Exercise 1 (Window Movie Lemma). We investigate the computation power of tile assembly at temperature $T^\circ = 1$. We allow mismatches, i.e. a tile can be added to the current aggregate as soon as it is attached by at least one side to the current aggregate for which the glues match (the other sides in contact can have mismatching glues). Unless specified explicitly otherwise, all assemblies take place at $T^\circ = 1$ in this exercise.

Let us first consider a (finite) tile set $T$ which only assembles unidimensional segments of size $1 \times \ell$ for some $\ell \geq 1$ starting from its seed tile. Let $\tau = |T|$ denote the number of tile types in $T$ in all of the following. Recall that the final productions of a tileset $T$ are the shapes corresponding to every possible assembly of tiles from $T$ starting from the seed tile of $T$ and where no more tile can be added.

▶ Question 1.1) Show (and explicit) that there is a constant $k(\tau)$, which depends only on $\tau$, such that if a segment of size $1 \times \ell$ with $\ell \geq k(\tau)$ is a final production of $T$, then there is an integer $1 \leq i < k(\tau)$ such that all the segments $1 \times (\ell + n \cdot i)$ are also final productions of $T$ for all $n \geq -1$. If so, we say that the tile set $T$ is pumpable.

▷ Hint. Pay attention to the order in which the tiles are attached, make sure that the pumped structure can indeed self-assemble.

Let us now consider a (finite) tile set $T$ whose final productions are $2$-thick rectangles of size $2 \times \ell$ for some $\ell \geq 1$.

▶ Question 1.2) Show (and explicit) that there is a constant $k_2(\tau)$, which depends only on $\tau$, such that if a $2$-thick rectangle of size $2 \times \ell$ with $\ell \geq k_2(\tau)$ is a final production of $T$, then $T$ is pumpable, i.e. that there is an integer $1 \leq i < k_2(\tau)$ such that all the $2$-thick rectangles $2 \times (\ell + n \cdot i)$ are also final productions of $T$ for all $n \geq -1$.

▷ Hint. Pay attention to the order in which the tiles are attached, make sure that the pumped structure can indeed self-assemble.

Let us now generalise and consider a (finite) tile set $T$ whose final productions are $q$-thick rectangles of size $q \times \ell$ for some $\ell \geq 1$.

▶ Question 1.3) Show (and explicit) that there is a constant $k_q(\tau)$, which depends only on $\tau$, such that if a $q$-thick rectangle of size $q \times \ell$ with $\ell \geq k_q(\tau)$ is a final production of $T$, then $T$ is pumpable, i.e. that there is an integer $1 \leq i < k_q(\tau)$ such that all the $q$-thick rectangles $q \times (\ell + n \cdot i)$ are also final productions of $T$ for all $n \geq -1$.

Consider the following tile set $U = \{ \star, A, B, C, A', B', C', D \}$ at $T^\circ = 2$ for which $\star$ is the seed tile:
The final productions of $\mathcal{U}$ at $T^0 = 2$ consist of two arms which are either 1) of different lengths and then don’t touch each other; or 2) of equal length and then there is a tile $D$ that makes contact between them:

![Diagram showing final productions of $\mathcal{U}$ at $T^0 = 2$.]

**Question 1.4)** Show that no tile set can simulate intrinsically at $T^0 = 1$, the dynamics of $\mathcal{U}$ at $T^0 = 2$.

> Hint. As a simplification, consider that in an intrinsic simulation, all megacell corresponding to an empty position in the simulated system must never be filled by more than 30% of tiles, and all megacell corresponding to a non-empty position in the simulated system must be filled at 100% by tiles. If you have time left: how would you waive these assumptions?

**Exercise 2 (Oritatami).** Let us first recall the definition of an Oritatami system:

**Triangular lattice.** Consider the triangular lattice defined as $\mathbb{T} = (\mathbb{Z}^2, \sim)$, where $(x, y) \sim (u, v)$ if and only if $(u, v) \in \bigcup_{e=\pm 1} \{(x + e, y), (x, y + e), (x + e, y + e)\}$. Every position $(x, y)$ in $\mathbb{T}$ is mapped in the Euclidean plane to $x \cdot X + y \cdot Y$ using the vector basis $X = (1, 0) \rightarrow$ and $Y = \text{RotateClockwise}(X, 120^\circ) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \sqrt{3}$.

**Oritatami systems.** Let $B$ denote a finite set of bead types. Recall that an Oritatami system (OS) $\mathcal{O} = (p, \heartsuit, \delta)$ is composed of:
1. a bead type sequence $p \in B^*$, called the transcript;
2. an attraction rule, which is a symmetric relation $\heartsuit \subseteq B^2$;
3. a parameter $\delta$ called the delay.

Given a bead type sequence $q \in B^*$, a configuration $c$ of $q$ is a self-avoiding path in $\mathbb{T}$ where each vertex $c_i$ is labelled by the bead type $q_i$.

We say that two bead types $a$ and $b \in B$ attract each other when $a \heartsuit b$. Furthermore, given a partial configuration $c$ of a bead type sequence $q$, we say that there is a bond between two adjacent positions $c_i$ and $c_j$ of $c$ in $\mathbb{T}$ if $q_i \heartsuit q_j$ and $|i - j| > 1$.

**Oritatami growth dynamics.** Given an OS $\mathcal{O} = (p, \heartsuit, \delta)$ and a seed configuration $\sigma$ of a seed bead type sequence $s$, the configuration at time 0 is $c^0 = \sigma$. The configuration $c^{t+1}$ at time $t+1$ is obtained by extending the configuration $c^t$ at time $t$ by placing the next bead, of type $p_{t+1}$, at the position(s) that maximize(s) the number of bonds over all the possible extensions of configuration $c^t$ by $\delta$ beads. When the maximizing position is always unique, we say that the OS is deterministic. We shall only consider deterministic OS in this exercise.

**Example.** The OS $\mathcal{O} = (p, \heartsuit, \delta = 2)$ with bead types $B = \{0, \ldots, 8\}$, transcript $p = (3, 4, 5, 6, 7, 8)$, rule $\heartsuit = \{0 \heartsuit 3, 0 \heartsuit 6, 1 \heartsuit 8, 2 \heartsuit 4, 2 \heartsuit 6, 2 \heartsuit 8, 4 \heartsuit 6, 5 \heartsuit 7\}$ and seed configuration $\sigma = (0@0(0,0); 1@1(1,1); 2@1(1,0))$, folds deterministically as follows:

![Diagram showing the growth dynamics of the Oritatami system.]

The seed configuration $\sigma$ is drawn in orange. The folded transcript is represented by a black...
line. The bonds made are represented by dotted black lines. The $\delta = 2$ beads currently folding at time $t$ are represented in bold. If there are several extensions that maximize the number of bonds, the freely moving part is represented translucently. Observe two remarkable steps:

- $t = 2$ and 3: the position of 5 is not determined when 4 is placed, but will be fixed when 5 and 6 are folded together.
- $t = 4$ and 5: 7 is initially placed next to 5 when 6 and 7 are folded together but will be finally placed to the right of 8 when 7 and 8 are folded together (because two bonds can be made there instead of only one) and 7 will thus remain there.

In the following, you are asked to propose a rule and a delay that folds the transcript in the desired configuration for the given seed.

**Question 2.1** Propose a delay-1 OS that, given the seed configuration $\langle 0, 1, 2 \rangle$ below, folds the transcript $\langle 3, 4, 3, 4, 3, 4, \ldots \rangle$ as: (just draw the bonds, no justification asked)

![Diagram of bond formation with positions 0 to 4 labeled]

**Question 2.2** Is there a delay-1 OS that, given the seed configuration $\langle 0, 1, 2 \rangle$ below, folds the transcript $p = \langle 3, 4, 5, 6, 7, 8 \rangle$ deterministically as below? Justify your answer.

Propose a delay-2 OS that folds $p$ as desired.

**Question 2.3** Propose a delay-2 OS that, given the seed configuration $\langle -13, \ldots, -1, 0 \rangle$ below, folds the transcript $p = \langle 1, 2, 3, 4 \rangle$ deterministically as below. Draw and explain the key steps of the folding.

![Diagram of bond formation with positions -1 to 13 labeled]

**Question 2.4** Prove that there is no delay-2 OS that, given the seed configuration $\langle 0, 1, 2 \rangle$ below, folds the transcript $p = \langle 3, 4, \ldots \rangle$ deterministically along the path $\langle \nearrow \nearrow \searrow \rangle^\infty$ as below.

![Diagram of bond formation with positions 0 to 10 labeled]

Conclude that there is no delay-2 deterministic OS with a fixed-size seed configuration that can cover exactly the positions of the following 3-arm star for large enough $n$: 