Universality in assembly models

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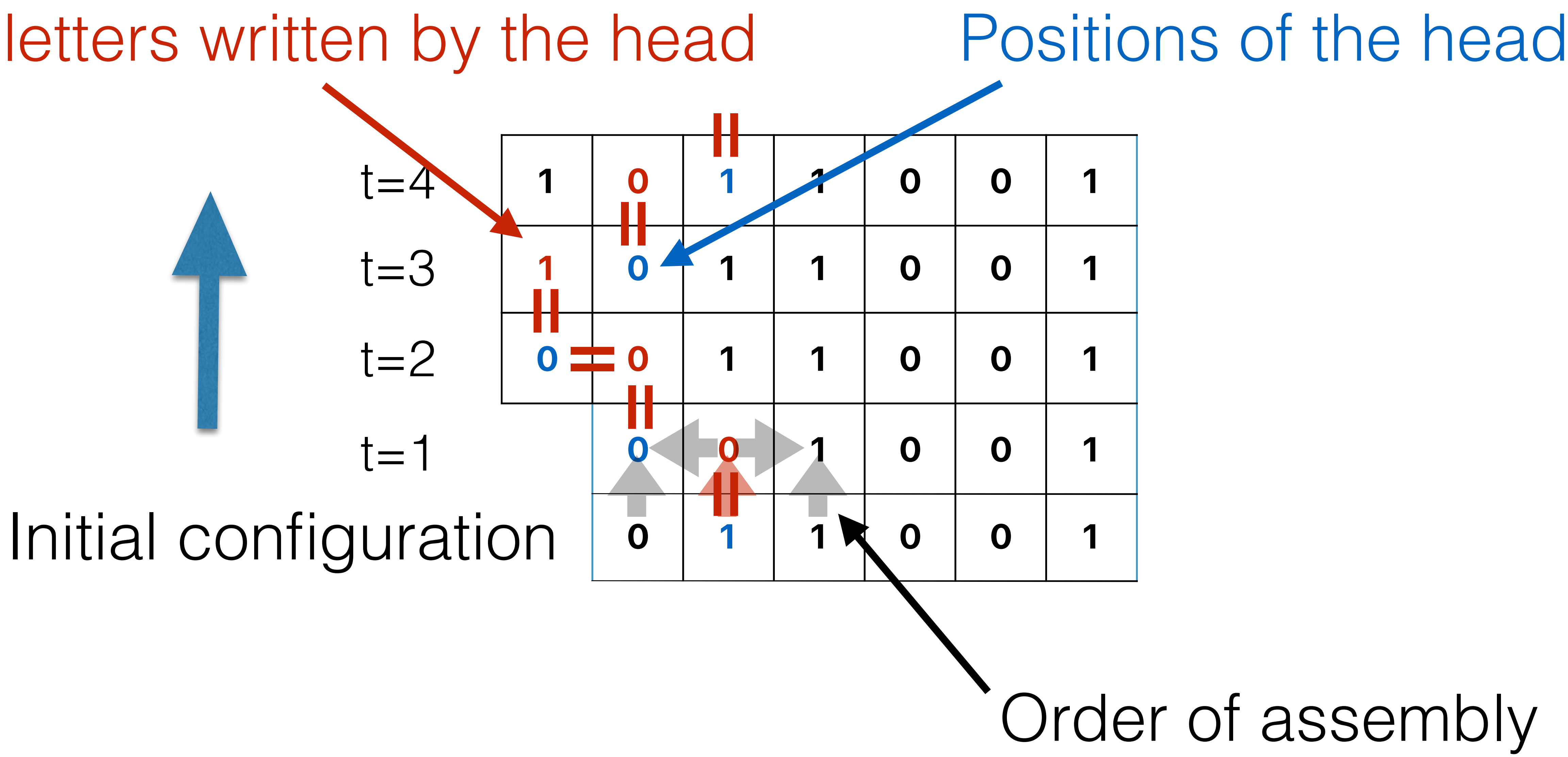
Universality in algorithmic self-assembly

- Can we decide if an assembly terminates?
- Can we design a tile system for a given shape?
- Do I need more than one tileset?
- Do I need more than one tile?
- Do I need more than one molecule?

Algorithmic self-assembly simulates any Turing machine at T°2 in 2D



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Position and state of the head

Current tape right end

t=4	0	0	1,Halt	1	★
t=3	0	0	0,q'''	1	★
t=2	0	0	0	0,q''	★
t=1	0	0	1,q'	★	
t=0	0	1,q	1	★	

Transitions:

$$(0,q''') \longmapsto (1,\mathrm{Halt},\bullet)$$

$$(0,q'') \longmapsto (1,q''',\leftarrow)$$

$$(1,q') \longmapsto (0,q'',\rightarrow)$$

$$(1,q) \longmapsto (0,q',\rightarrow)$$

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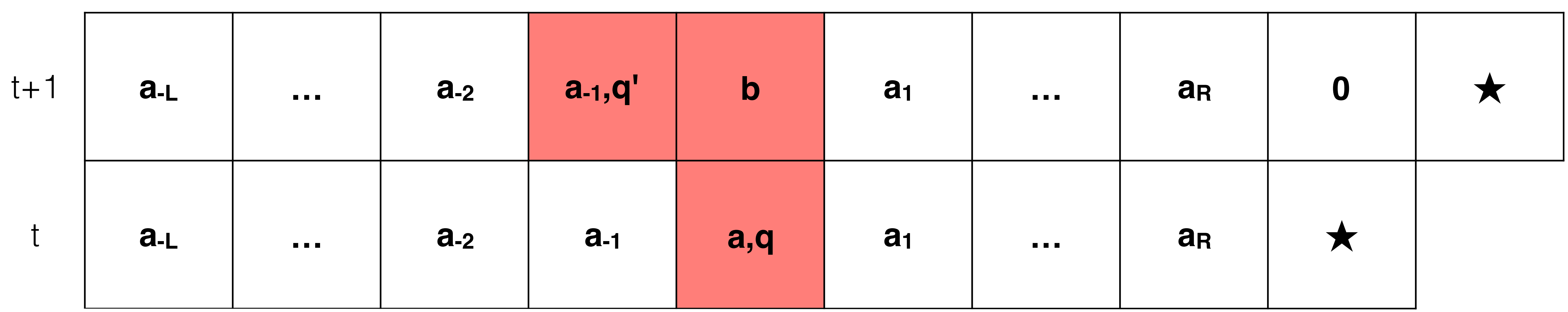
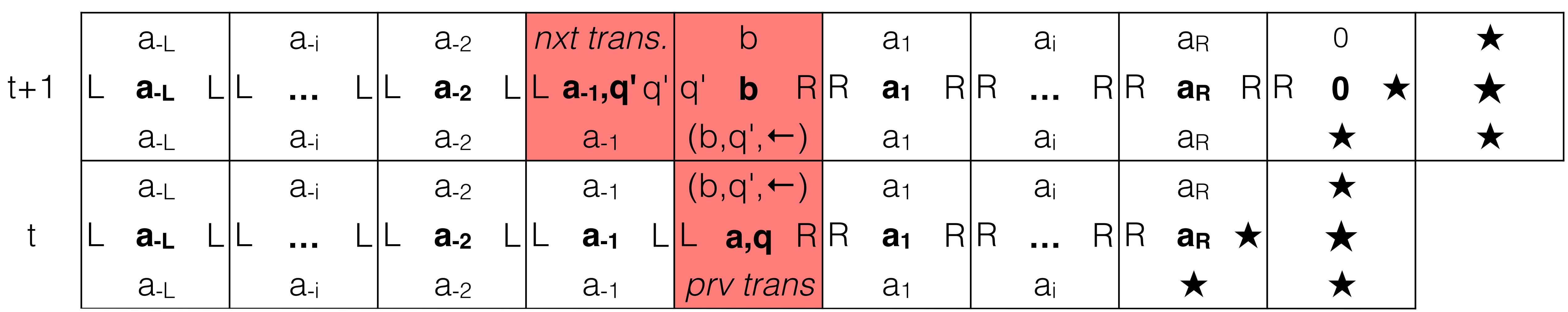


Tiles for (a,q) $\mapsto$ (b,q',$\leftarrow$)

Algorithmic self-assembly simulates any Turing machine at T°2 in 2D

1) Organize the information flow passing through the glues on the sides



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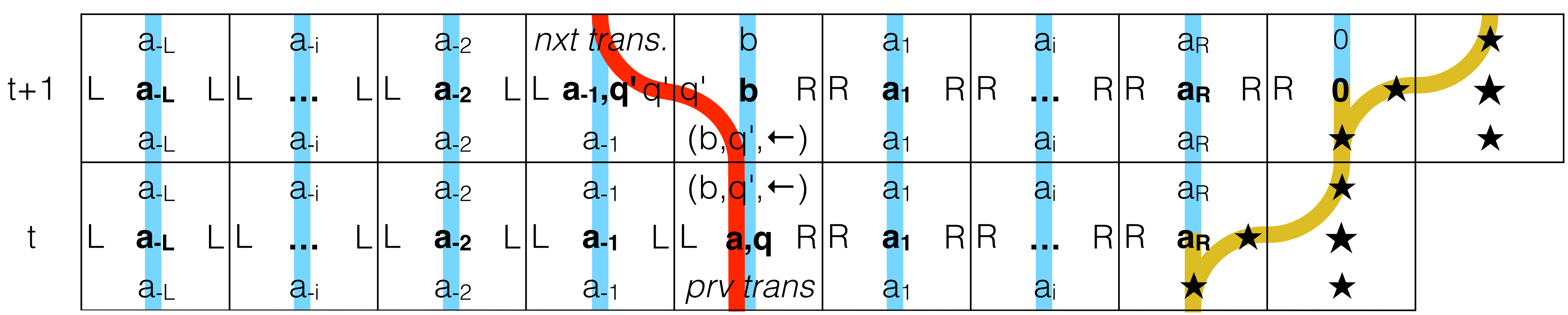
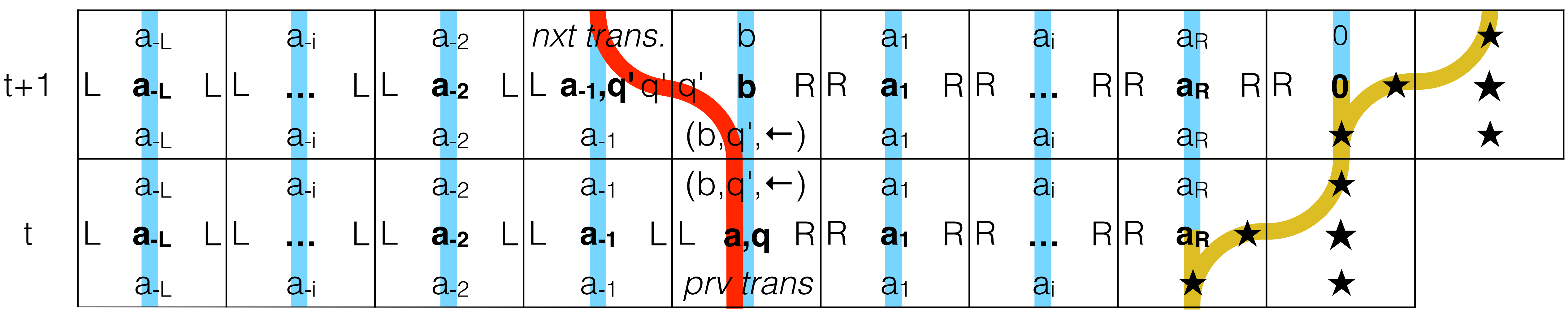


Tiles for $(a,q) \mapsto (b,q',\leftarrow)$

Algorithmic self-assembly simulates any Turing machine at T°2 in 2D

1) Organize the information flow passing through the glues on the sides

2) Identify each type of tiles



Tiles for (a,q) ⟼ (b,q',←)

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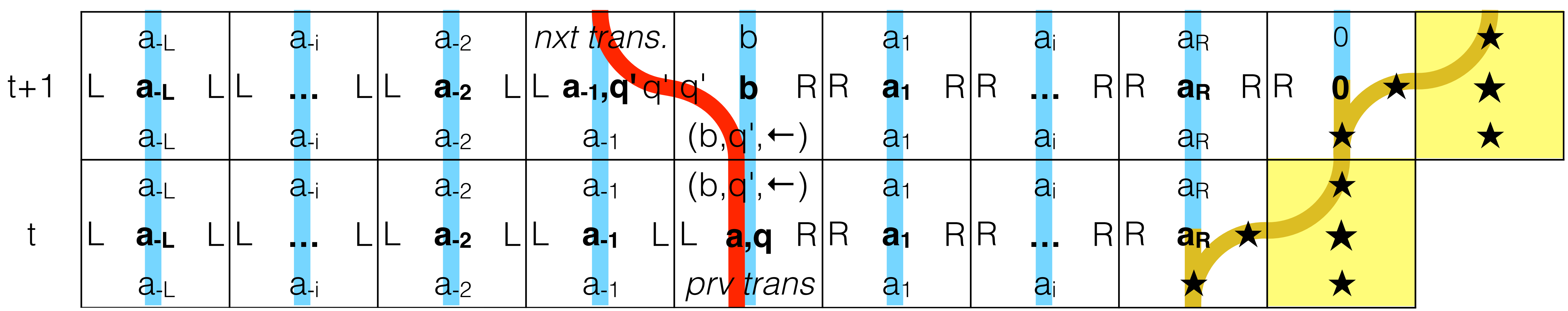
2) Identify each type of tiles

The skeleton



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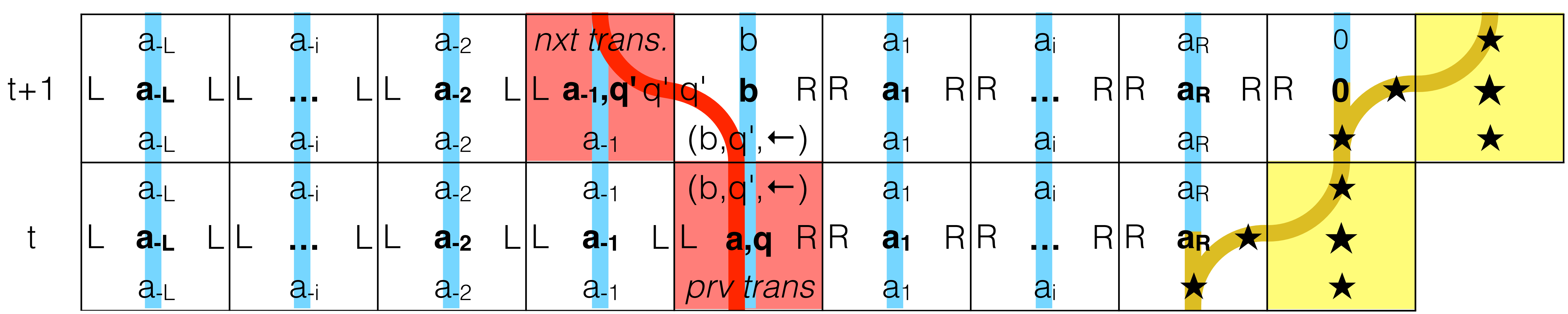

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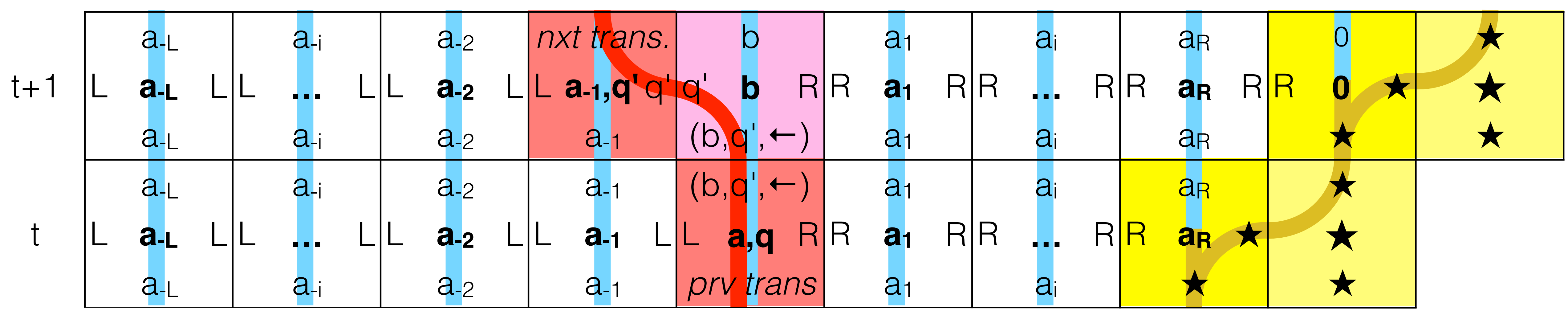
The carriers



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1) Organize the information flow passing through the glues on the sides

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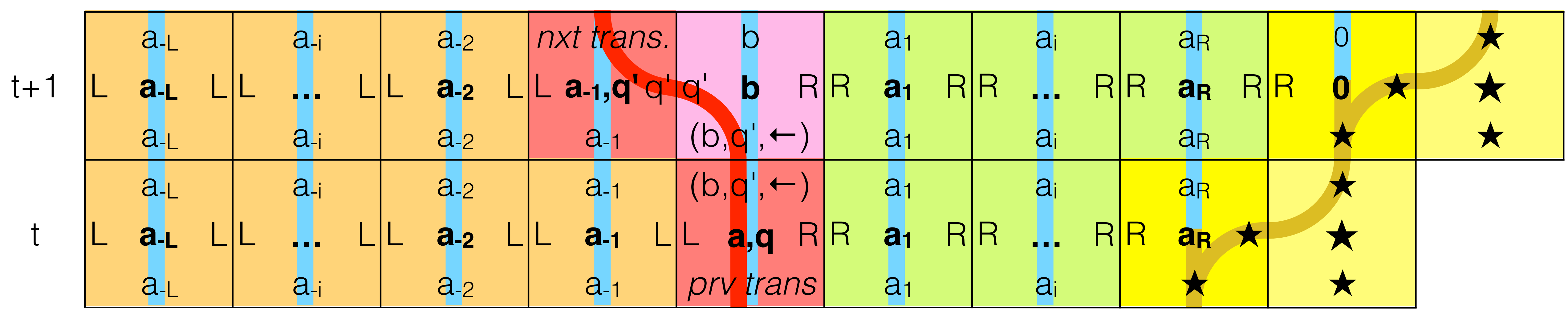
The filling tiles



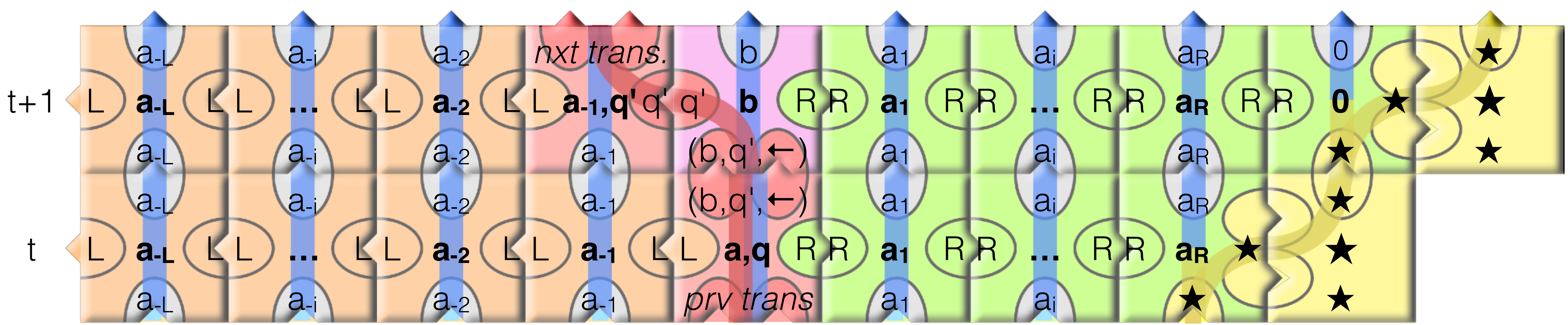
Tiles for (a,q) ⟼ (b,q',←)

Algorithmic self-assembly simulates any Turing machine at T°2 in 2D

1) Organize the information flow passing through the glues on the sides

2) Identify each type of tiles

3) Set the order, glue strength & colors to match the flow

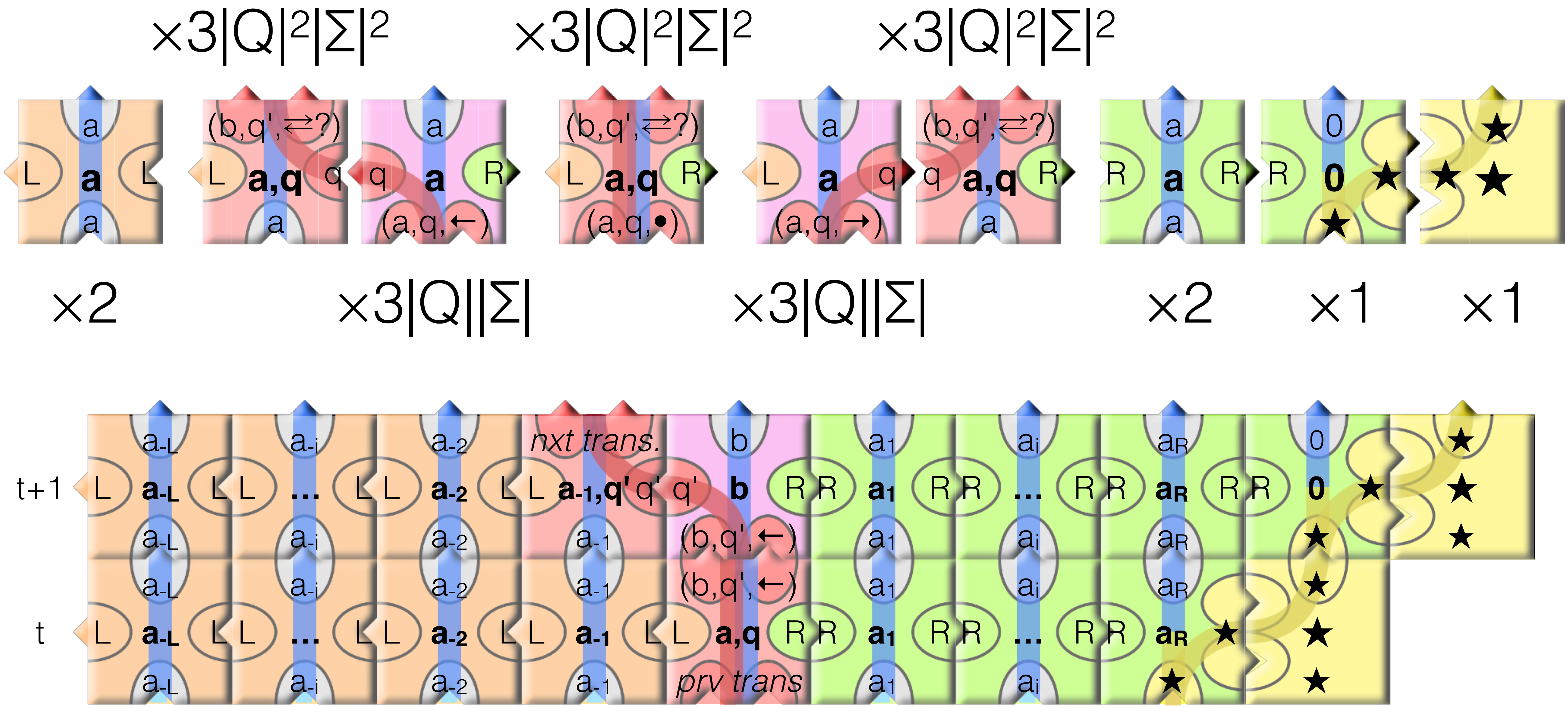


Tiles for (a,q) ↦ (b,q',←)

The tileset

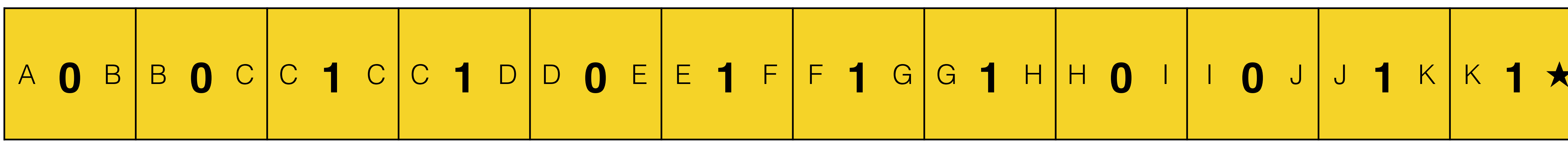
with size $O(|Q|^2|\Sigma|^2)$

The tiles for each transition $(a,q) \mapsto (b,q',\rightleftarrows?)$



Encoding the input as a seed

- **Solution 1: hardcoding**
  Uses **n** tiles for n bits



- Each tile requires **4 log n bits** to encode its glues, thus this encoding uses **4n log n bits** in total
- Can we do better, with less tiles ?

Kolmogorov complexity

- Given a universal Turing machine U:

  $$K_U(x) = \min \{ |p| : U(p,\varepsilon) = x \}$$

  is the size of the smallest program in U that outputs x

- **Fact.** $\forall U, \exists A$ s.t. $\forall x$, $K_U(x) \leq |x| + A$

  Proof. A is the size of the program "print" in U.

- **Theorem. $K_U(x)$ is independent of U**, indeed:

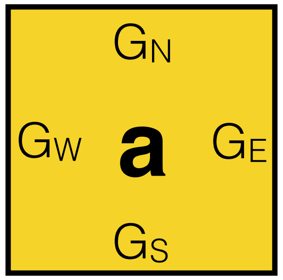
  $$\forall U, U' \; \exists A, B \text{ s.t. } \forall x,\; K_{U'}(x) - A \leq K_U(x) \leq K_{U'}(x) + B$$

  Proof. A and B are the sizes of the programs that execute U and U' respectively in U' and U.

Lower bounding the required number of tiles to encode the seed

- The Kolmogorov complexity **K(x)** is the size of the smallest program that outputs **x**

- Bit size of the encoding with **T** tiles is $\boldsymbol{\leq 4\ T \log_2 T}$

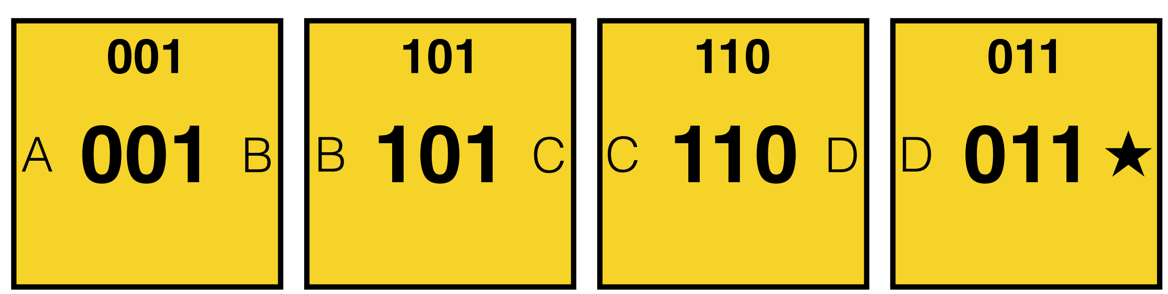


- A tileset that self-assembles x is a program that outputs x,

Thus: $\boldsymbol{4T \log T \geq K(x)}$, I.e. $\boldsymbol{T \geq K(x) / 4 \log K(x)}$

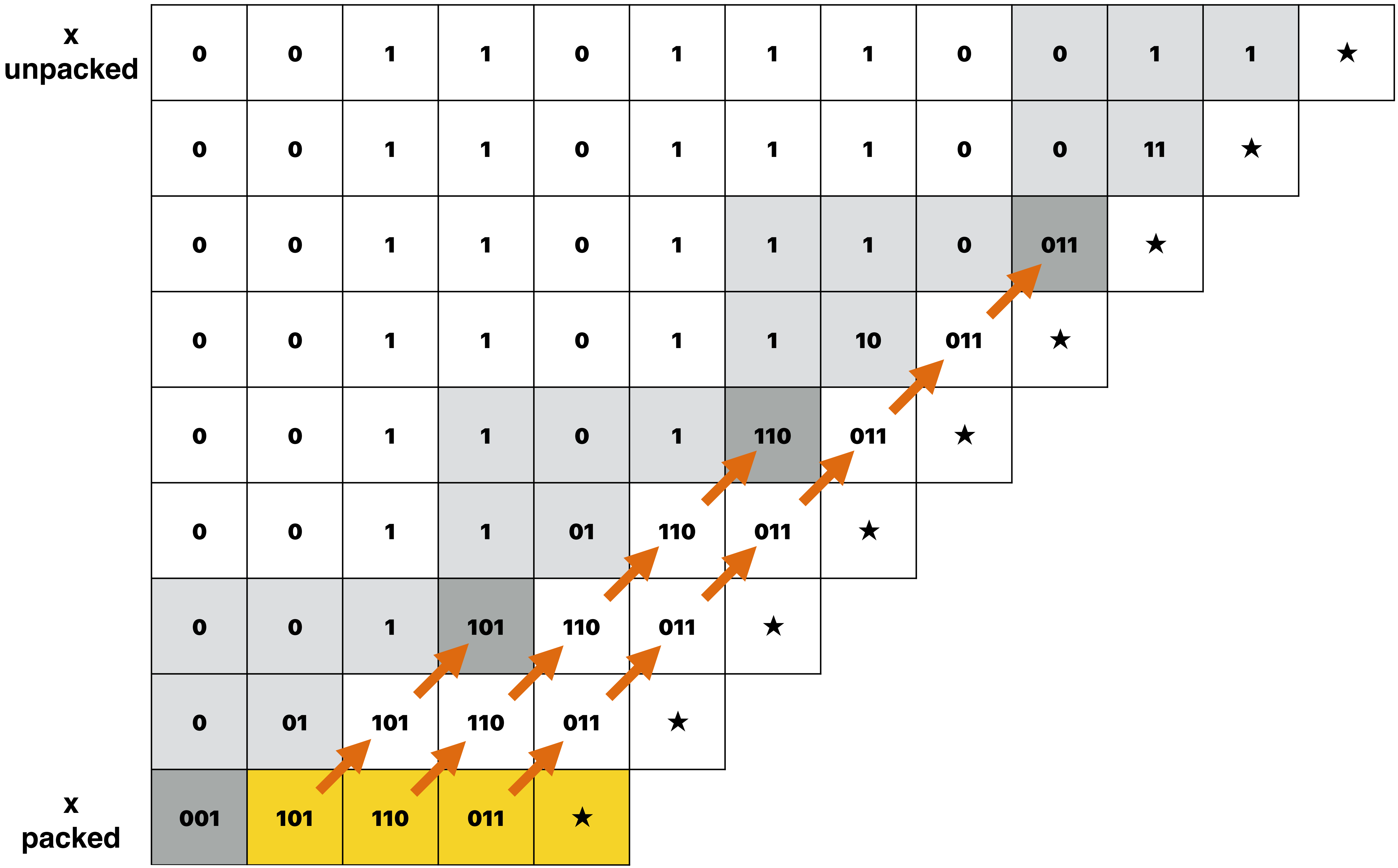
Unpacking a binary string

- Cut string **x** in **n / b** chunks of **b** bits and uncompress it
- Example: **x = 001 101 110 011** and **b = 3**

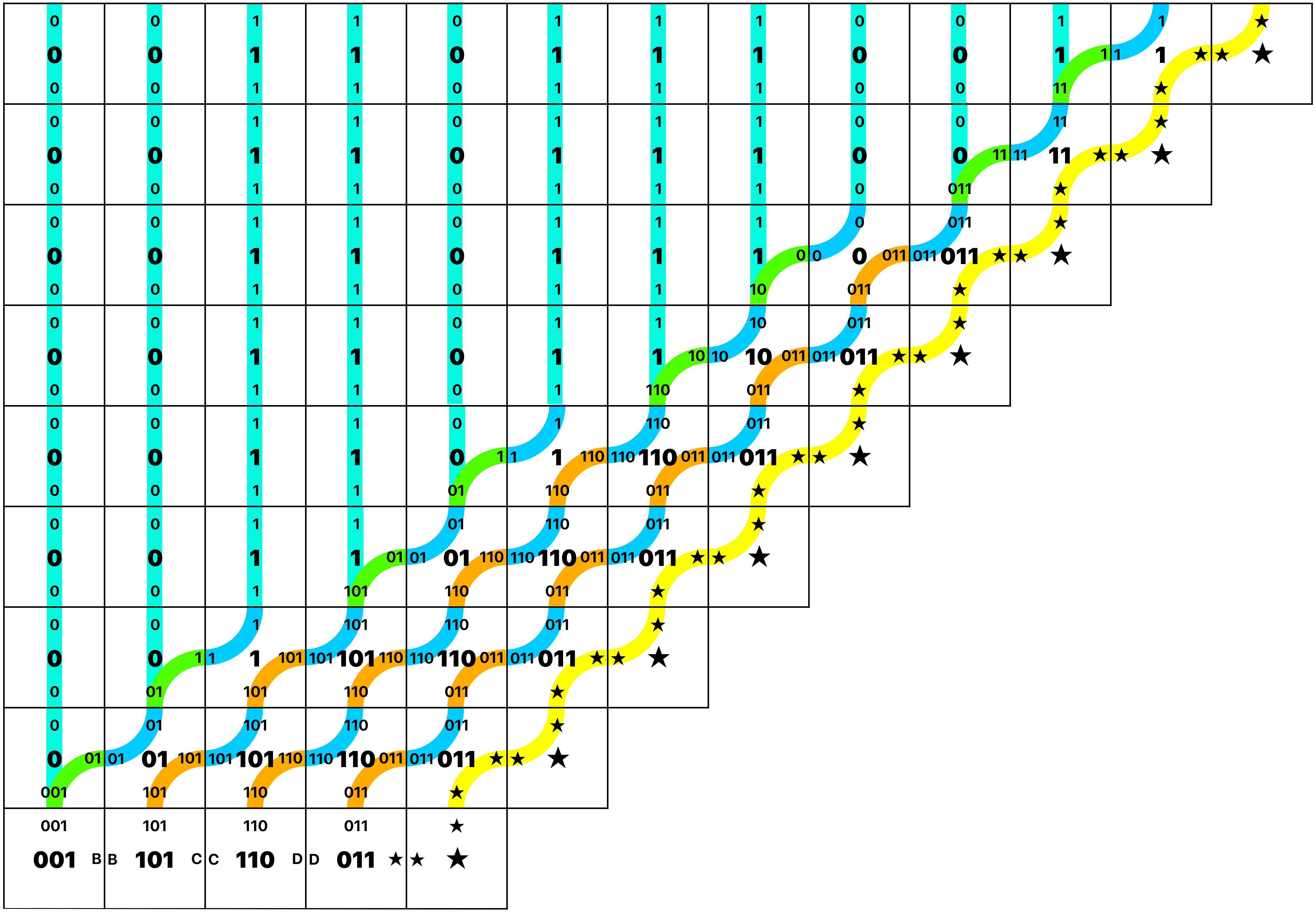


n / b tiles

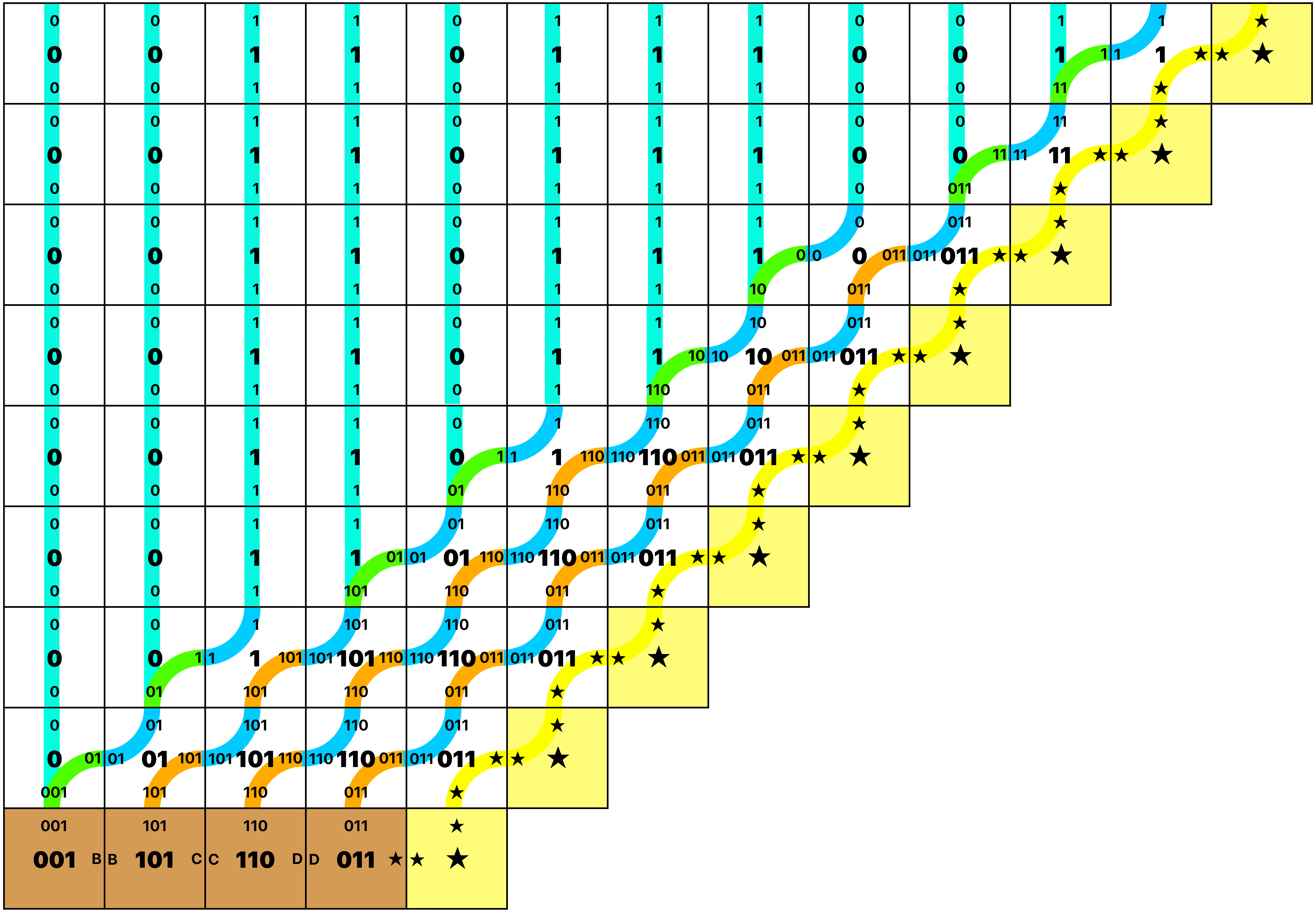
Unpacking a binary string



Determine the flow of information



Determine the various tile types



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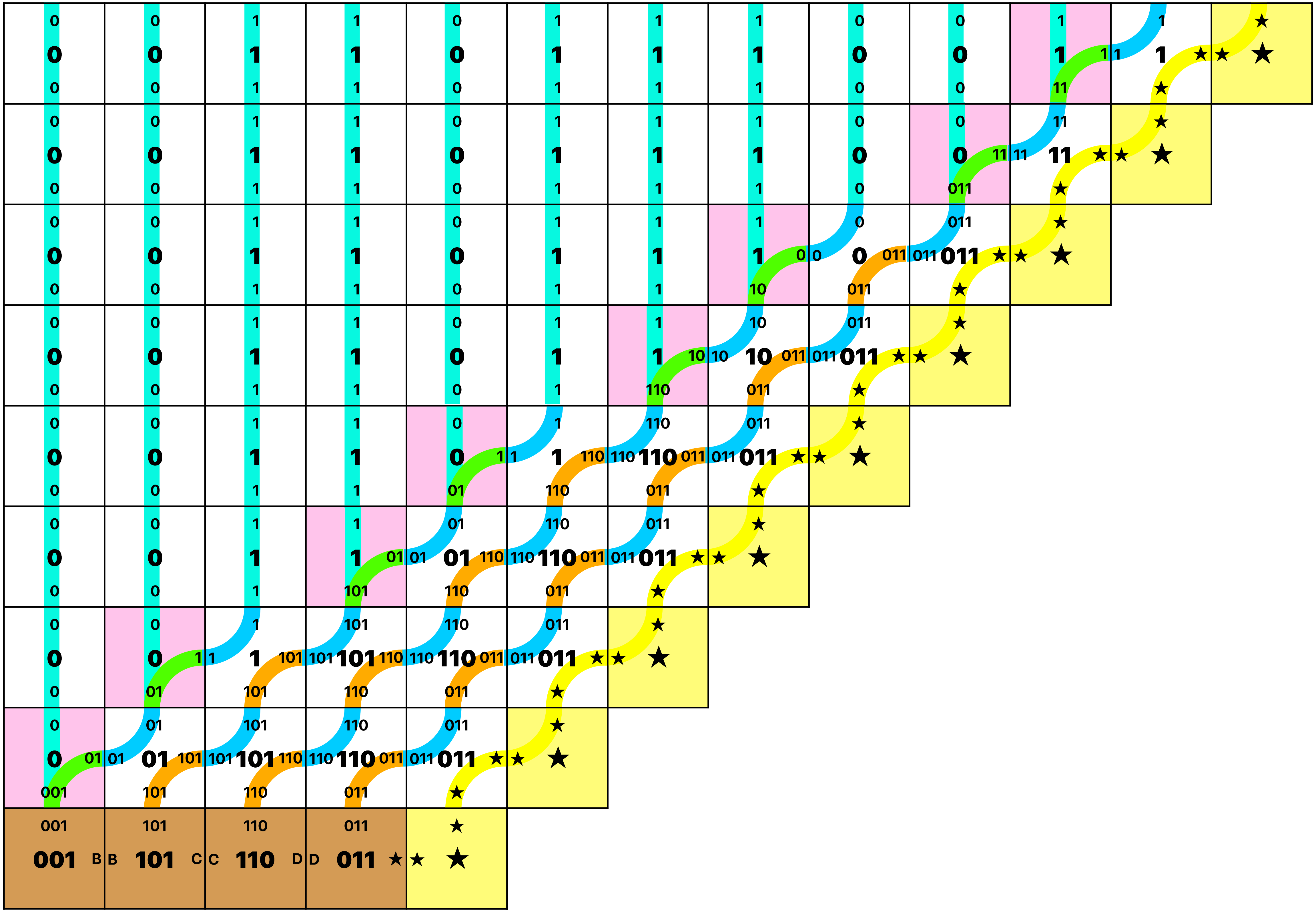

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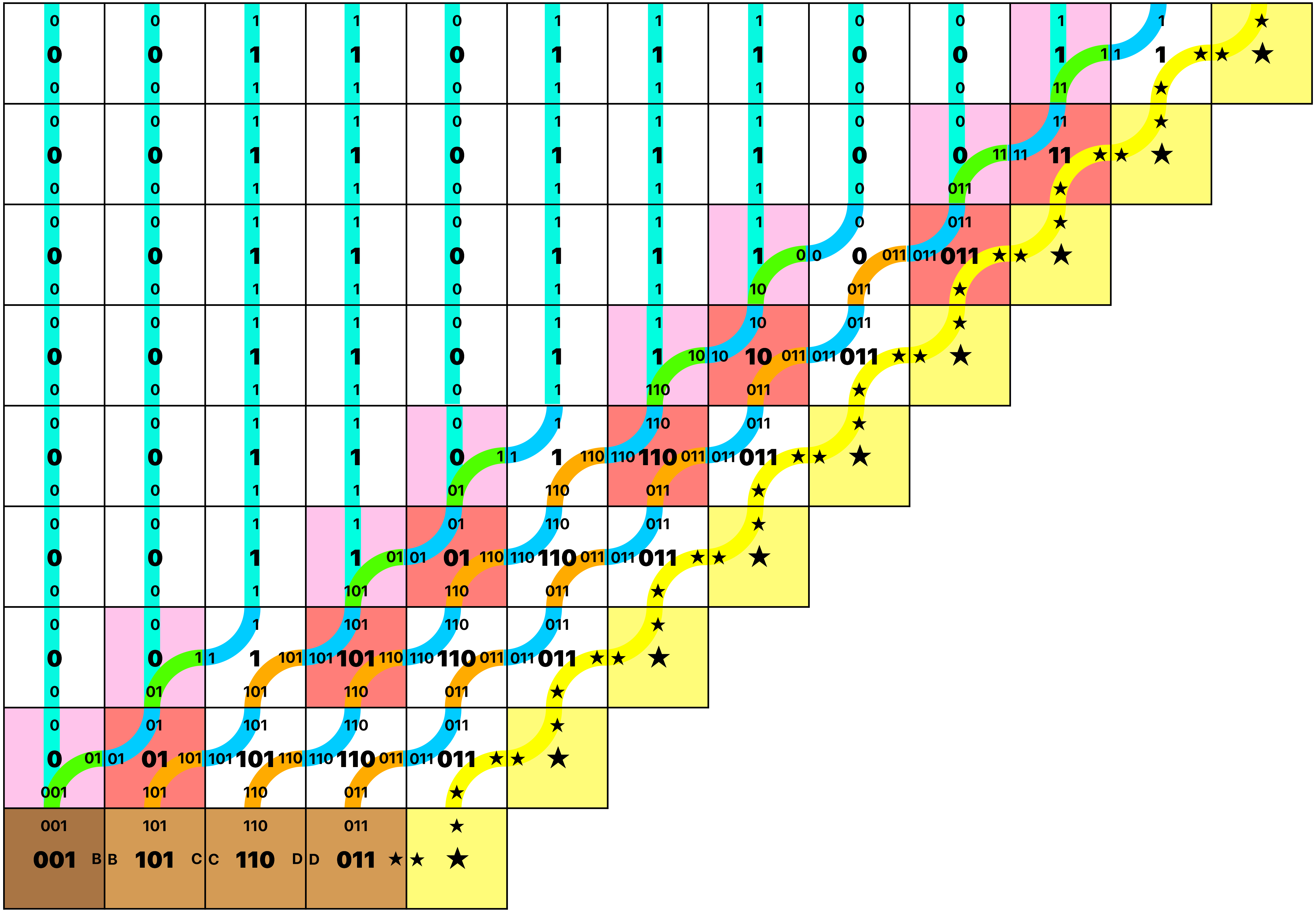
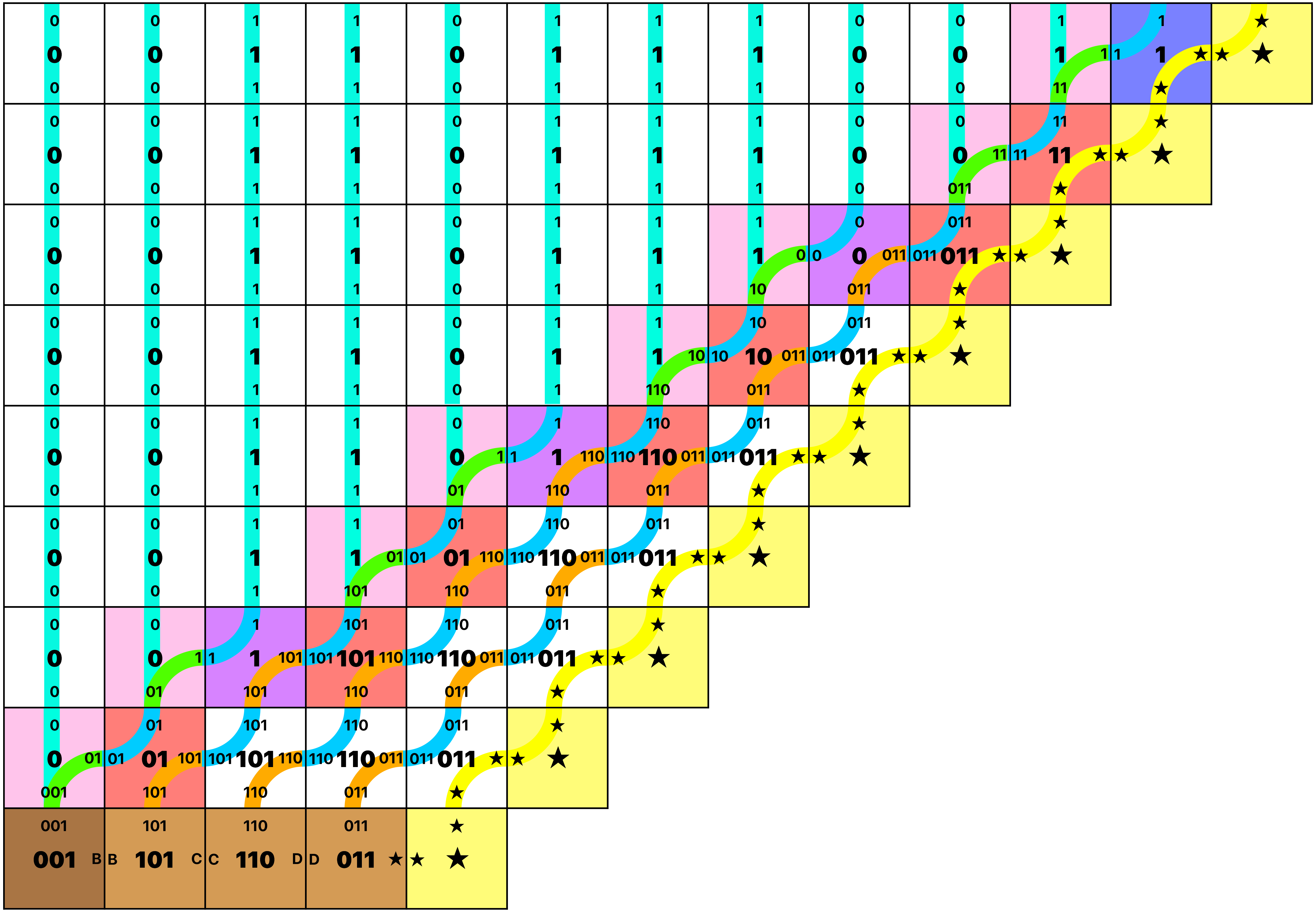
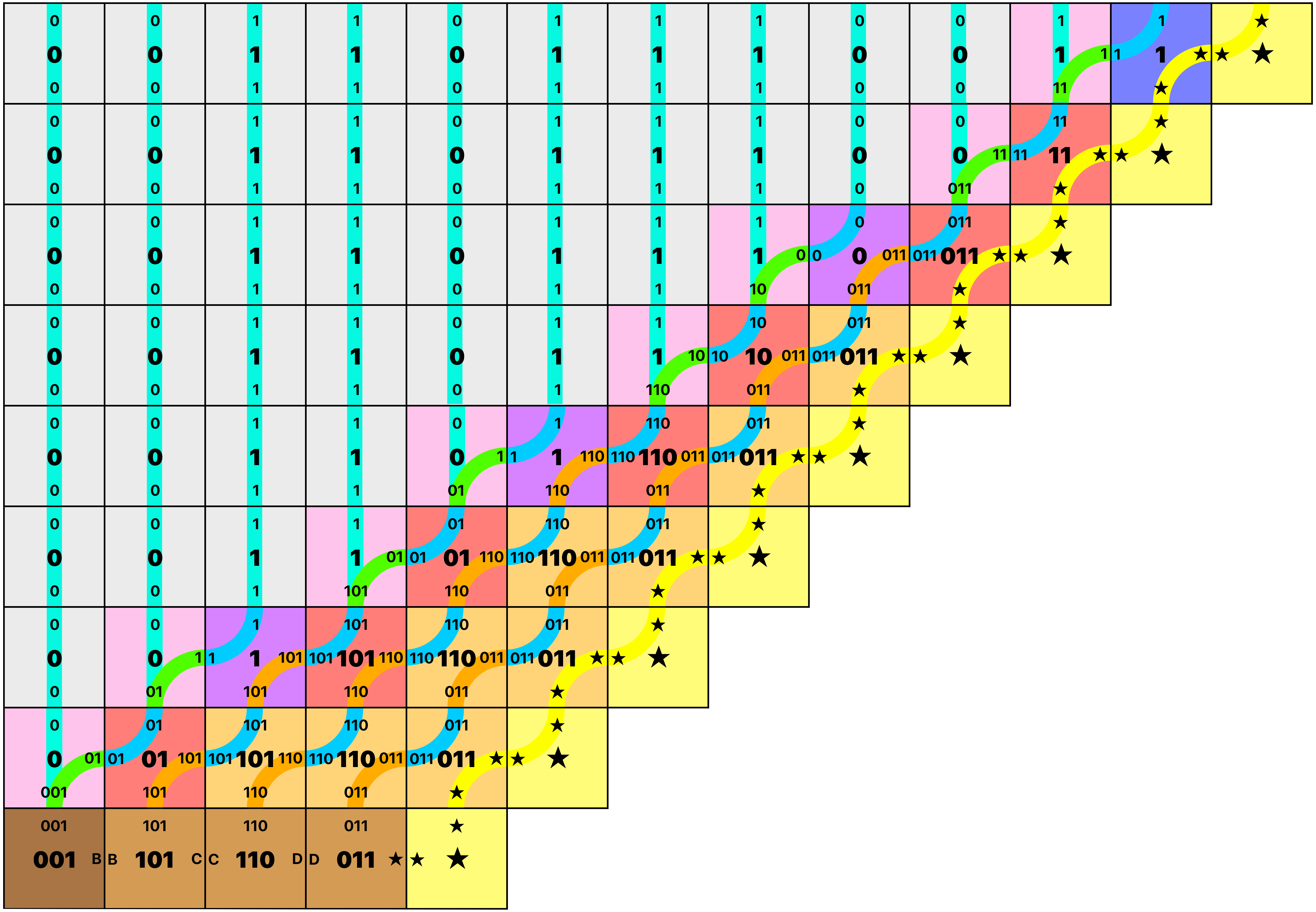
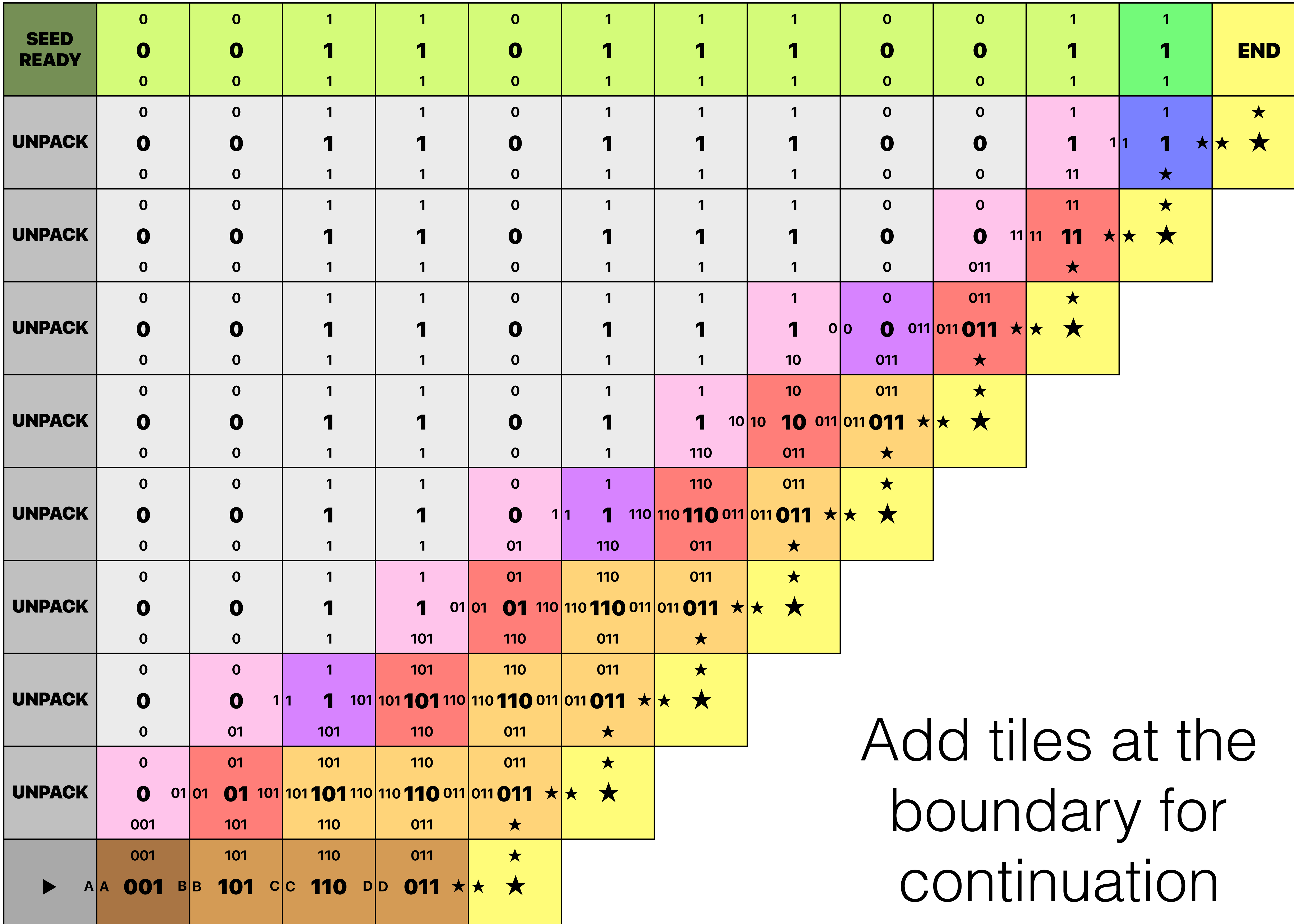


Determine the various tile types

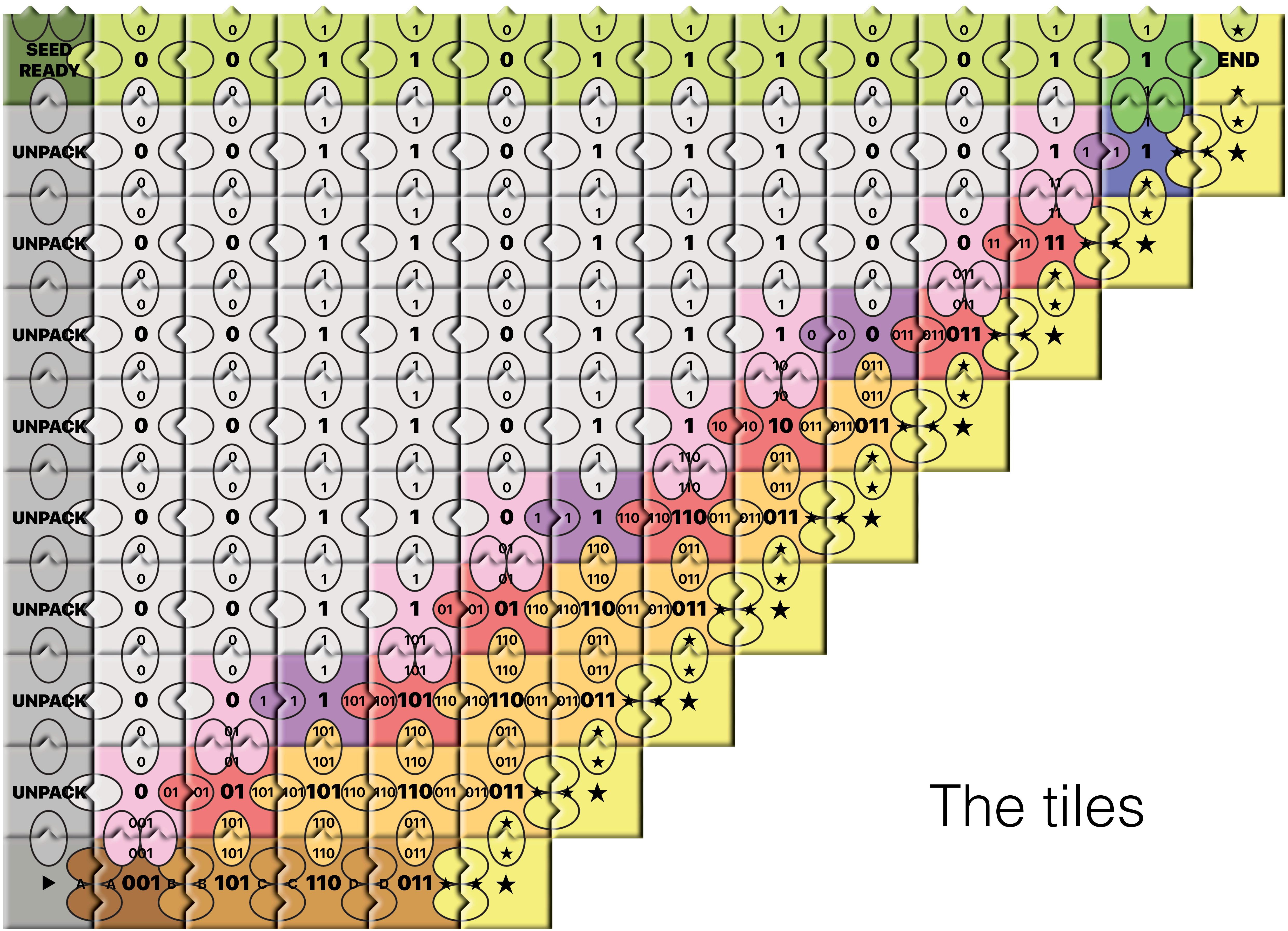


Determine the various tile types

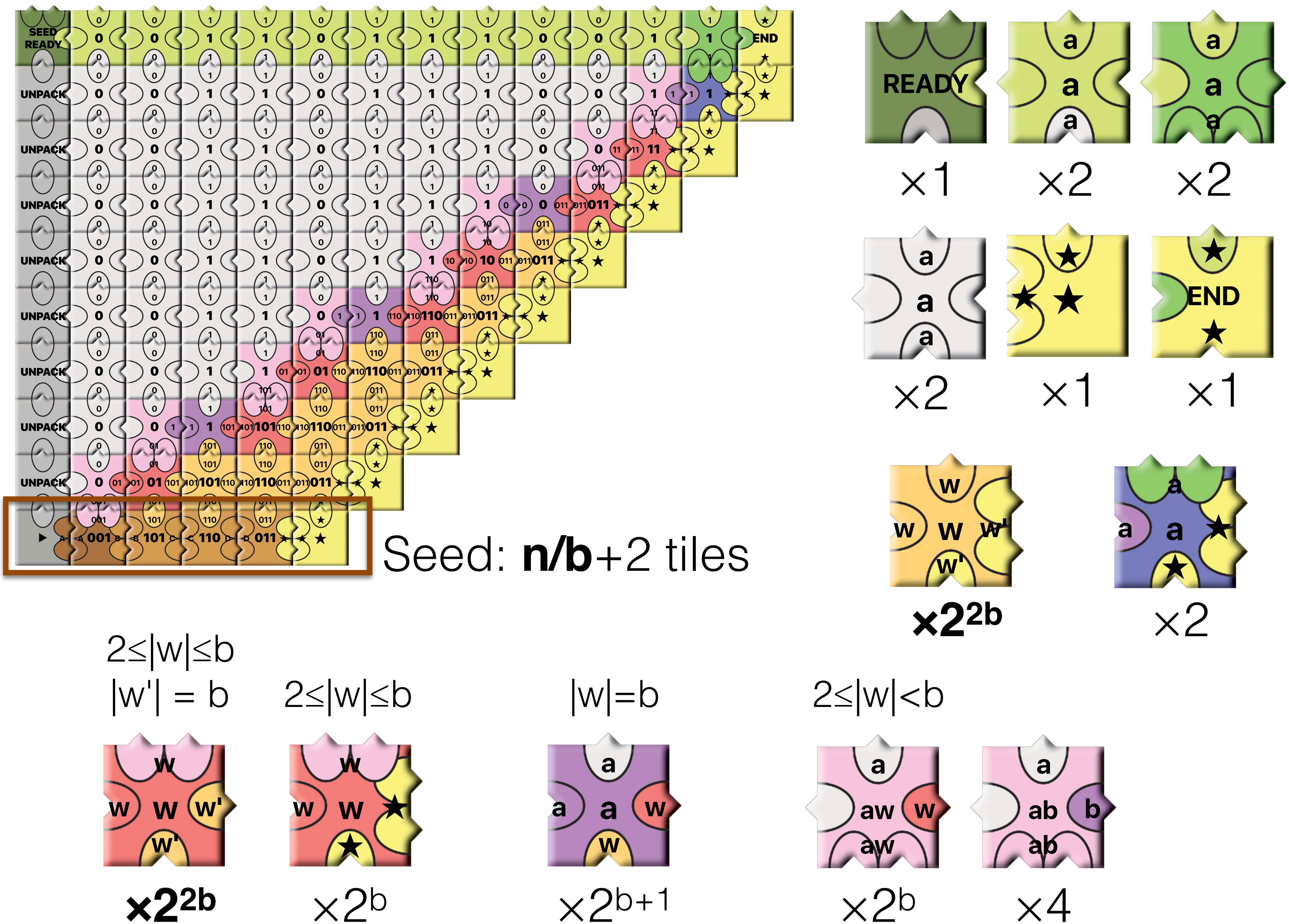




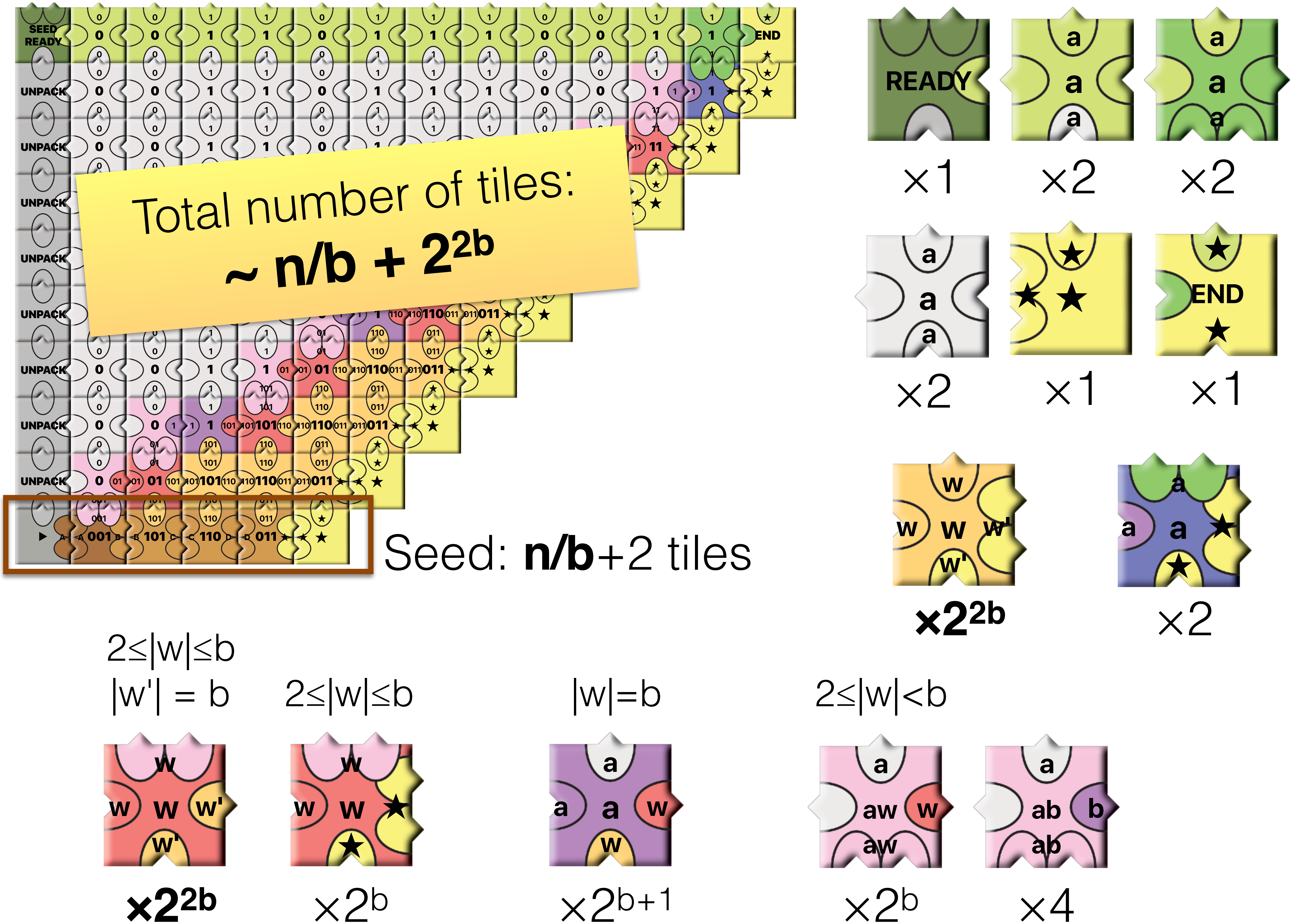
Add tiles at the boundary for continuation

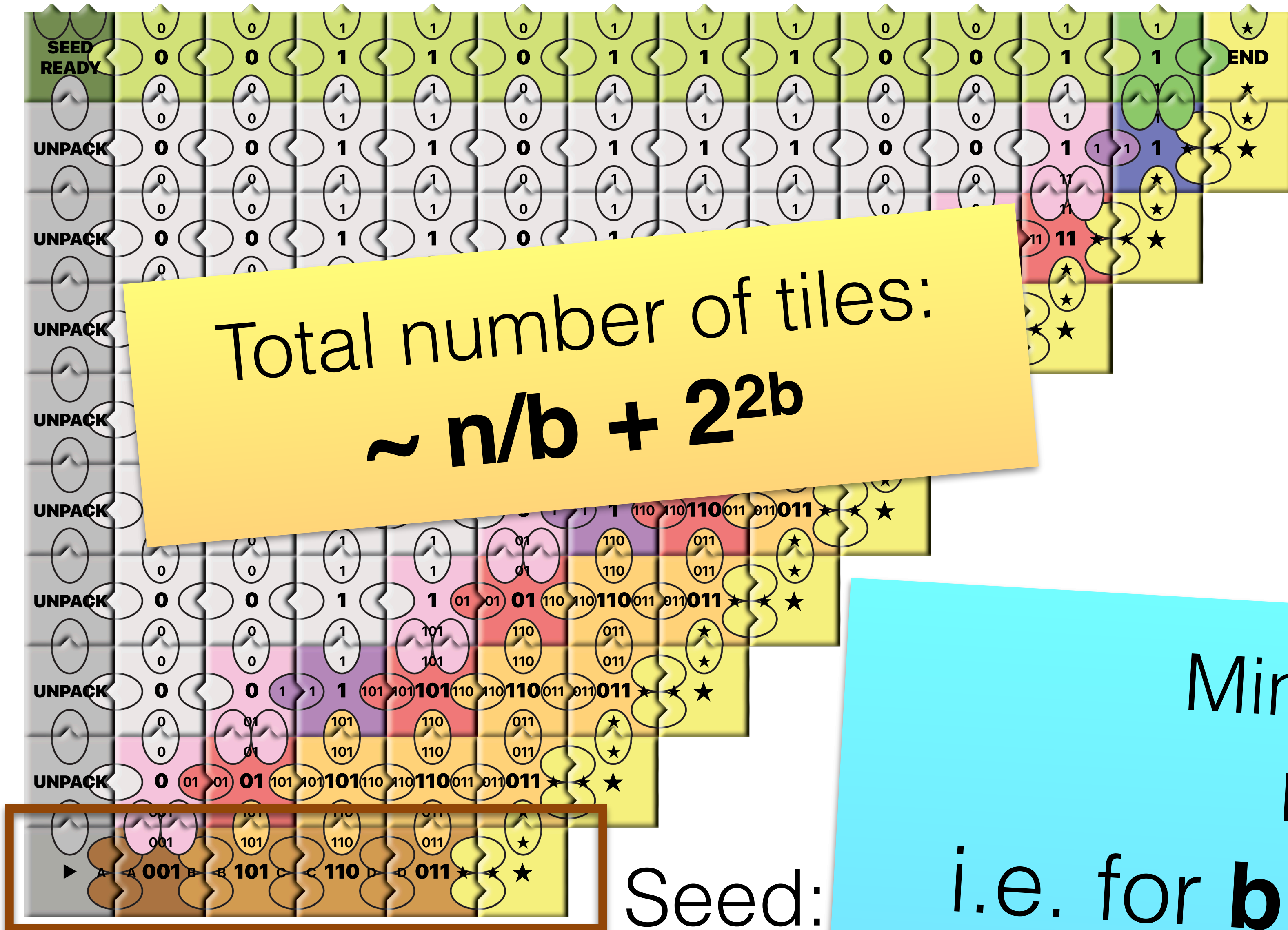


The tiles

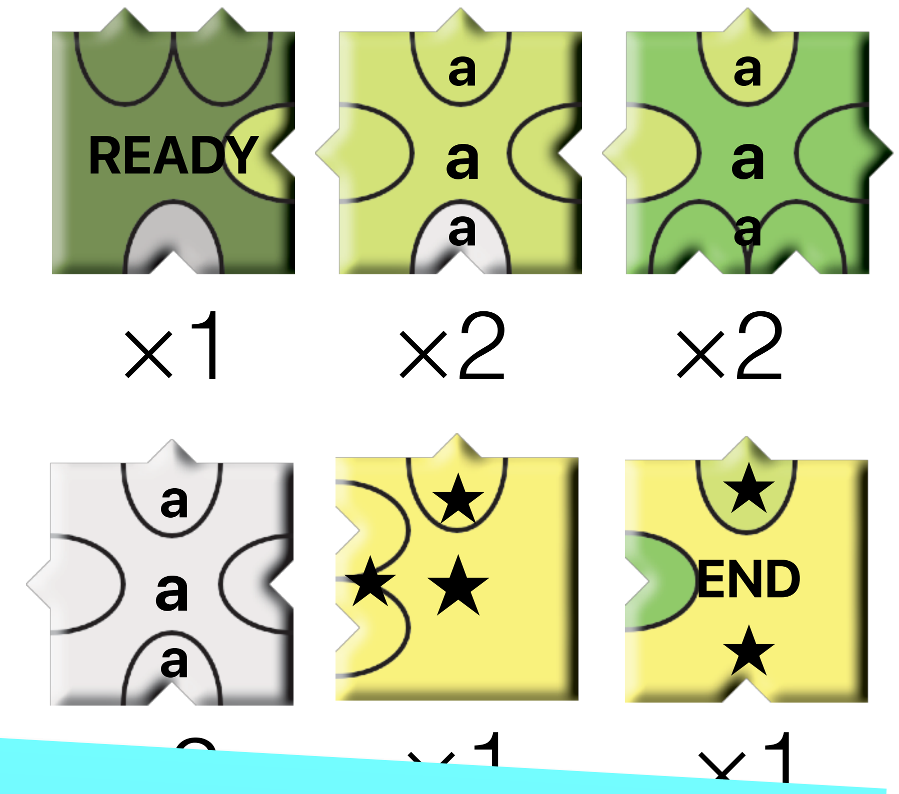


SEED READY
UNPACK
END
READY
×1
×2
×2
×2
×1
×1
Seed: **n/b**+2 tiles
×2^{2b}
×2
2≤|w|≤b
|w'| = b
2≤|w|≤b
|w|=b
2≤|w|<b
×2^{2b}
×2^b
×2^{b+1}
×2^b
×4

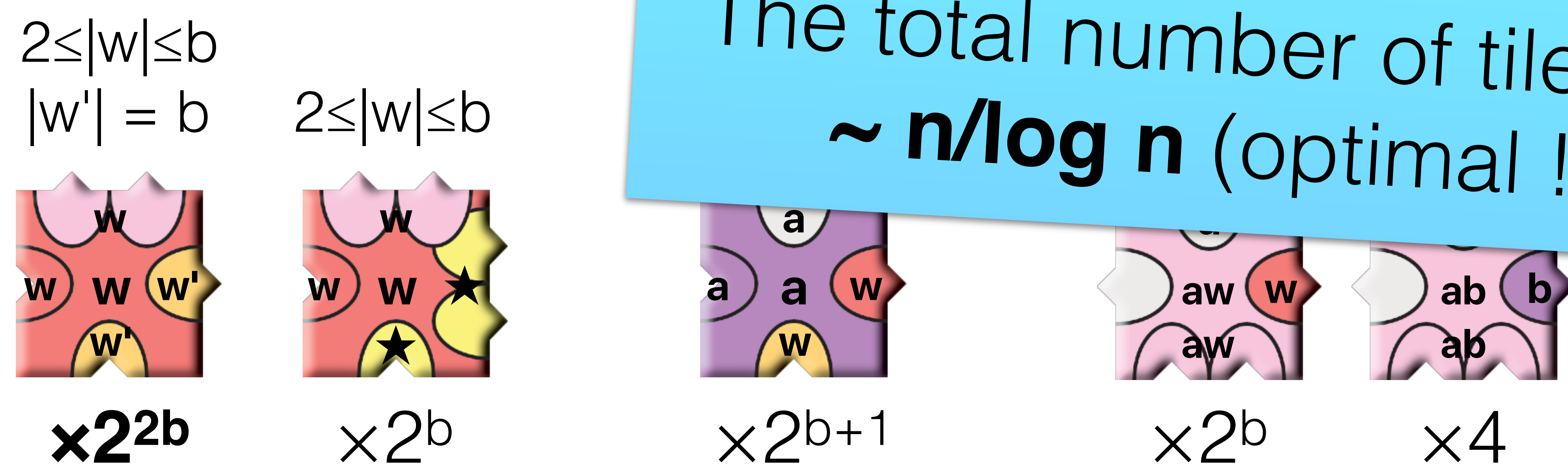

Total number of tiles:
~ n/b + 2^2b
Seed: n/b+2 tiles
SEED READY
UNPACK
END
READY
×1
×2
×2
×2
×1
×1
×2^2b
×2
2≤|w|≤b
|w'| = b
2≤|w|≤b
|w|=b
2≤|w|<b
×2^2b
×2^b
×2^b+1
×2^b
×4



Total number of tiles:
~ n/b + 2^{2b}



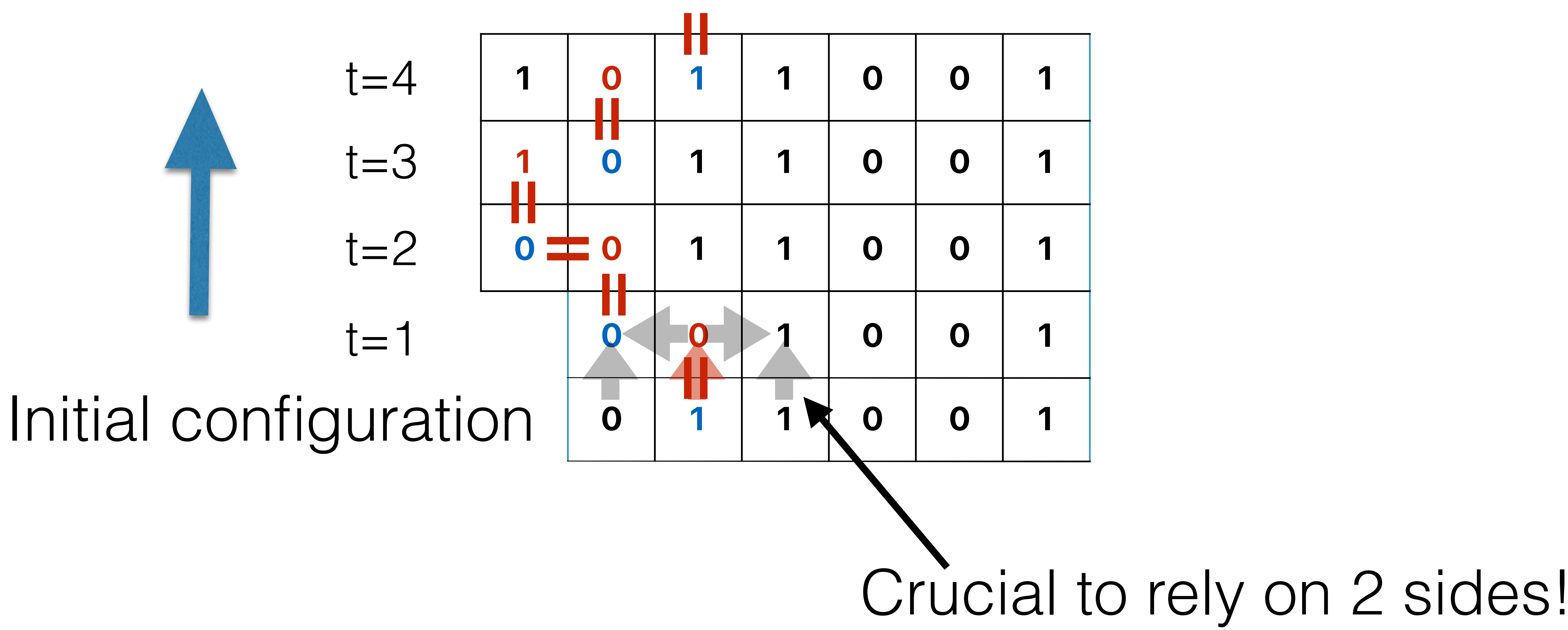
Minimized when
$n/b = 2^{2b}$
i.e. for **b ~ log(n/log n)/2**
The total number of tiles is:
~ n/log n (optimal !)



What about $T^{\circ}=1$?

Does algorithmic self-assembly simulate Turing machine at **T°1** in 2D?

- How to read and propagate the position of the head?



Does algorithmic self-assembly simulate Turing machine at **T°1** in 2D?

- **Theorem.** [Meunier, Regnault, 2015]
  Any deterministic T°1 tile set can be pumped outside a fixed radius in 2D
- **Corollary.** No T°1 tilesystem is Turing complete in 2D

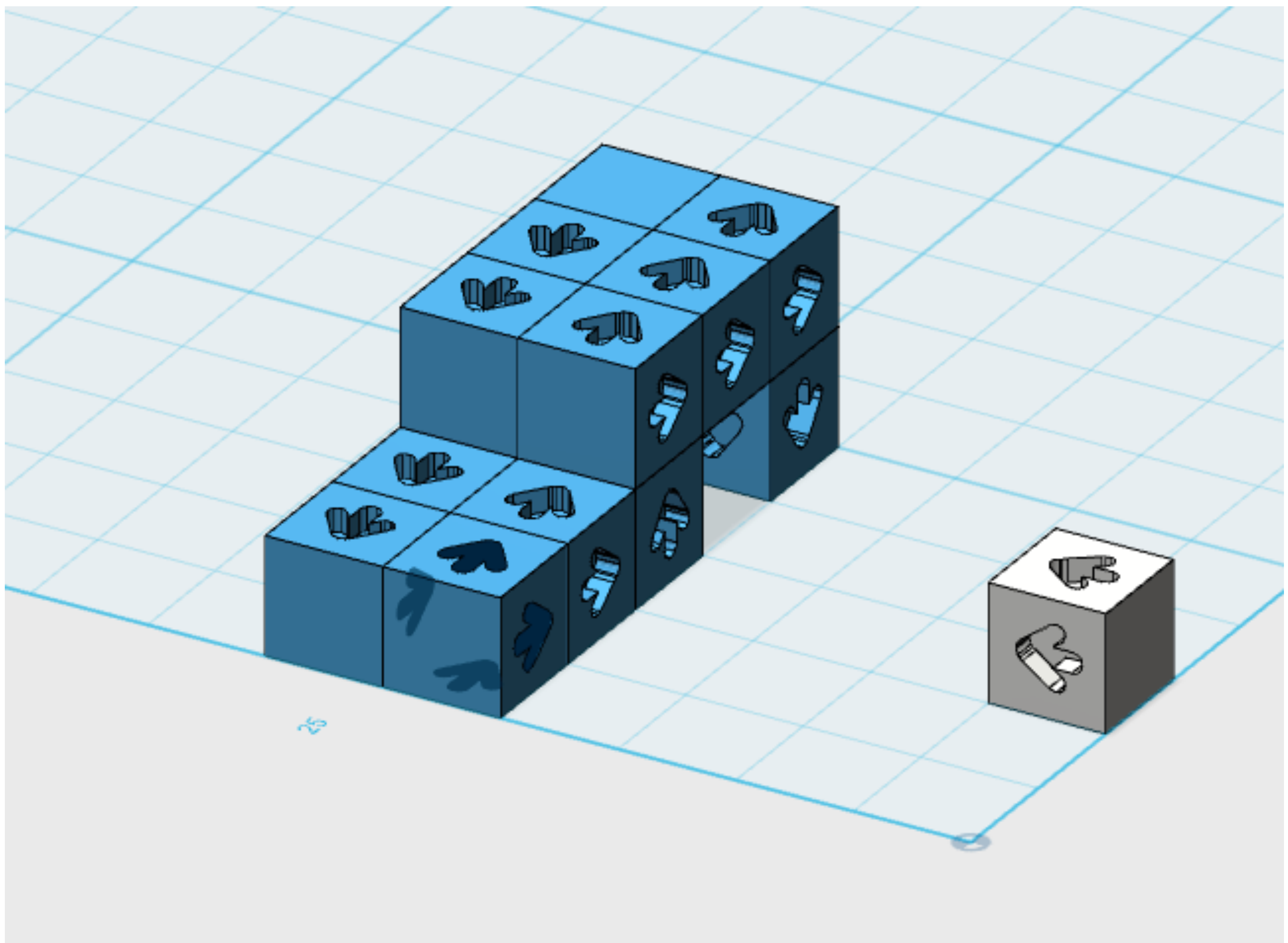
Algorithmic self-assembly simulates any Turing machine at T°1 in 3D

- **Theorem.** *[Cook, Fu, Schweller, 2011]*
  There is a T°1 tile set that simulates any Turing machine with 2-layers in 3D.

- *Key beautiful idea:* Blocking probe crystal

Algorithmic self-assembly simulates any Turing machine at T°1 in 3D

- *Key beautiful idea:* Blocking probe crystal



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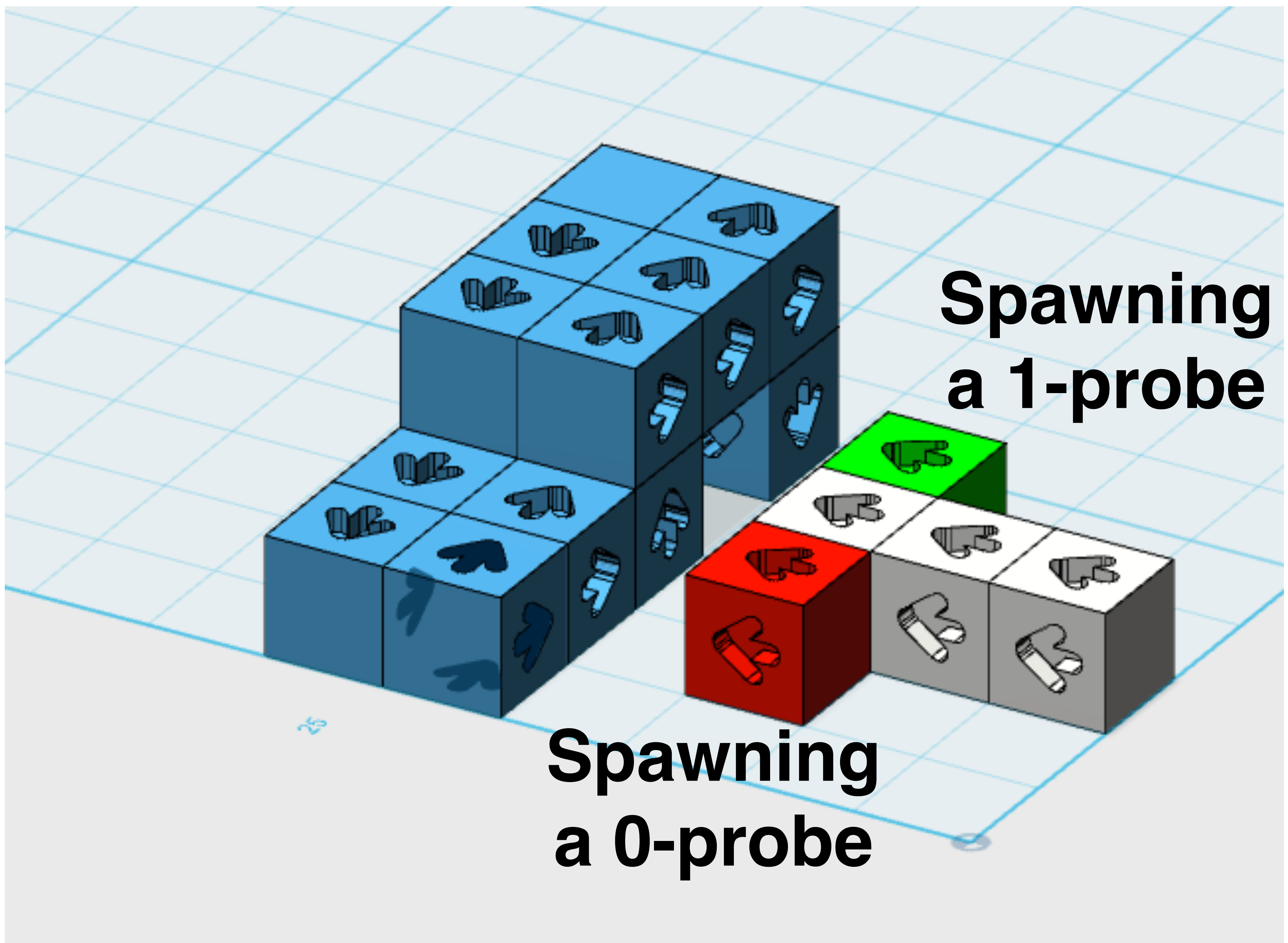

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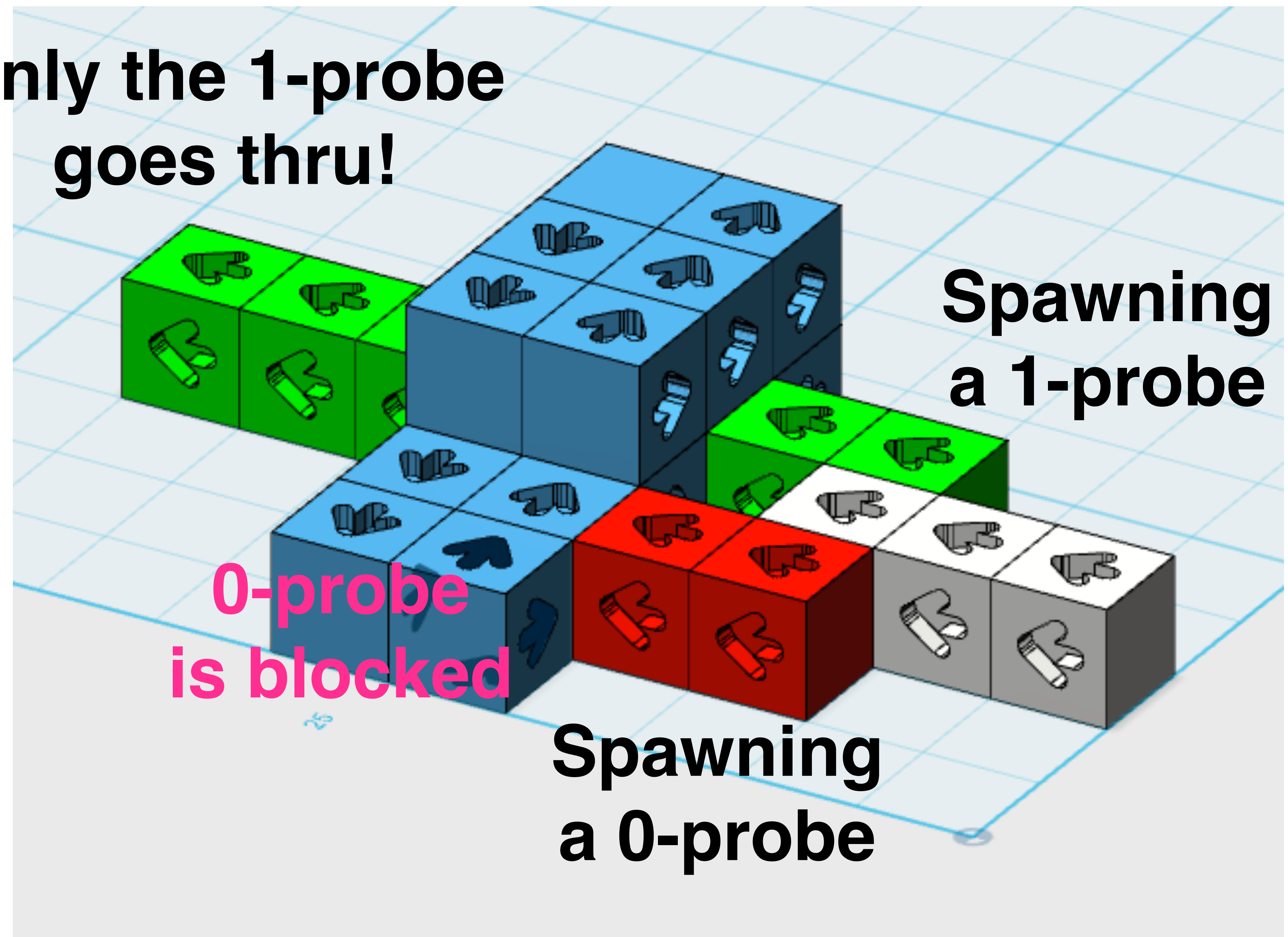

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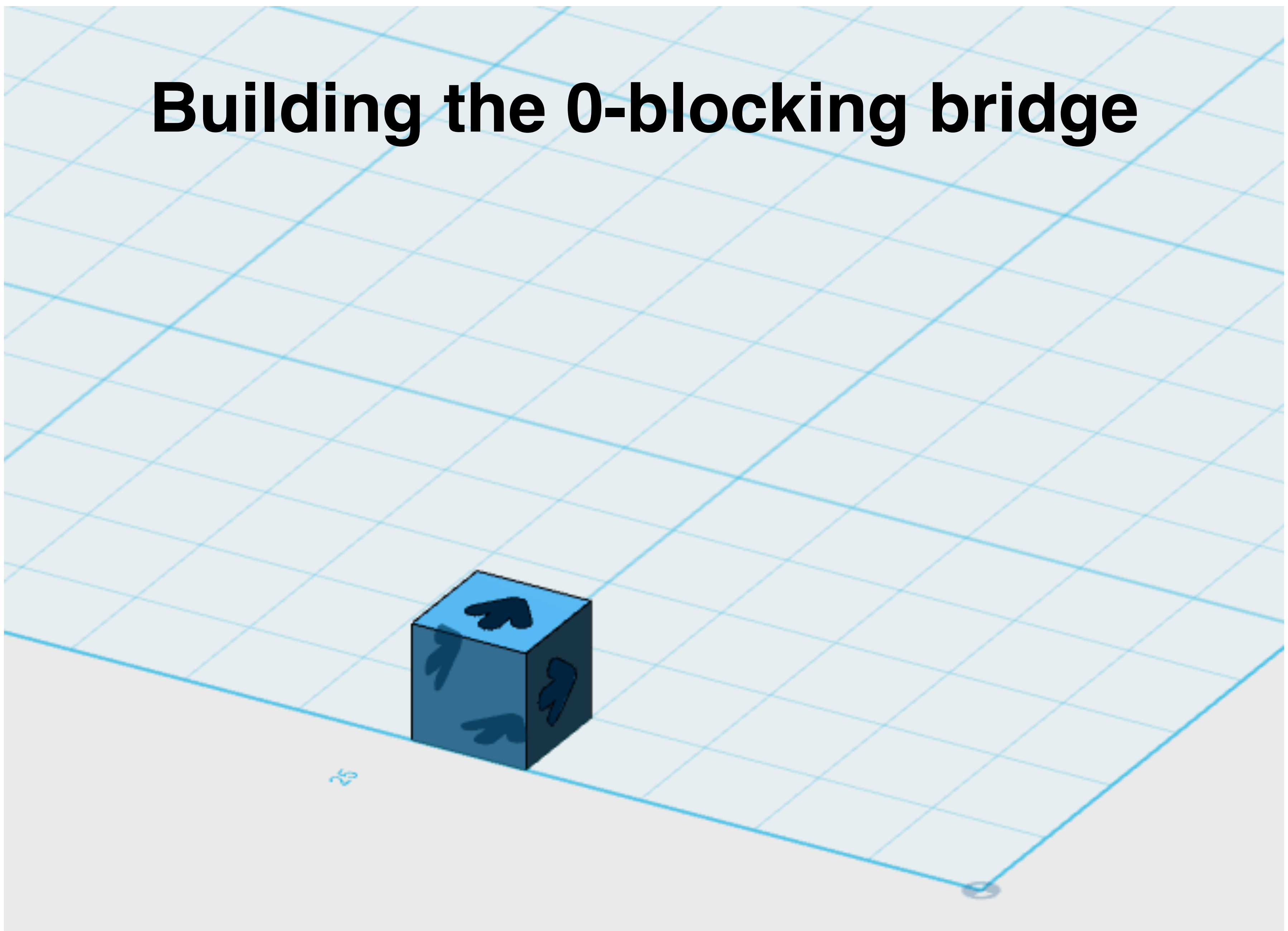

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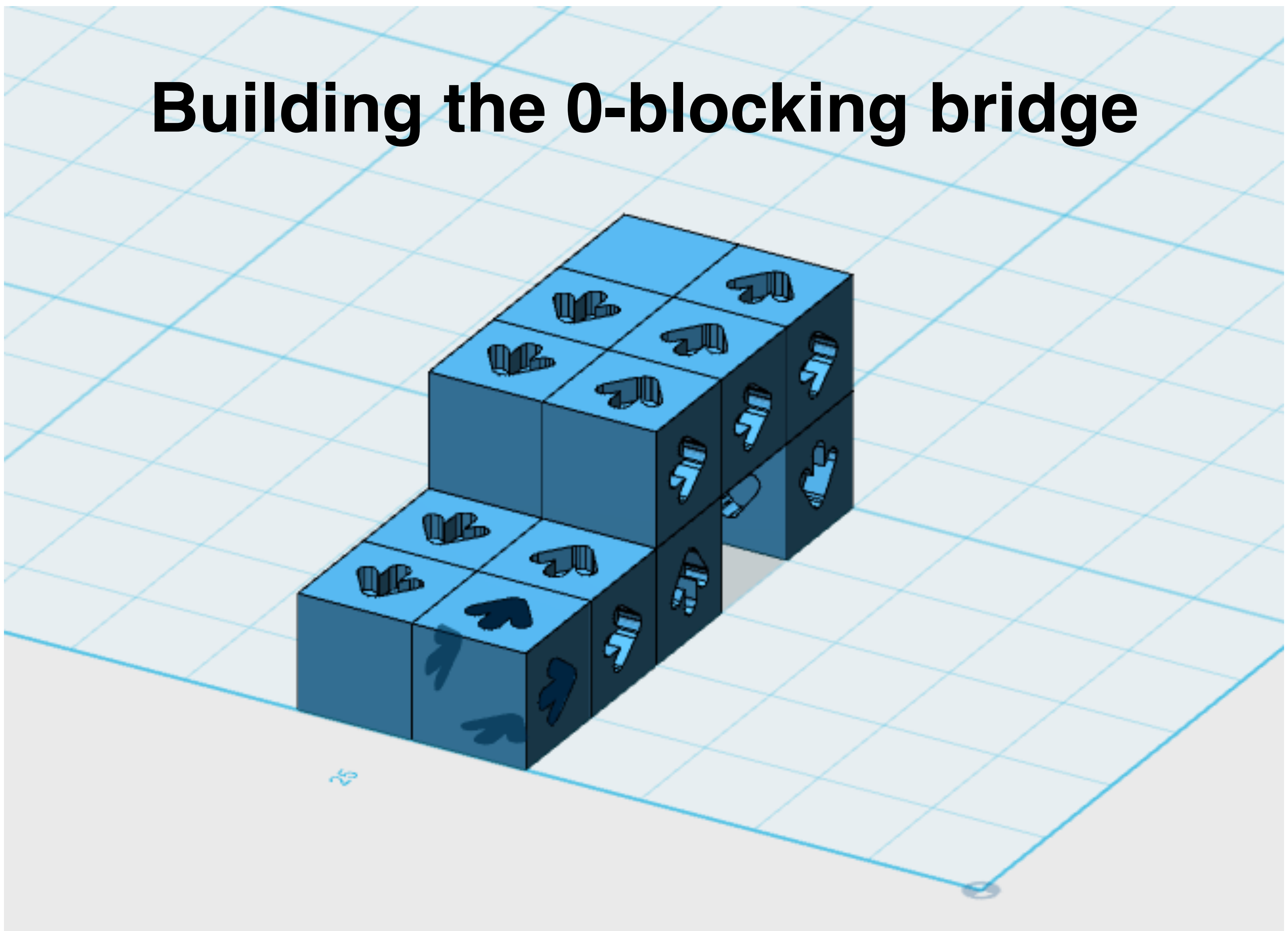


Algorithmic self-assembly simulates any Turing machine at T°1 in 3D

- *Key beautiful idea:* Blocking probe crystal

We can thus read and write 0 and 1 on the "next tape" !

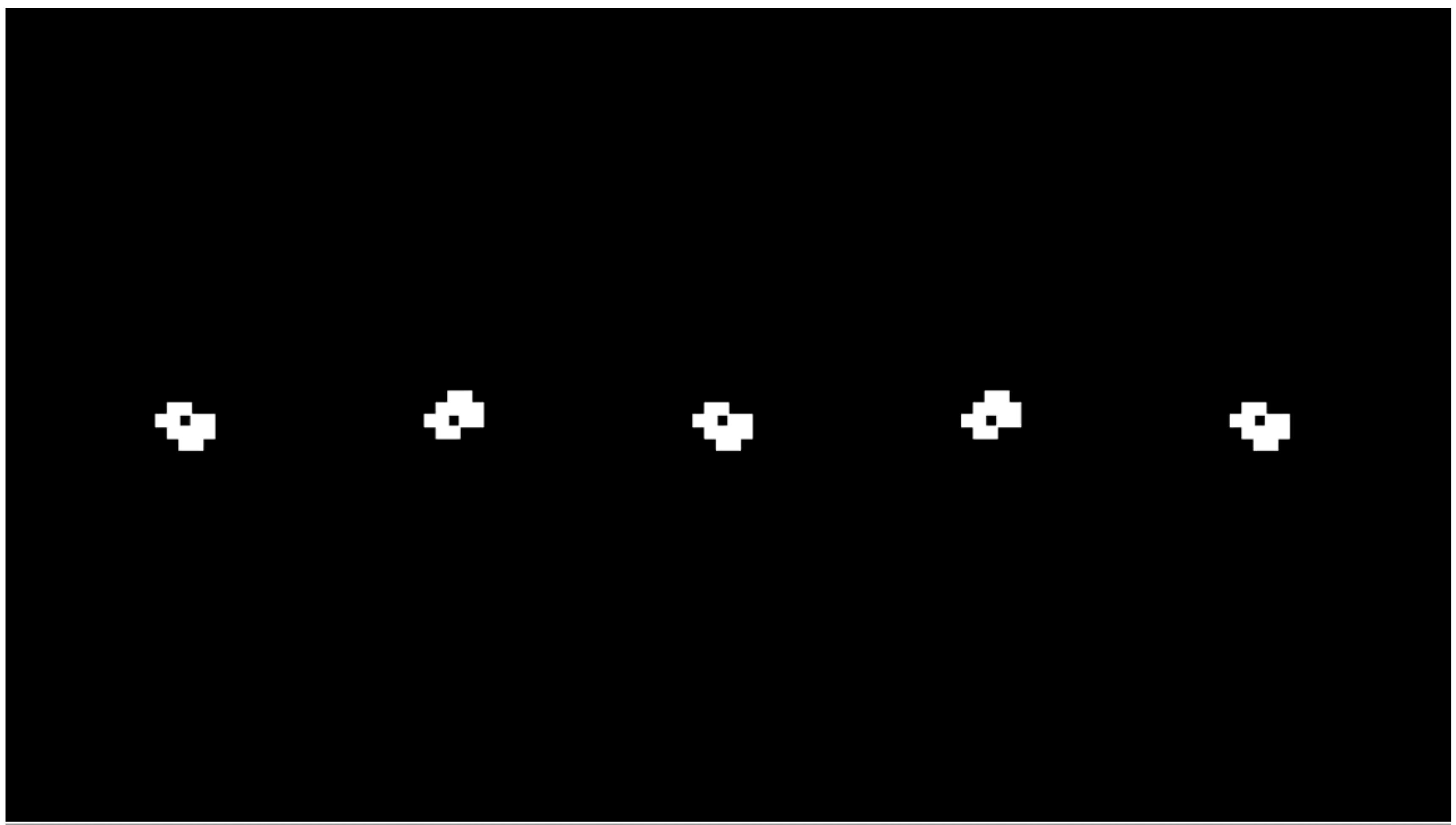
Crystals are Turing complete !!!

Universal Tileset at T°2

A universal tile set to build any (assemblable) shape

Is there a universal tileset at T°=2 ?

- Rescaling : *intrinsic simulation*



Brice Due 2006

The Game of Life self-simulating itself intrinsically:
Smaller cells simulate macro-cells

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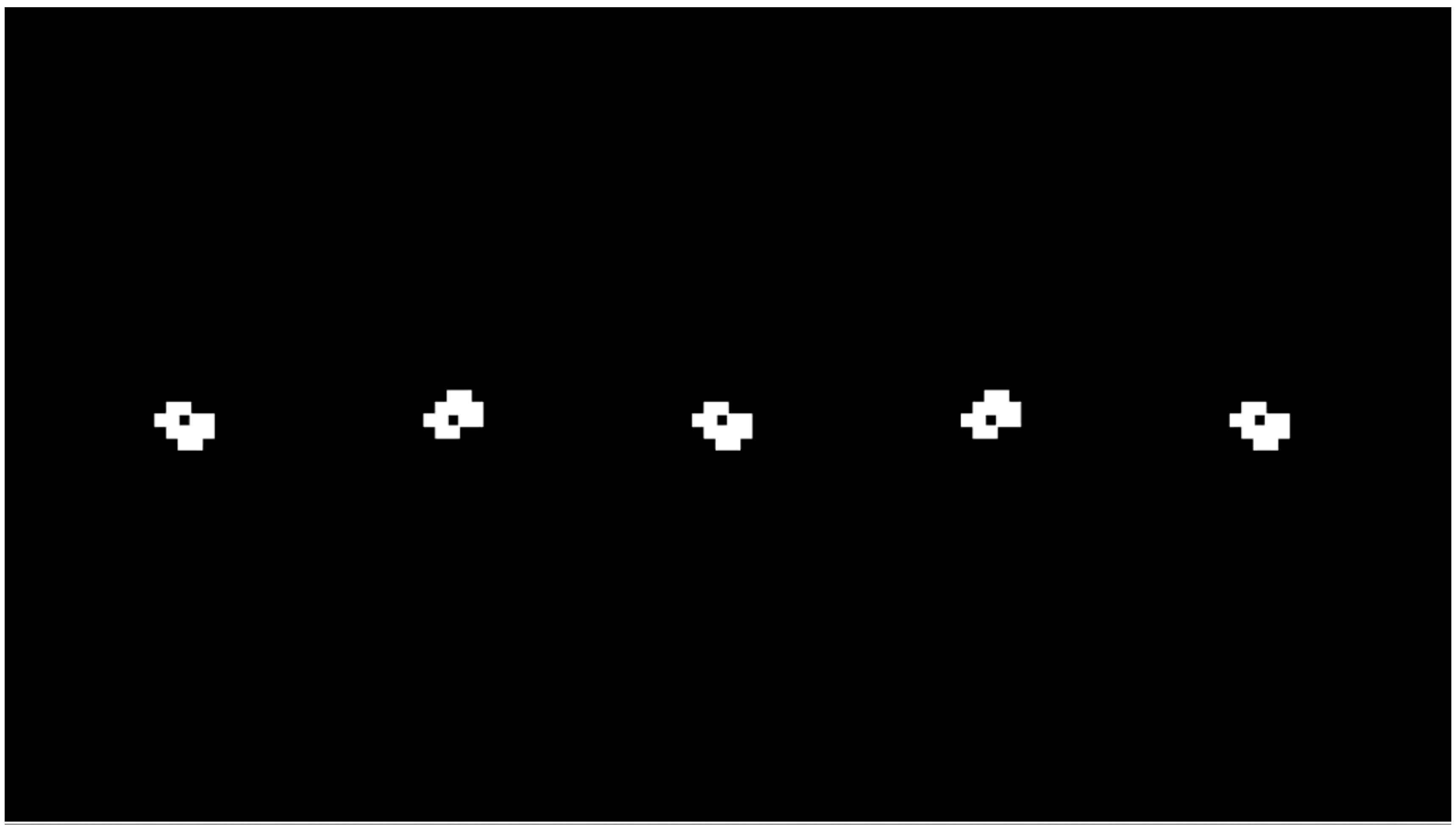

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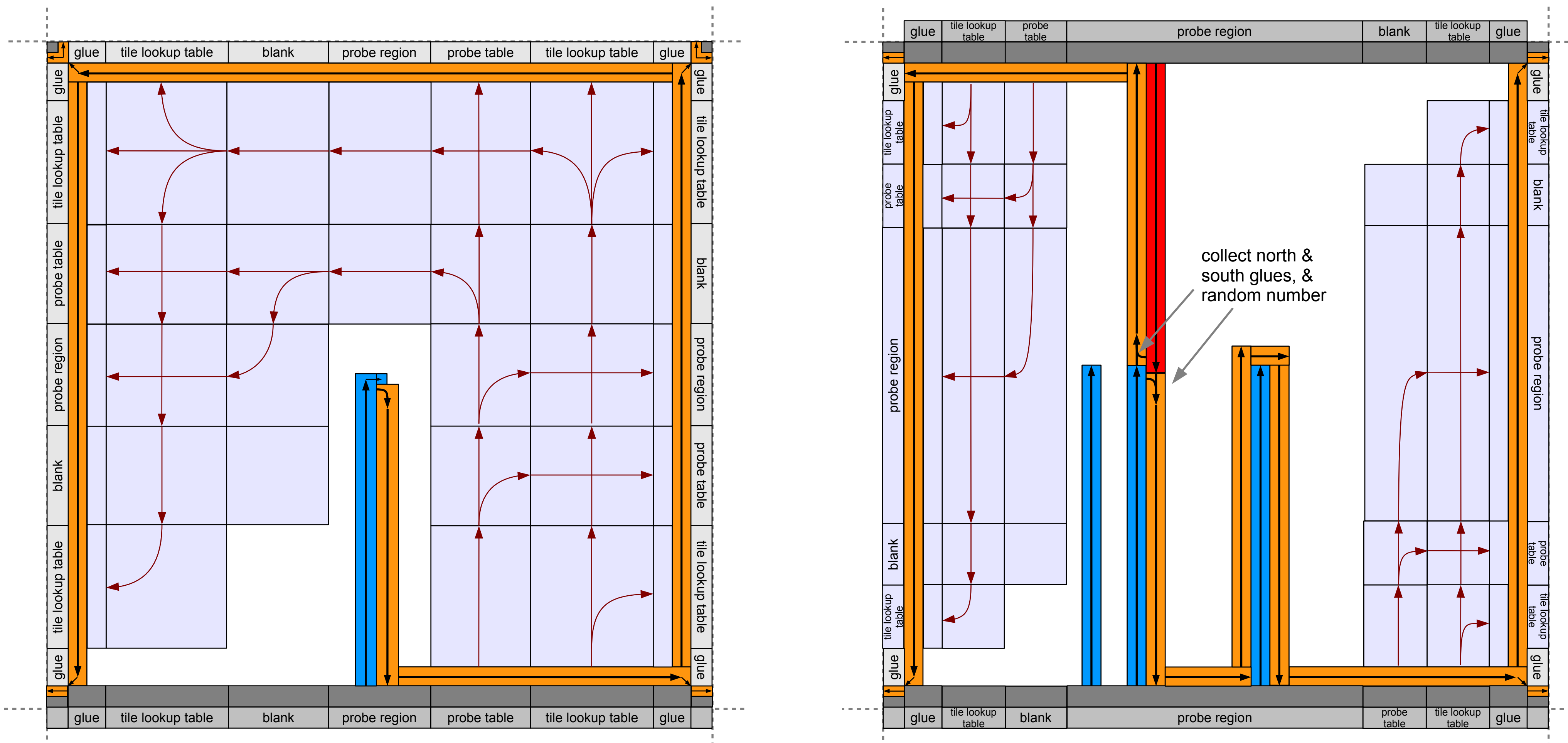
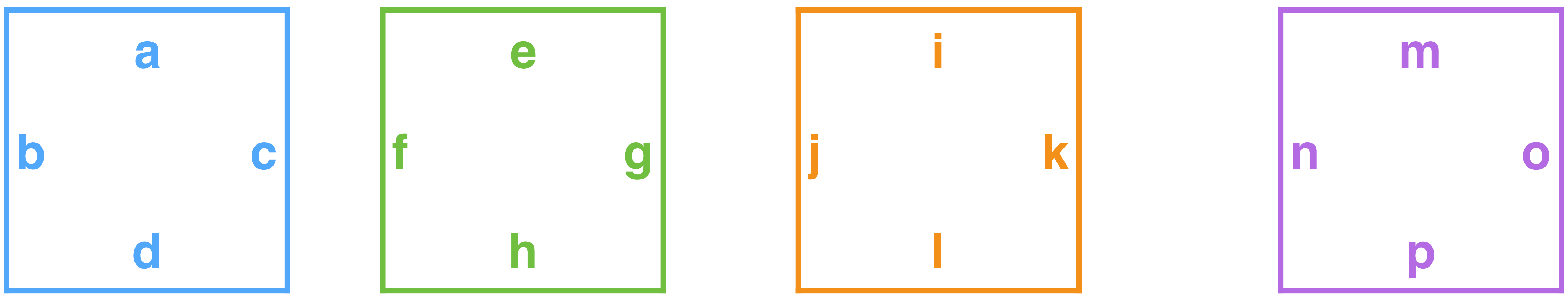


Doty Lutz Patitz Schweller Summers Woods 2012

Examples of macro-tiles

How many tiles do we need ?

- Just ONE with rotation!… What?!?… But a *polygonal* one



Demaine Demaine Fekete Patitz Schweller Winslow Woods 2012

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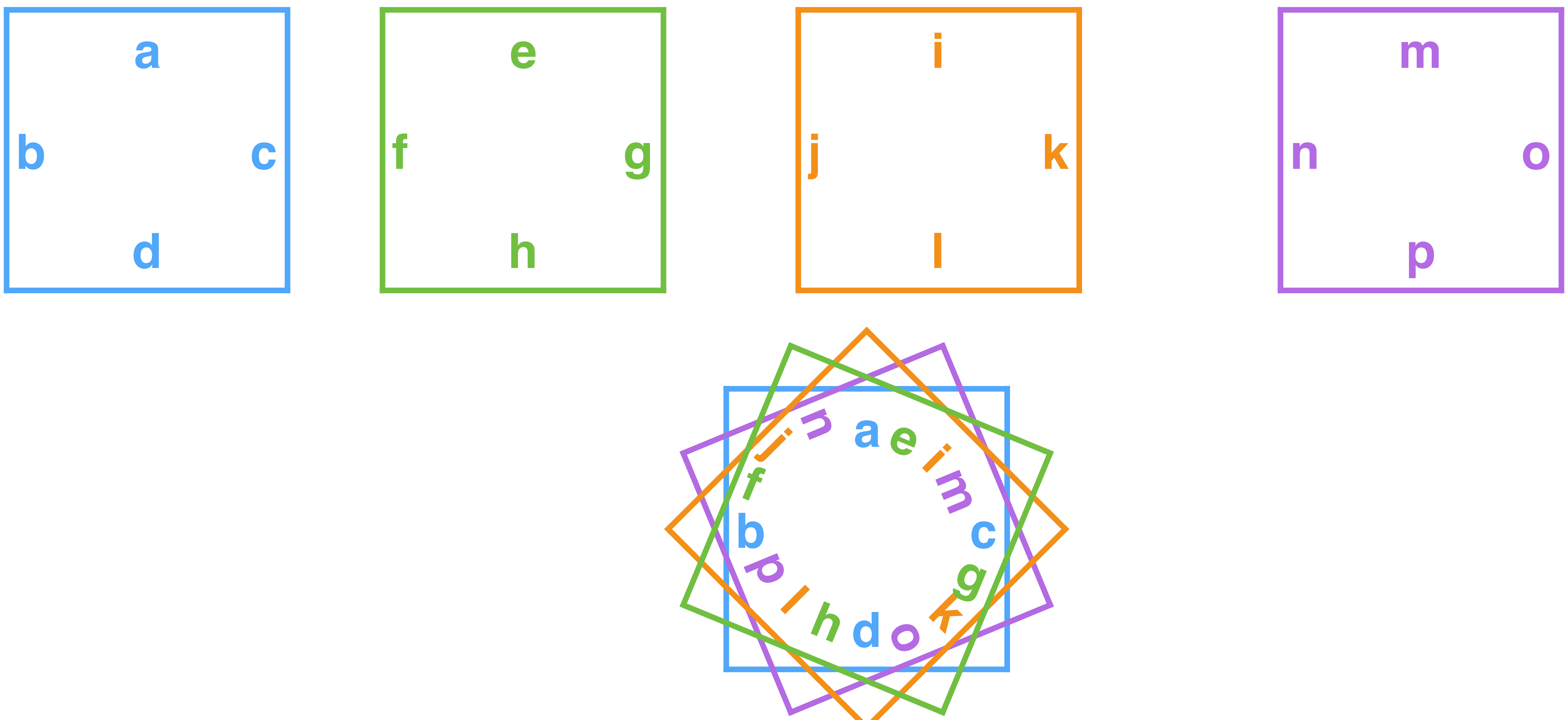

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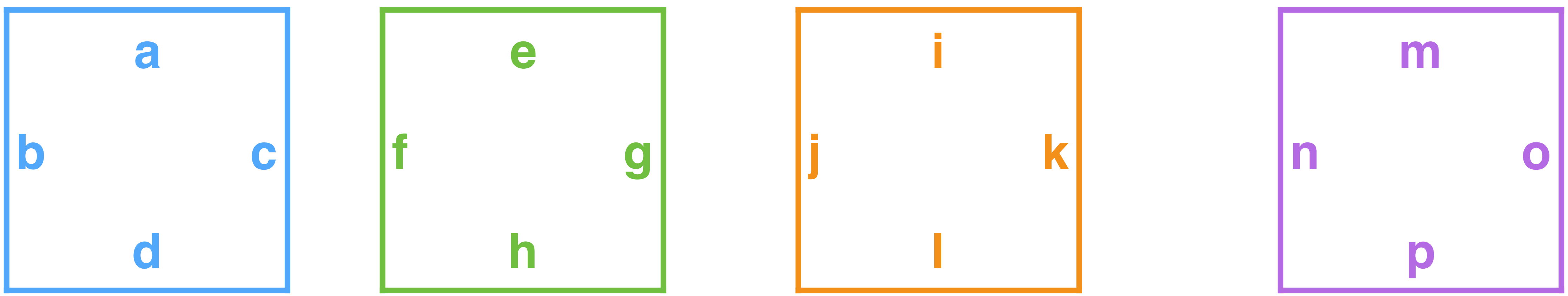

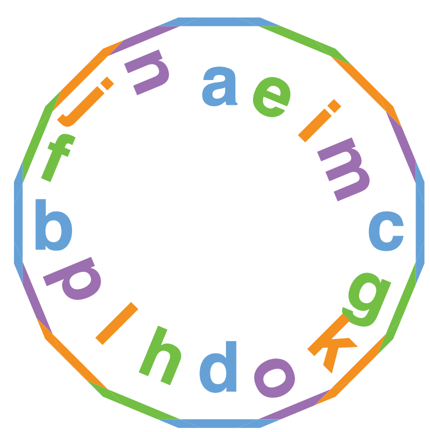

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How many tiles do we need ?

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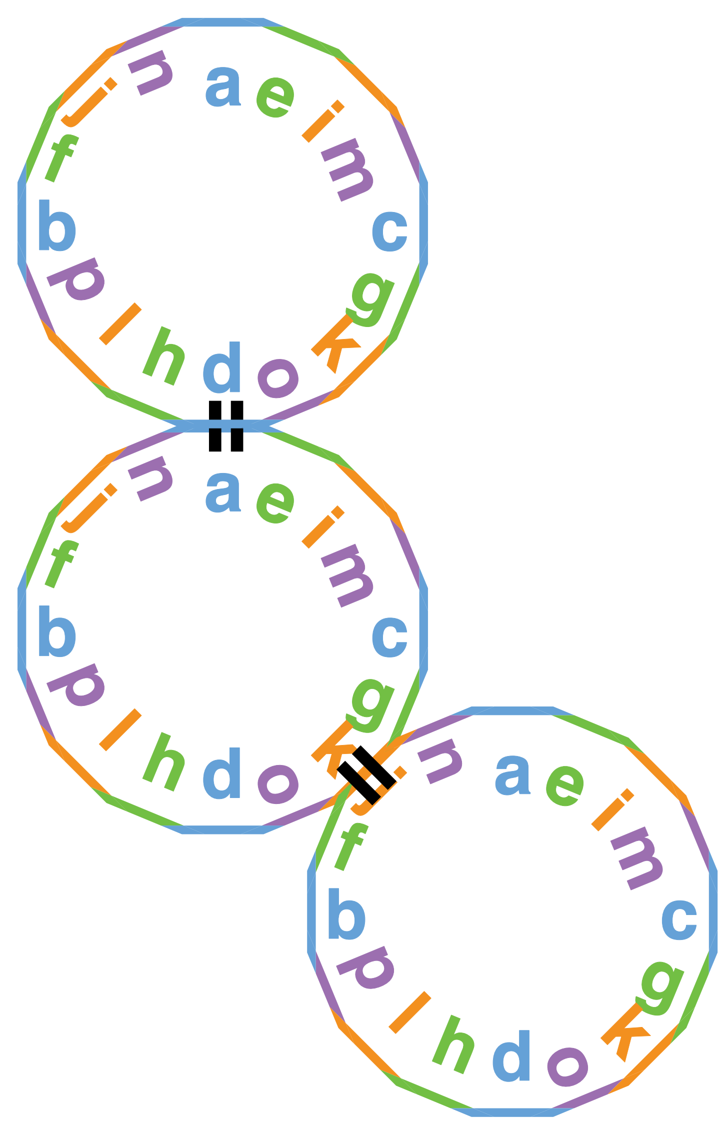
Problem with glue of strength 2 !!!



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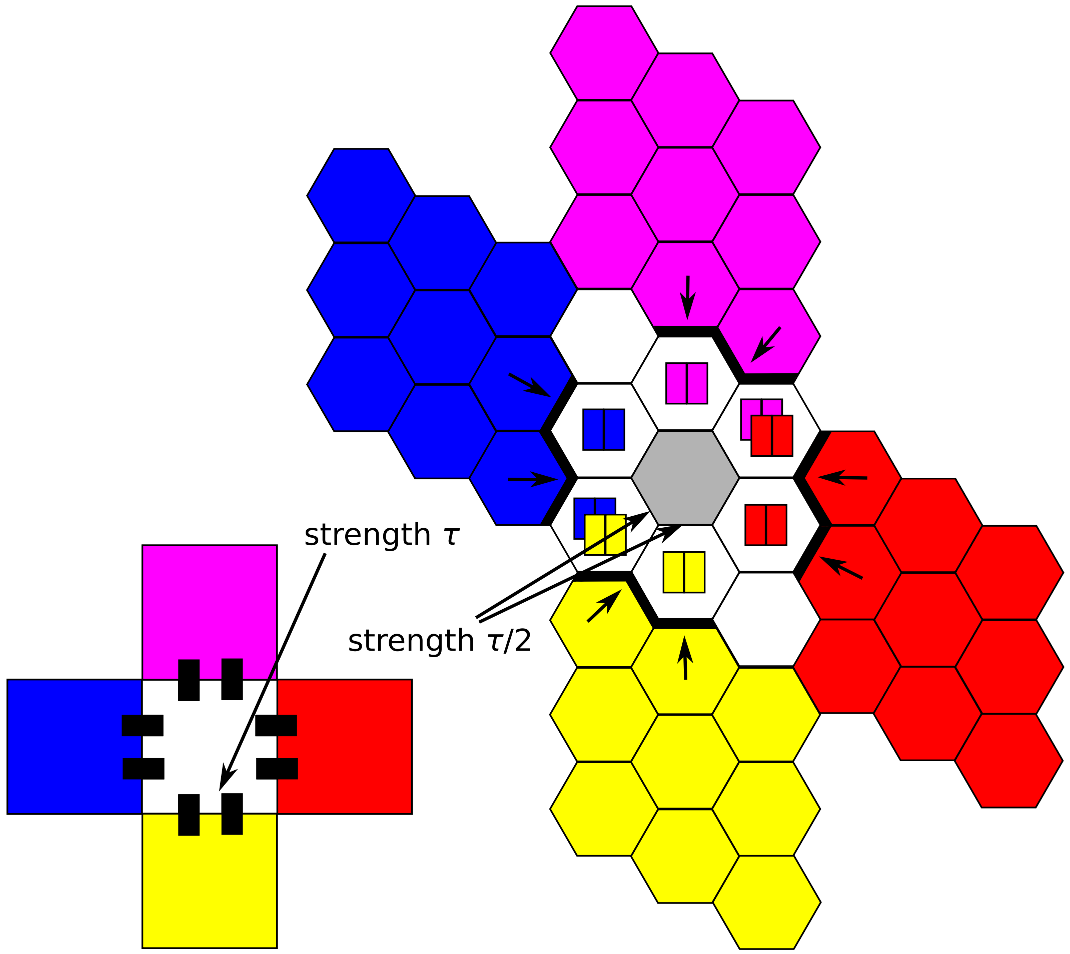
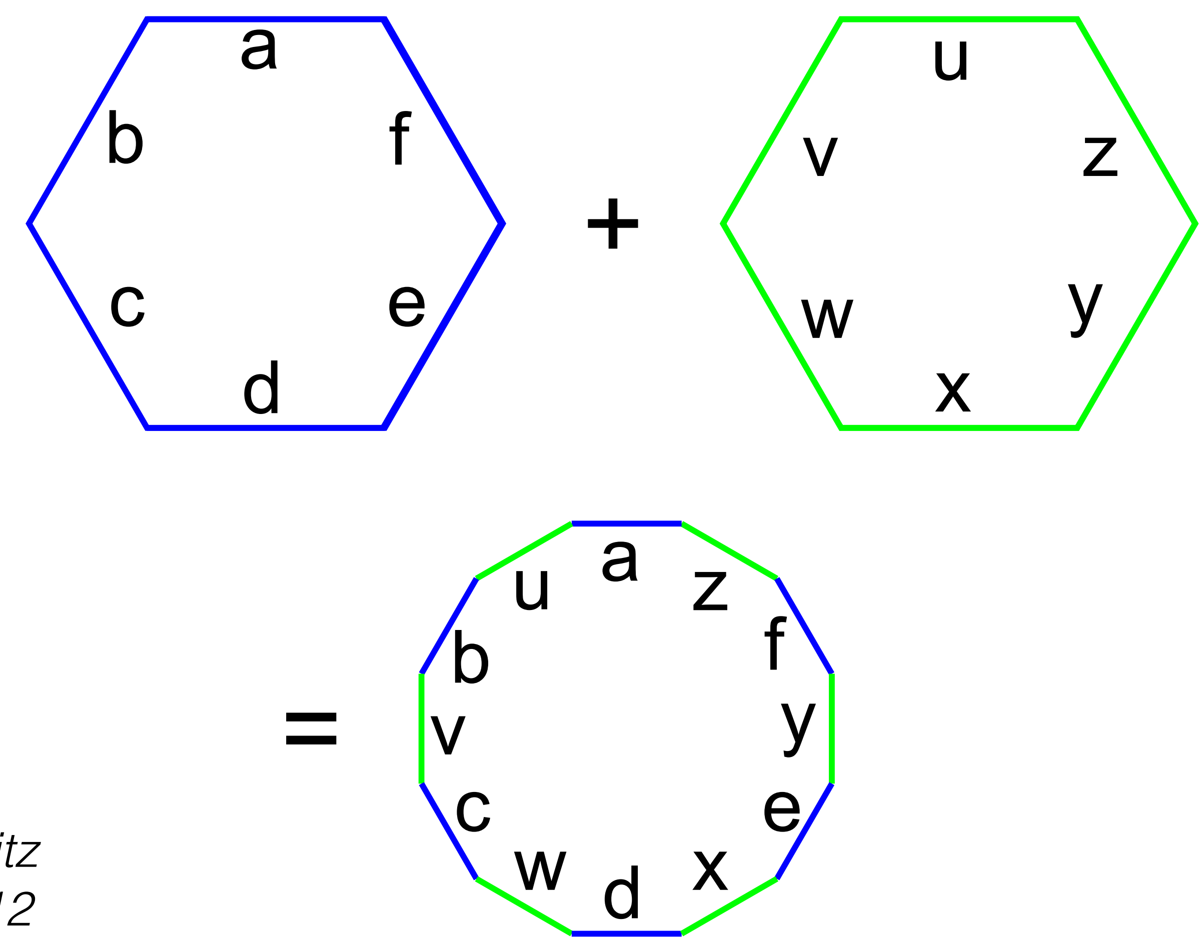
Encoding strength 2 glues into strength 1 glue in hexagonal tiles



A **single** (polygonal) tile is enough !



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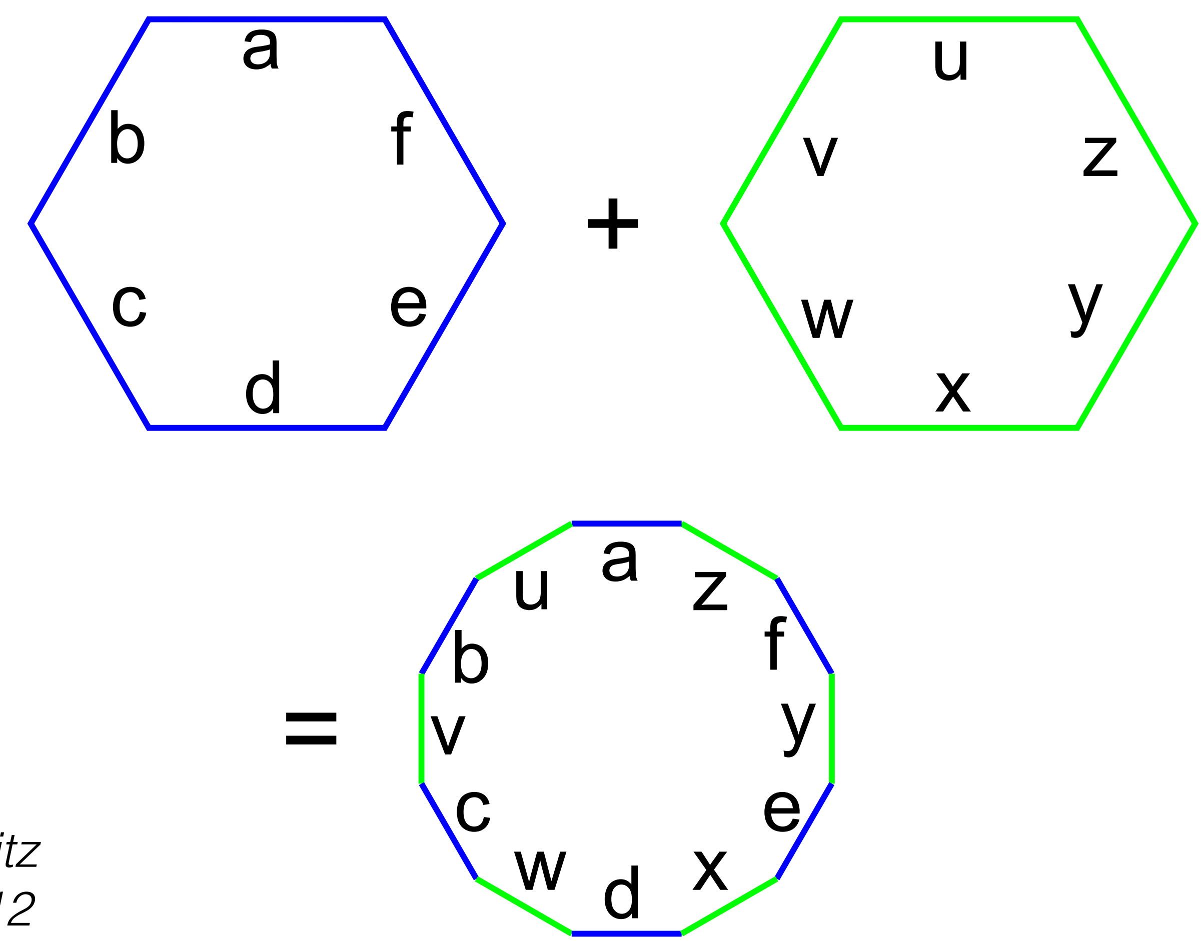

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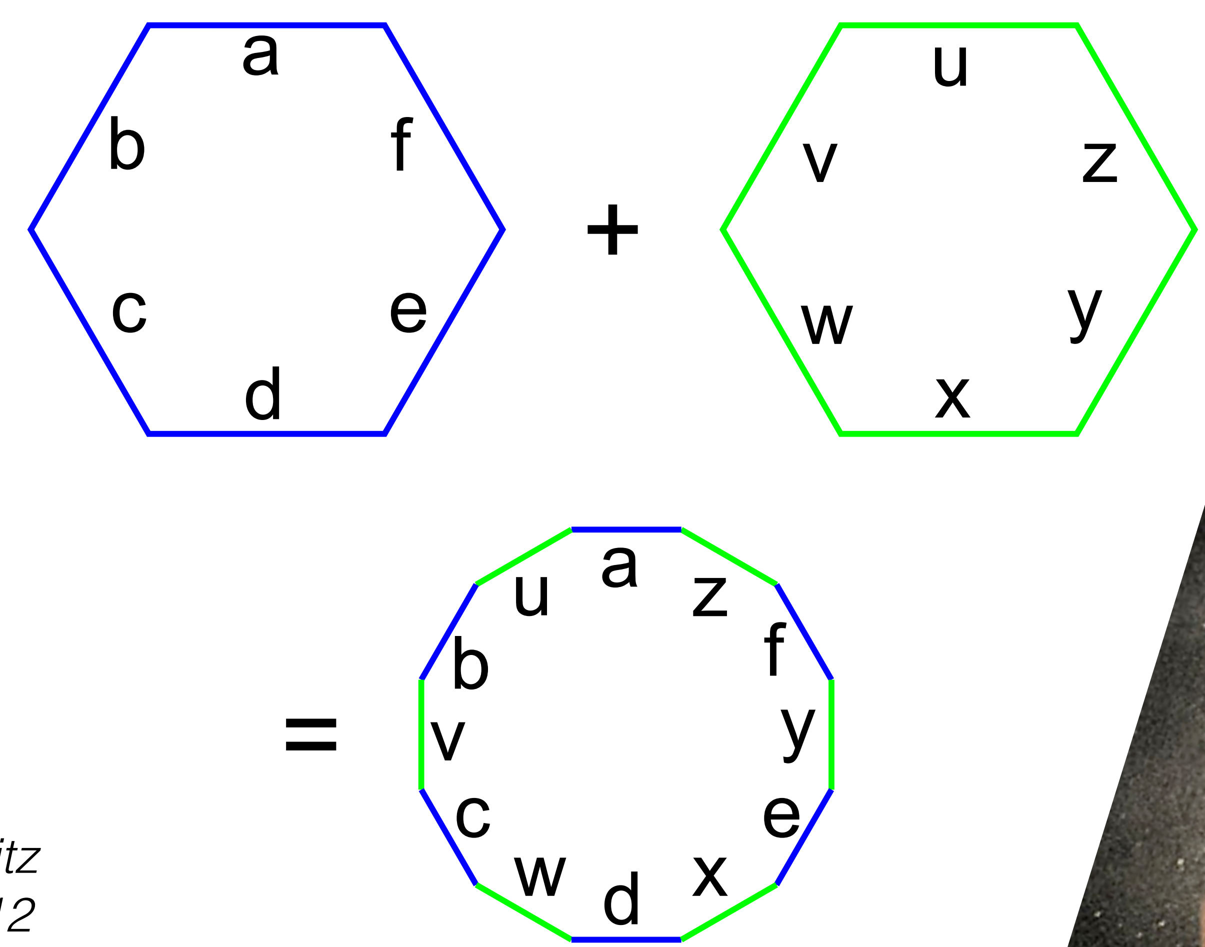

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The magic powder can assemble anything!

One molecule is enough !
(Next week)

Co-transcriptional folding

Joint work with Cody Geary Pierre-Étienne Meunier and Shinnosuke Seki