Parallel Algorithm for the Optimization of the Span of an Hexagonal Frequency Planing Graph
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Absract

Wireless telecommunication systems share the spectral congestion problem and user capacity management problem. To solve both problem the cellular concept and frequency reuse have been introduced. In this paper the frequency planning constraints are modeled in terms of graph theory.

For each planning function f let us call $sp(f)$ - or the span *of the frequency planning f - the difference between the largest and the smallest frequency used. Let the* Order *of the graph be $Or(G) = sp(G) + 1$ and the* maximal local order *of the graph the maximum order of a clique of G, i.e. $Mlo(G) = \max_{X \text{ clique of } G} Or(X)$. We show that for a triangular lattice G:*

$$Mlo(G) \leq sp(G) \leq \alpha Mlo(G)$$

where $\alpha = 4/3$, $3/2$ or 1 depending on co-site interference constraints.

Each lower bound is obtained using a parallel algorithm.

1 Introduction

Wireless telecommunication systems come from various contexts, i.e. military (command systems) or civil (numerical TV, mobile phone, paging systems...), fixed (TV broadcasting) or Mobile (cellular phone), half-duplex or full-duplex. The next generation of wireless telecommunication systems - referred as Universal Mobile Telecommunication System - will provide a wide variety of services combining a wide variety of telecommunication technologies. All this systems share the spectral congestion problem and user capacity management problem.

The cellular concept was a major breakthrough in solving such problems. It is a system level idea which calls for replacing a single, high power transmitter (large cells) with many low power transmitters (small cells). Each transmitter is allocated a portion of the total number of channels available to the entire system.

Because the next generation of cellular network will introduce much more dynamic optimization we need some **parallel** and **distributed** algorithm for channel allocation.

1.1 Hexagonal model

The conceptual hexagonal model is a model where each cell has a hexagonal shape with the corresponding transmitter in the center of it. This model is simple but it has been universally adopted since it permits easy and manageable analysis of cellular system.

The cellular planing area is now tilled with hexagons. Radio parameters are mapped onto this hexagonal grid. The usual parameters are *demands* and *interference constraints*. Each cell of the network receives a demand, i.e. the number of frequencies needed to fulfill the forecasted services in this area.

Interference is a major limiting factor in the performance of cellular radio systems. The two major types of system-generated cellular interference are co-channel interference and adjacent channel interference.

Frequency reuse means that in a given coverage area there are several cells that use the same set of frequencies. To reduce co-channel interference, co-channel cells must be physically separated by a minimum distance to provide sufficient isolation due to propagation.

Interference resulting from signals which are adjacent in frequency to the desired signal is call adjacent channel interference. Adjacent channel interference results from imperfect receiver filters which allow nearby frequencies to leak into the passband.

1.2 Generalized coloring

In the hexagonal model the neighbors of a cell are simply the 6 neighboring hexagons in the grid. The re-

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sulting neighboring graph $G(V, E)$ is a triangular lattice. An integer $d(v)$ is assigned to each node $v \in V$ of graph. Parameters are attached to the graph: the co-site interference constraint K_0 and the adjacent interference set of constraints K_i, where K_i is the interference constraint for pair of vertices at distance i (usually $K_i = 0$ for i after the "reuse distance").

We assume for the rest of the paper that frequencies are taken from the interval $[1, F_{max}]$. Let us call a *planning function* or a *frequency assignment* of this weighted graph $G(V, E, \{K_i\})$ a function $f : V \to \mathcal{P}([1, F_{max}])$, which assigns a subset of frequencies to each vertex of a graph. Planning function must respect the following constraints:

C0) $\forall v \in G, card(f(v)) = d(v)$

C1) $\forall v \in G,\ \forall f_1, f_2 \in f(v),\ f_1 \neq f_2, \quad |f_1 - f_2| \geq K_0$

C2) $\forall v, u \in G, distance(v, u) = i,\ \forall f_1 \in f(v), f_2 \in f(u), \quad |f_1 - f_2| \geq K_i$

In this paper we will be interested in the case $K_0 = k \geq 1$, $K_1 = 1$ and $\forall i > 1,\ K_i = 0$.

For each planning function f let us call $sp(f)$ - or the *span* of the frequency planning f - the difference between the largest and the smallest frequency used. The goal of the frequency assignment problem is to find the planning function with the minimal span, i.e to find f^* for which $sp(f^*) = sp(G) = \min_f sp(f)$. We also define $Or(G)$, the *Order* of the graph G as $Or(G) = sp(G) + 1$; and $Mlo(G)$, the *maximal local order* of the graph G as the maximum order of a clique of G, i.e. $Mlo(G) = \max_{X \text{ clique of } G} sp(X)$.

Let us denote by ω and χ respectively the clique number and the chromatic number of the graph G. We define the *weighted clique number* ω_W as the maximal sum of the demand of a clique of G. Note that if $\forall v \in G, d(v) = 1$ then $\omega(G) = \omega_W(G)$. We also define the *weighted chromatic number* $\chi_W(G)$ as the chromatic number of the graph G' obtained by the following *blowup* operation. When the demand is not less or equal to one everywhere, the blowup operation consists of expanding each vertex v with $d(v) > 1$ to a clique of size $d(v)$ (if $d(v) = 0$, then v is deleted). Note that $\omega_W(G) = \omega(G')$. Furthermore, for $k = 1$ we get $Or(G) = \chi_W(G)$.

If $\forall v \in G, d(v) = 1$ and $K_1 = K_0 = 1$ ($\forall i > 1,\ K_i = 0$) we obtain a graph coloring problem on a triangular lattice. For arbitrary demand and $K_1 = K_0 = 1$ we obtain a graph multicoloring problem on a triangular lattice.

In this paper we give tight upper bounds for χ_W in terms of $Mlo(G)$ for case K_0 set to an arbitrary integer $k \geq 1$, and $K_1 = 1$ and $\forall i > 1,\ K_i = 0$. The main result of our paper is that for:

- $k = 1, \quad \omega_W(G) = Mlo(G) \leq \chi_W(G) \leq 4\lceil \frac{Mlo(G)}{3} \rceil$
- $k = 2, \quad Mlo(G) \leq \chi_W(G) \leq 3\lceil \frac{Mlo(G)}{2} \rceil$
- $k \geq 3, \quad Mlo(G) \leq \chi_W(G) \leq Mlo(G) + 2$

For $k = 1$, there is another proof of the upper bound [3]. A linear distributed algorithm which guarantees the $4\chi/3$ is reported in [2]. In fact their algorithm guarantees the $4\omega/3$. We are not aware of any work on upper bound for general case.

There was a lot of work done on celebrated Philadelphia examples (see [4] and the references there). In this examples there are constraints at distance 2 and 3 (i.e. K_2 and K_3 are not zero). The results on Philadelphias include tight lower bounds while upper bounds were given by planing functions constructed. Therefore no upper bounds were derived as far as we know. Although instances are relatively small (21 cells) they have proved to be extremely difficult. This anticipates the general problem to be very challenging.

In the next section we start with an example and give some definitions and observations. In section 3 we prove a tight upper bound for χ_W obtained by a linear time sequential algorithm. We also give a parallelization of the algorithm. In sections 4 and 5 analogous results are given for cases $k = 2$ and $k \geq 3$.

The problem for arbitrary $K_1 > K_0 > 1$ is still open!

2 Preliminaries

2.1 A simple case: $k = 1$, $\omega_W = 3$

We start with an illustrative example. Although simple, the general idea will be used in some other proofs later.

Proposition 1 *If $k = 1$ and $\omega(G) = 3$ then $\chi(G) \leq 4$.*

Proof: Let G be a triangular lattice graph with $\omega = 3$. First we apply the red-blue-green coloring to the graph G. We will show that it is possible to cover all other cliques by using only one additional color.

Clearly, the red-blue-green coloring reduces all demands by one.

It is easy to see that there are no triangles left in the graph. Even more, the graph induced on vertices, which still have positive demand is a graph of isolated vertices.

If demand of a vertex after red-blue-green coloring is 2, then the vertex was an isolated vertex in the original graph. If not, then there was a clique of demand > 3, which is in contradiction with $\omega = 3$. Hence we can color it by the other two colors.

If demand of a vertex after red-blue-green coloring is 1, we can use the fourth color. This can be done, because these vertices are clearly independent. If not, then there were two adjacent vertices with demand 2 in G, which is in contradiction with $\omega = 3$. $\square$

The bound just proved is best possible as the example of Fig. 1 shows.

There is only one vertex with demand 2 in the graph (this is the vertex of degree 2) All other vertices have demand 1. It is easy to see that the maximal clique size of the expanded graph is 3 and that the chromatic number of this graph is 4. $\square$

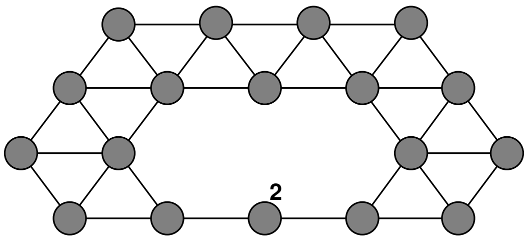


Figure 1: Example of a graph with $\omega = 3$ and $\chi = 4$

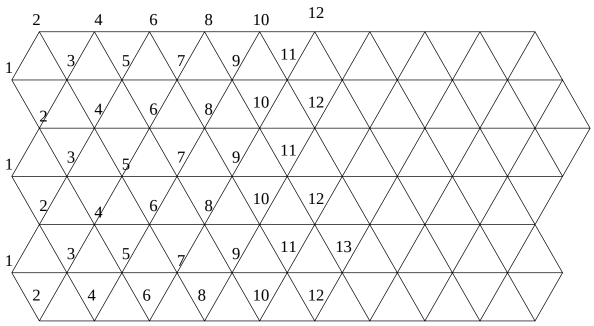


Figure 2: Coordinates

2.2 Triangular lattice graphs

In this paper we are interested only in graphs which are induced subgraphs of triangular lattice. We therefore can label each vertex of G with coordinates which are defined by embedding in the infinite triangular lattice.

Given a graph G we will always assume that there is an embedding into the infinite triangular lattice given. Equivalently, we will assume that each vertex of G is assigned two coordinates (one for each "column" and one for each "line" ; the figure 2 show only how the column are numbered.

Furthermore, because the infinite triangular lattice has essentially unique 3 coloring, or in other words its vertex set has a unique partition into three independent sets, we assume this partition is given, and according to this partition each vertex of G is called *red, blue* or *green vertex*.

(Note that each node can compute its color only with the two coordinates describing it position in the lattice.)

2.3 Red-blue-green colorings

For any triple (i, j, k), $i, j, k \geq 0$ we define a (i, j, k) *r-b-g coloring* as follows: assign a set of i colors to each of the red vertices of G, another set of j colors to each of the blue vertices of G and another set of k colors to each of the green vertices of G.

When saying that we have applied a (i, j, k) r-b-g coloring to G, we will assume that we get a new graph with reduced demands. This graph is a subgraph of G, because there may be some vertices, for which the demand was already fulfilled by the r-b-g coloring.

Lemma 2 *Let H be a graph obtained from G after application of (i, j, k) r-b-g coloring, $i+j+k \geq \omega - 2$. There are no triangles in H.*

Proof: if not, then $\omega \geq (i+1)+(j+1)+(k+1) > \omega$, contradiction. □

Definition 1 *A vertex v in H with color $c_1 \in \{r, g, b\}$ is said to be* c-free *$(c \neq c_1, c \in \{r, g, b\})$ iff it has no c neighbors in H.*

Lemma 3 *Let H be a graph obtained from G after application of (i, j, k) r-b-g coloring, $i+j+k \geq \omega - 2$. Each vertex of H is either free with respect to two colors, or free with respect to one color or is not free.*

(a) If a vertex is free with respect to two colors, then it is an isolated vertex of H.

(b) The set of c-free vertices is bipartite.

(c) If a vertex v is not free, then it may have at most two neighbors in H. In this case, the two edges incident to v are on a straight line.

(d) Let K be the graph induced on the vertices which are not free in H. The connected components of K are paths or isolated vertices.

Proof:

(a) clear.

(b) any such set is colored by the other two colors (as defined at end of 2.2).

(c) Assume v is not free and has two neighbors u and w. We now look at the angle between edges vu and vw. The angle can not be $\frac{\pi}{3}$, because then the vertices u, v, w would induce a triangle in H. The angle can not be $\frac{2\pi}{3}$, as the following argument shows. Assume for a moment the angle is $\frac{2\pi}{3}$. Then u and w are of the same color, say c. Since v is not free with respect to any color this implies that there must be a neighbor of v in H of color different from c. But then this neighbor, the vertex v and u or w induce a triangle in H. Contradiction. The only possibility left is hence angle π.

(d) clear using (c).

□

2.4 Formulas for $Mlo(G)$

Let D_{max} be the maximal demand of a vertex of G.

Denote by Q the set of all cliques of G. These are only triangles, edges or vertices. We assume that every element of Q is a triangle adding "virtual" vertices with demand zero adjacent to isolated vertices and edges.

For $k = 1$, the demand of a clique is simply the sum of demands, since the only constraint is that all colors must be distinct. Hence

Lemma 4 *If $k = 1$ then,*

$$Mlo(G) = \max\{d_u + d_v + d_w \mid uvw \in Q\}$$

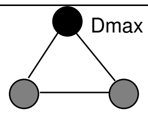
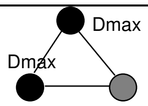
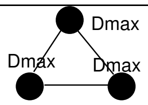
Assume $k = 2$ and let $d_1 \geq d_2 \geq d_3$ be demands on vertices of an arbitrary triangle of Q. Then the order of any planning function for this triangle is at least
$2(d_1 - 1) + 1$ if $d_1 > d_2 + d_3$
or
$d_1 + d_2 + d_3$ if $d_1 \leq d_2 + d_3$ (see Fig. 3).

Therefore,

Lemma 5 *If $k = 2$ then,*

$$Mlo(G) = \max\{2D_{max} - 1, \max\{d_u + d_v + d_w \mid uvw \in Q\}\}$$

Let now $k \geq 3$. We compute the maximal order needed to fulfill demand of any triangle. The order of any planning function fulfilling the demand of a triangle only depends on the number of vertices with with demand D_{max} in this triangle:

	$k(D_{max} - 1) + 1$
	$k(D_{max} - 1) + 2$
	$k(D_{max} - 1) + 3$

Lemma 6 *If $k \geq 3$ then depending of cases,*

$$\begin{cases} C1 & \textit{if all demands } D_{max} \textit{ isolated} \\ C2 & \textit{if at most two maximal demands adjacent} \\ C3 & \textit{if there is a triangle with three demands } D_{max} \end{cases}$$

$$Mlo(G) = \begin{cases} k(D_{max} - 1) + 1 & (C1) \\ k(D_{max} - 1) + 2 & (C2) \\ k(D_{max} - 1) + 3 & (C3) \end{cases}$$

3 Lower bound and algorithm for $k = 1$

Recall that for $k = 1$, $\omega_W(G) = Mlo(G)$ and $\chi_W(G) = Or(G)$. This is the usual multicoloring of triangular lattice with demand.

Proposition 7 *If $\omega_W(G) \geq 3$ then $\omega_W(G) \leq \chi_W(G) \leq \lceil \frac{4}{3}\omega_W(G) \rceil$.*

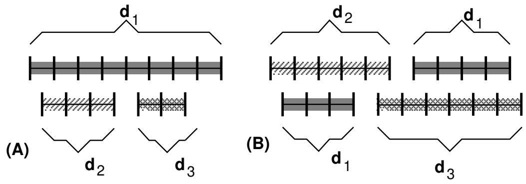


Figure 3: Optimal frequency planning for a triangle, $k = 2$.

The case $\omega_W = 1$ is trivial. If $\omega_W = 2$, then the connected components of G are isolated vertices, paths and cycles. Since odd cycles can be induced subgraphs of triangular lattice, $\chi_W \leq 3$.

Proof: We give an algorithm which multi-colors any triangular lattice graph with at most $\lceil \frac{4}{3}\omega_W(G) \rceil$ colors. The algorithm has two steps.
step 1. We apply a $(\omega_1, \omega_2, \omega_3)$ r[ed]-b[lue]-g[reen] coloring to the graph G. where $\lceil \frac{\omega_W}{3} \rceil = \omega_1 \geq \omega_2 \geq \omega_3 = \lfloor \frac{\omega_W}{3} \rfloor$ so that $\omega_1 + \omega_2 + \omega_3 = \omega$.

We will show that it is possible to cover all other cliques by using only ω_{max} additional colors.

Let H to be the graph obtained after the $(\omega_1, \omega_2, \omega_3)$ r[ed]-b[lue]-g[reen] coloring to the graph G, using low colors in each of the 3 colors sets.

By Lemma 2, we know that there is no triangle in H. Furthermore, the vertices of H are either isolated or free with respect to one color or not free. In the second step we show, that in each case the vertex can be colored by at most ω_{max} additional colors.
Step 2.1. Each isolated vertex v can borrow colors from its neighbors and if it is not enough it is easy to see that we can fulfill it with ω_{max} additional (white) colors (if v cannot borrow any color then $d(v) \leq 2\omega_{max}$). Isolated vertex are the only vertex combining both "borrowing" strategy and using the white colors (*borrowing* means stealing an unused color assigned to one of its neighbor by the RGB algorithm). Note that since v is isolated in H, there is no possible conflict for using white colors at the rest of H.
Step 2.2. All remaining vertex have degree 1 or 2. Some of them are free according to one color while the other are not. Inegalities on triangles give that a c-free vertex can without conflict fulfill its remaining demand by borrowing the highest c-colors as soon as it has at least two neighbors of the third primary color. But the other vertices (not free or free with only one neighbor) must be fulfilled with the white colors. Therefore, the problem is that two adjacent vertex free according to the same colors may not both borrow from this color set (see Fig. 4).

Therefore we label each remaining vertex with one of the 3 strategies :

Bo The node is c free and is fulfilled by borrowing c colors

W+ The node is fulfilled using colors from the white set starting from the higher colors

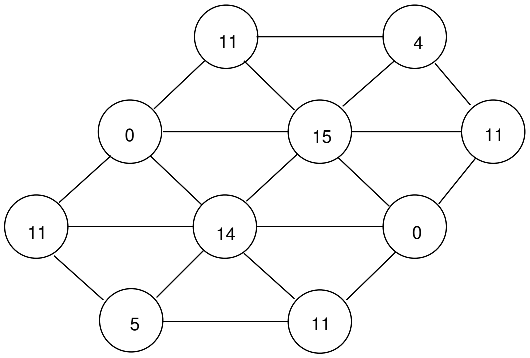


Figure 4: In this example, we suppose that $\omega = 30$; therefore, the only vertex surviving after the RGB step are the one with demand greater than 10; the two neighbors with demand 14 and 15 are both free according to the same color; therefore, they need to borrow $4 + 5$ colors; 5 colors are free at the vertex with demand 5 and 6 at the vertex with demand 4; because this conflict can append everywhere in the graph t cannot be solve locally.

W- The node is fulfilled using colors from the white set starting from the lowers colors

We have the following constraints :

c1) only a c free vertices can receive the F strategy

c2) any two adjacent vertex cannot received the same strategy

We scan the grid column by column according to the numbering presented in Fig. 2. At each step, all vertex in the current column choose one of the 3 strategies according to the set N of strategies already chosen by its neighbors (0, 1 or 2 neighbors).

N	$\emptyset$	Bo	W+	W-	Bo,W+	Bo,W-	W+,W-
Strat.	W-	W-	W- or Bo*	W+ or Bo*	W-	W+	Bo

* only if the vertex is c free with at least two neighbors of the third primary color.

The resulting labeling fulfills the constraints.

After that step, each vertex fulfills its demand according to its own strategy. If two adjacent neighbors have to take colors in the white set, the size of this set is enough to fulfill both demands (because the sum of the two demands and ω_{max} cannot be greater than ω). If a vertex has to borrow colors, there is always enough colors to fulfill its demand (if not, the sum of the demand would be locally greater than ω_W).

Therefore, all demands are fulfilled by $\omega_W + \omega_{max}$ colors. $\square$

4 Lower bound and algorithm for $k = 2$

Proposition 8 *Let $k = 2$ and $Mlo(G) \geq 3$. Then* $Mlo(G) \leq Or(G) \leq 3\lceil \frac{Mlo(G)}{2} \rceil$

This result comes from the following lemma derived from lemma 5:

Lemma 9 $D_{max} \leq \lceil \frac{Mlo(G)}{2} \rceil$

The algorithm consists of a single parallel r-g-b coloring step with 3 primary interleaved color sets R, G and B where the size of each set is $\lceil \frac{Mlo(G)}{2} \rceil$.

5 Case $k \geq 3$

In this case a direct assignment is sufficent to solve the problem.

Assignment
$1, k+1, 2k+1, \ldots, d_{max}k$ to red,
$2, k+2, 2k+2, \ldots, d_{max}k+1$ to blue and
$3, k+3, 2k+3, \ldots, d_{max}k+2$ to green vertices
is always proper.

Therefore, the difference between order $Or(G)$ and $Mlo(G)$ is always 0,1 or 2.

Proposition 10 *Let $k \geq 3$. Then $Mlo(G) \leq Or(G) \leq Mlo(G) + 2$*

The resulting algorithm is fully parallel (only local information is needed to decide which color to use).

6 Conclusion

In this paper we prove lower bound of the frequency planing problem is the hexagonal with co-site constraints. The algorithm is fully parallel if the co-site constraint is greater than 3 and is "partially" parallel if $k < 3$ (a parallel scan over the column in the grid has to be done).

We conclude with a couple of open problems.

Problem I find good upper bounds for $Or(G)$ in term of $Mlo(G)$ for more general K_i. For example:

- $K_0 > K_1 > 1, \forall i \geq 2, K_i = 0$
- $K_0 \geq K_1 \geq K_2 = 1, \forall i \geq 3, K_i = 0$ (some Philadephias fall in this case).
- etc...

Problem II let G be arbitrary k-colorable graph. Is there a bound for $\chi_W(G)$ in terms of $\omega_W(G)$; for example, if $k = 3$ we have by simple generalization of our methods $\chi_W(G) \leq \lceil \frac{3}{2}\omega_W(G) \rceil$.

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