheme at temperature $T^* = 1$ of the shape family $E$ of candelabrums with $n$ branches of length $n$.

Describe the tiles, glues, their number, the number of stages and the number of different bechers needed. Give an illustration of the stages to build a generic production.