Fully Asynchronous Behavior of Double-quiescent Elementary Cellular Automata

Nazim Fatès[1], Michel Morvan[1,2], Nicolas Schabanel[1], and Éric Thierry[1]
{Nazim.Fates},{Michel.Morvan},{Nicolas.Schabanel},{Eric.Thierry}@ens-lyon.fr

[1] ENS Lyon - LIP (UMR CNRS - ENS Lyon - UCB Lyon - INRIA 5668)
46 allée d'Italie, 69 364 Lyon Cedex 07, France.
[2] Institut universitaire de France, École des hautes études en sciences sociales and Santa Fe Insitute

Abstract. In this paper we propose a probabilistic analysis of the fully asynchronous behavior (i.e., two cells are never simultaneously updated, as in a continuous time process) of elementary finite cellular automata (i.e., $\{0,1\}$ states, radius 1 and unidimensional) for which both states are quiescent (i.e., $(0,0,0) \mapsto 0$ and $(1,1,1) \mapsto 1$). It has been experimentally shown in previous works that introducing asynchronism in the global function of a cellular automaton may perturb its behavior, but as far as we know, only few theoretical work exist on the subject. The cellular automata we consider live on a ring of size n and asynchronism is introduced as follows: at each time step one cell is selected uniformly at random and the transition rule is applied to this cell while the others remain unchanged. Among the sixty-four cellular automata belonging to the class we consider, we show that fifty-five other converge almost surely to a random fixed point while nine of them diverge on all non-trivial configurations. We show that the convergence time of these fifty-five automata can only take the following values: either 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, or $\Theta(n 2^n)$. Furthermore, the global behavior of each of these cellular automata can be guessed by simply reading its code.

1 Introduction

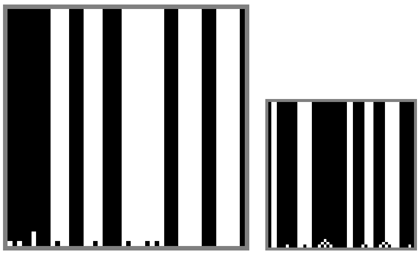
The aim of this article is to analyze theoretically the asynchronous behavior of unbounded finite cellular automata. During the last two decades, several empirical studies [3,12,9,1,13,4] have shown that certain cellular automata behavior change drastically under asynchronous behavior. In particular, [1,5] observe that finite size Game of Life space-time diagrams under synchronous and asynchronous updating differ qualitatively. For instance, fixed size Game of Life exhibits convergence to cycles of arbitrary length under synchronous updating, while appears to converge towards a random fixed point under asynchronous dynamics [1].

Cellular automata are widely used to model systems involving a huge number of interacting elements such as agents in economy, particles in physics, proteins in biology, etc. In most of these applications, in particular in many real system models, agents are not synchronous. Interestingly enough, in spite of this lack of synchronism, real living systems are very resilient over time. One might then expect the cellular automata used to model these systems to be robust to asynchronism and other kind of failure as well (such as misreading the state of the neighbors). Surprisingly enough, it turns out that the resilience to asynchronism widely varies from one automata to another (e.g., [1,4]). In particular, the aspect of asynchronous space-time diagrams of cellular automata may differ radically from their synchronous ones.

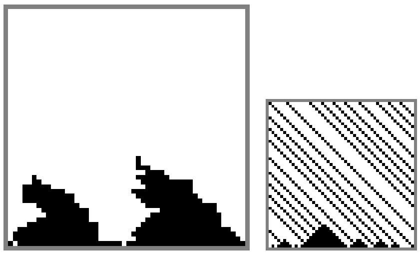
As far as we know, the question of the importance of perfect synchrony on the behavior of a cellular automaton is not yet understood theoretically. To our knowledge, only Gács shows in [8] undecidability results on the invariance with respect to the update history. Studies have also been led in the more general context of probabilistic cellular automata regarding the question of the existence of stationary distribution on infinite configurations (see [10] for a state of the art).

In this paper, we quantify the convergence time and describe the space-time diagrams for a class of cellular automata under fully asynchronous updating, where two cells are not updated simultaneously. This asynchronous regime, also known as step-driven asynchronous dynamics [13], arises for instance in continuous time updating processes. We focus on double-quiescent elementary automata. We show that among these sixty-four automata, nine diverge on all non-trivial configurations (see Theorem 13), and the fifty-five other converge almost surely to a random fixed point (see Theorem 1). Furthermore, the convergence time of these fifty-five automata on (spatially) periodic configurations, can only take the following values: either 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, or $\Theta(n 2^n)$, where n is the size of the configurations. One of the most striking results is that the fully asynchronous global behavior of double quiescent elementary automata is obtained simply by reading the code of their local transition rules (see Tab. 1), which is known to be a difficult problem in general. Moreover, the asynchronous behavior of all automata is in a certain sense *characterized* by this convergence time: all automata within the same convergence time present the same kind of space-time diagrams (see Tab. 1 and Fig. 1). Remark that the asynchronous behavior of some very simple automata like the shift (Wolfram rule code 170) actually simulates intricate stochastic processes that are currently under investigation in mathematics and physics, such as annihilating random walks, studied for instance in [11]. Our results rely on coupling the automata with a proper random process.

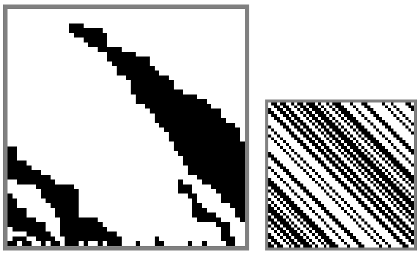
Definitions and our main result are given in Section 2. In section 3, we present basic but useful properties of the automata we consider. Section 4 is a technical section that develops probabilistic tools used to analyze the automata. Section 5 finally analyzes in details the asynchronous behavior of each automaton.



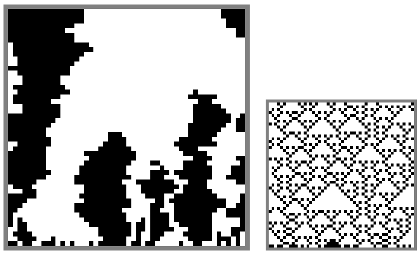
(a) DE 232



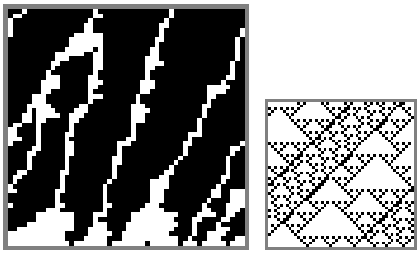
(b) BEFG 130



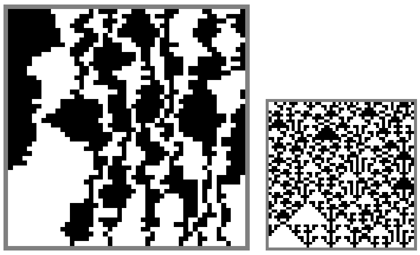
(c) BDEG 170



(d) BCEFG 146



(e) BCEF 210



(f) BCFG 150

Fig. 1. Examples of space-time diagrams under fully asynchronous and synchronous dynamics for each type of convergence, with $n = 50$. For each automaton, the larger left and the smaller right diagrams are respectively examples of asynchronous and synchronous dynamics. White and black pixels respectively stand for states 0 and 1. The k-th line from bottom is the configuration at time $t = 50\,k$ for the asynchronous dynamics, and at time $t = k$ for the synchronous one. Note that automata (a) and (c) are respectively the classic Majority and Shift rules. Each automata is described by two codes: a number, which is the classic Wolfram's number, and a sequence of letters, which will be introduced later in the paper.

2 Definitions, Notations and Main Results

In this paper, we consider two-state cellular automata on finite size configurations.

Definition 1. *An* Elementary Cellular Automata (ECA) *is given by its* transition function $\delta : \{0,1\}^3 \to \{0,1\}$. *We denote by* $Q = \{0,1\}$ *the set of* states. *A state* q *is* quiescent *if* $\delta(q,q,q) = q$. *An ECA is* double-quiescent (DQECA) *if both states* 0 *and* 1 *are quiescent.*

We denote by $U = \mathbb{Z}/n\mathbb{Z}$ *the set of* cells. *A* finite configuration *with periodic boundary conditions* $x \in Q^U$ *is a word indexed by* U *with letters in* Q. *For a given pattern* $w \in Q^U$, *we denote by* $|x|_w = \#\{i \in U : x_{i+1} \dots x_{i+|w|} = w\}$ *the number of occurrences of* w *in configuration* x.

We consider two kinds of dynamics for ECAs: the synchronous dynamics and the fully asynchronous dynamics. The synchronous dynamics is the classic dynamics of cellular automata, where the transition function is applied at each (discrete) time step on each cell simultaneously.

Definition 2 (Synchronous Dynamics). *The* synchronous dynamics $S_\delta : Q^U \to Q^U$ *of an ECA* δ, *associates to each configuration* x *the configuration* y, *such that for all* i *in* U, $y_i = \delta(x_{i-1}, x_i, x_{i+1})$.

The asynchronous regime studied here can be seen as the most extreme asynchronous regime as two cells are never updated simultaneously.

Definition 3 (Fully Asynchronous Dynamics). *The* fully asynchronous dynamics AS_δ *of an ECA* δ *associates to each configuration* x *a random configuration* y, *such that* $y_j = x_j$ *for* $j \neq i$, *and* $y_i = \delta(x_{i-1}, x_i, x_{i+1})$, *where* i *is uniformly chosen at random in* U. AS_δ *could equivalently be seen as a function with two arguments, the configuration* x *and the random index* $i \in U$. *For a given ECA* δ, *we denote by* x^t *the random variable for the configuration obtained by* t *applications of the asynchronous dynamics function* AS_δ *on configuration* x, *i.e.,* $x^t = (AS_\delta)^t(x)$.

Definition 4 (Fixed point). *We say that a configuration* x *is a* fixed point *for* δ *under fully asynchronous dynamics if* $AS_\delta(x) = x$ *whatever the choice of* i *(the cell to be updated) is.* $\mathfrak{F}_\delta$ *denotes the set of fixed points for* δ.

The set of fixed points of the asynchronous dynamics is clearly identical to $\{x : S_\delta(x) = x\}$ the set of fixed points of the synchronous dynamics. Note that every DQECA admits two *trivial fixed points*, 0^n and 1^n.

Definition 5 (Worst Expected Convergence Time). *Given an ECA* δ *and a configuration* x, *we denote by* $T_\delta(x)$ *the random variable for the time to reach a fixed point from configuration* x *under fully asynchronous dynamics, i.e.,* $T_\delta(x) = \min\{t : x^t \in \mathfrak{F}_\delta\}$. *The* worst expected convergence time T_δ *of ECA* δ *is :*

$$T_\delta = \max_{x \in Q^U} \mathbb{E}[T_\delta(x)].$$

We can now state our main theorem.

Theorem 1 (Main result). *Under fully asynchronous dynamics, among the sixty-four DQECAs,*

– *fifty-five converge almost surely to a random fixed point on any initial configuration, and the worst expected convergence times of these fifty-five convergent DQECAs are* 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, *and* $\Theta(n2^n)$*;*
– *the nine others diverge almost surely on any initial configuration that is neither* $\mathtt{0}^n$, *nor* $\mathtt{1}^n$ *nor, when* n *is even,* $(\mathtt{01})^{n/2}$.

Furthermore, the behaviors of the different DQECAs are similar within each class, and are obtained by simply reading its code as illustrated in Tab. 1.

Figure 1 gives examples of the asynchronous space-time diagrams of a representative of each class (but Identity). It is interesting to notice that except for the first diagram (Fig. 1(a)), the asynchronous space-time diagrams (the larger ones) considerably differ from the corresponding synchronous ones (the smaller ones).

3 Basic properties of DQECAs

The transition function δ of an ECA is given by the set of its eight transitions $\delta(\mathtt{000}), \delta(\mathtt{001}), \ldots, \delta(\mathtt{111})$, traditionally written $\frac{\mathtt{000}}{\delta(\mathtt{000})}, \ldots, \frac{\mathtt{111}}{\delta(\mathtt{111})}$. The following code describes each ECA by its differences to the Identity automaton. We use this notation rather than the classic Wolfram's one [14] since it is not immediate to infer the local behavior of the cellular automaton just by looking at its Wolfram code. In order to allow comparison with other work we still indicate the classic Wolfram number in Tab. 1.

Notation 1 We say that a transition is *active* if it changes the state of the cell where it is applied. Each ECA is fully determined by its active transitions. We label each active transition by a letter as follow:

A	B	C	D	E	F	G	H
000 1	001 1	100 1	101 1	010 0	011 0	110 0	111 0

We label each ECA by the set of its active transitions.

Note that with these notations, the DQECAs are exactly the ECAs having a label containing neither A nor H. By 0/1 and horizontal symmetries of configurations, we shall w.l.o.g. only consider the 24 DQECAs listed in Tab. 1 among the 64 DQECAs. For each of these 24 DQECAs, the number of the equivalent automata under symmetries is written within parentheses after their classic ECA code in the table.

From now on, we only consider the fully asynchronous dynamics (with uniform choice); this will be implicit in all the following propositions. Our results

Table 1. Behavior of DQECA under fully asynchronous dynamics. WECT stands for worst expected convergence time. See Section 2 for explanations.

Behavior	**ECA**	**(#)**	**Rule**	**01**	**10**	**010**	**101**	**WECT**
Identity	204	(1)	$\varnothing$	·	·	·	·	0
Coupon collector	200	(2)	E	·	·	+	·	$\Theta(n \ln n)$
	232	(1)	DE	·	·	+	+	
Monotonic	206	(4)	B	$\leftarrow$	·	·	·	$\Theta(n^2)$
	222	(2)	BC	$\leftarrow$	$\rightarrow$	·	·	
	234	(4)	BDE	$\leftarrow$	·	+	+	
	250	(2)	BCDE	$\leftarrow$	$\rightarrow$	+	+	
	202	(4)	BE	$\leftarrow$	·	+	·	
	192	(4)	EF	$\rightarrow$	·	+	·	
	218	(2)	BCE	$\leftarrow$	$\rightarrow$	+	·	
	128	(2)	EFG	$\rightarrow$	$\leftarrow$	+	·	
Biased Random Walk	242	(4)	BCDEF	$\leftrightsquigarrow$	$\rightarrow$	+	+	
	130	(4)	BEFG	$\leftrightsquigarrow$	$\leftarrow$	+	·	
Random Walk	226	(2)	BDEF	$\leftrightsquigarrow$	·	+	+	$\Theta(n^3)$
	170	(2)	BDEG	$\leftarrow$	$\leftarrow$	+	+	
	178	(1)	BCDEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	+	
	194	(4)	BEF	$\leftrightsquigarrow$	·	+	·	
	138	(4)	BEG	$\leftarrow$	$\leftarrow$	+	·	
	146	(2)	BCEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	·	
Biased Random Walk	210	(4)	BCEF	$\leftrightsquigarrow$	$\rightarrow$	+	·	$\Theta(n2^n)$
Divergent	198	(2)	BF	$\leftrightsquigarrow$	·	·	·	Divergent
	142	(2)	BG	$\leftarrow$	$\leftarrow$	·	·	
	214	(4)	BCF	$\leftrightsquigarrow$	$\rightarrow$	·	·	
	150	(1)	BCFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	·	·	

rely on the study of the evolution of the "regions" in the space-time diagram (i.e., of the intervals of consecutive 0s or 1s in configuration x^t). The key observation is that for DQECAs, under fully asynchronous dynamics, the number of regions is non-increasing since no new region can be created; furthermore, only regions of length one can disappear (see Fig. 1). We denote by $Z(x) = |x|_{\mathtt{01}}$ $(= |x|_{\mathtt{10}})$ the *number of alternations* from 0 to 1 in configuration x, which will be our counter for the number of regions.

Fact 2 *For any DQECA, $Z(x^t)$ is a non-increasing function of time. Furthermore, $Z(x^{t+1}) < Z(x^t)$ if and only if x^{t+1} is obtained from x^t by applying a transition* D *or* E *at time t, and then $Z(x^{t+1}) = Z(x^t) - 1$.*

On the one hand, transitions D and E are thus responsible for decreasing the number of regions in the space-time diagram: D "erases" the 1-regions and E the 0-regions. On the other hand, transitions B and F act on patterns 01. Intuitively, transition B moves a pattern 01 to the left, and transition F moves it to the right. In particular, patterns 01 perform a kind of random walk for DQECA with both transitions B and F. Similarly, transitions C and G act on patterns 10. Transition C moves a pattern 10 to the right, and transition G moves it to the left. The arrows in Tab. 1 represent the different behavior of the patterns: $\leftarrow$ or $\rightarrow$, for left or right moves of the patterns 01 or 10; $\leftrightsquigarrow$, for random walks of these patterns.

The following lemma characterizes the fixed points of a given DQECA according to its code.

Fact 3 *If a DQECA δ admits a non-trivial fixed point x, then:*
- *if δ contains transition* B *or* C*, then all* 0*s in x are isolated;*
- *if δ contains transition* F *or* G*, then all* 1*s in x are isolated;*
- *if δ contains transition* D*, then none of the* 0*s in x is isolated;*
- *if δ contains transition* E*, then none of the* 1*s in x is isolated.*

The next section is devoted to analyzing particular random walk-like processes that will be used as tools to obtain our bounds on the convergence time.

4 Probabilistic toolbox

Notation 2 *For a given random sequence $(X_t)_{t\in\mathbb{N}}$, we denote by $(\Delta X_t)_{t>0}$ the random sequence $\Delta X_t = X_t - X_{t-1}$.*

Quadratic DQECA toolbox. Consider $\epsilon > 0$, a non-negative integer m, and $(X_t)_{t\in\mathbb{N}}$ a sequence of random variables with values in $\{0,\dots,m\}$ given with a suitable filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$. In probability theory, $\mathcal{F}_t$ represents intuitively the σ-algebra (the "set") of the events that happened up to time t and is the formal tool to condition relatively to the past (see [7, Chap. 7]). In the sequel, $\mathcal{F}_t$ will either be the values of the previous random variables $X_0,\dots,X_t$, or in some cases, the set of past configurations $x^0,\dots,x^t$. The following lemma bounds the convergence time of a random variable that decreases by a constant on expectation.

Lemma 4 *Assume that if $X_t > 0$, then $\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] \leqslant -\epsilon$. Let $T = \min\{t : X_t \leqslant 0\}$ denote the random variable for the first time t where $X_t \leqslant 0$. Then, if $X_0 = x_0$,*

$$\mathbb{E}[T] \leqslant \frac{m + x_0}{\epsilon}.$$

Cubic DQECA toolbox. Let $\epsilon > 0$ and $(X_t)_{t\in\mathbb{N}}$ a sequence of random variables with values in $\{0,\dots,m\}$, given with a suitable filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$.

Definition 6. *The following two types of process will be extensively used in the next section:*
- *We say that $(X_t)_{t\in\mathbb{N}}$ is* of type I *if for all t:*
  - $\mathbb{E}[X_{t+1}|\mathcal{F}_t] = X_t$ *(i.e., (X_t) is a martingale), and*
  - *if* $0 < X_t < m$*, then* $\Pr\{\Delta X_{t+1} \geqslant 1\} = \Pr\{\Delta X_{t+1} \leqslant -1\} \geqslant \epsilon$.

– *We say that* $(X_t)_{t\in\mathbb{N}}$ *is* of type II *if for all* t*:*
  - *if* $X_t < m$*, then* $\mathbb{E}[X_{t+1}] = X_t$ *(i.e.,* (X_t) *behaves as a martingale when* $X_t < m$*), and*
  - *if* $0 < X_t < m$*, then* $\Pr\{\Delta X_{t+1} \geqslant 1\} = \Pr\{\Delta X_{t+1} \leqslant -1\} \geqslant \epsilon$*, and*
  - *if* $X_t = m$*, then* $\Pr\{X_{t+1} \leqslant m-1\} \geqslant \epsilon$ *(i.e.,* X_t *"bounces on* m*").*

Note that when (X_t) is of type I, if for some t, $X_t \in \{0, m\}$, then $X_{t'} = X_t$ for all $t' \geqslant t$, because (X_t) is a martingale bounded between 0 and m. Thus, $\{0, m\}$ are the (only) fixed points of any type I sequence. When (X_t) is of type II, if for some t, $X_t = 0$, then $X_{t'} = X_t$ for all $t' \geqslant t$, because (X_t) is a martingale lower bounded by 0. Thus, 0 is the (only) fixed point of any type II sequence.

Definition 7. *The* convergence time *of a type I sequence* (X_t) *is defined as the random variable* $T = \min\{t : X_t \in \{0, m\}\}$*. The convergence time of a type II sequence* (X_t) *is similarly defined as the random variable* $T = \min\{t : X_t = 0\}$*.*

The following lemmas bound the convergence time of these two types of random processes.

Lemma 5 *For sequence* (X_t)*, if* $X_0 = x_0$*, the expectation of* T *satisfies:*

$$\mathbb{E}[T] \leqslant \frac{x_0(m-x_0)}{2\epsilon} \qquad \text{if } (X_t) \text{ is of type I,}$$
$$\mathbb{E}[T] \leqslant \frac{x_0(2m+1-x_0)}{2\epsilon} \qquad \text{if } (X_t) \text{ is of type II.}$$

5 Convergence

In this section, we evaluate the worst expected convergence time for each of the twenty-four representative automata in Tab. 1. Our results rely on studying the evolution of quantities computed on the random configurations (x^t), whose convergence implies the convergence of the automaton. The upper bounds on the convergence time of these quantities are obtained by coupling them with one of the integer random processes analyzed in the previous section. The lower bounds are obtained by analyzing the exact expected convergence time for a particular initial configuration (most of the time, a configuration with a single 0-region and a single 1-region). This involves building suitable variants measuring progress towards fixed points. One of the main difficulties is to handle correctly the mergings of the regions, i.e., the applications of transitions D and E.

We introduce the following convenient functions that simplify the evaluation of the quantities that are used to bound the convergence time. These function will spare us tedious parsings of the patterns in the configurations. For a given configuration x, we denote by $a(x), \ldots, h(x)$ the number of cells where transitions A, …, H are applicable, i.e.:

$$a(x) = |x|_{000},\ b(x) = |x|_{001},\ c(x) = |x|_{100},\ d(x) = |x|_{101},$$
$$e(x) = |x|_{010},\ f(x) = |x|_{011},\ g(x) = |x|_{110},\ h(x) = |x|_{111}.$$

For instance, consider rule BCG. For convenience, we denote by $p = 1/n$ the probability that a given cell is updated under fully asynchronous dynamics. Applying the transitions $\mathsf{A}, \ldots, \mathsf{D}$ increases the number of $\mathtt{1}$s by one and applying $\mathsf{E}, \ldots, \mathsf{H}$ decreases it by one. The expected variation of the number of $\mathtt{1}$s for configuration x in one step is then immediately $p \cdot (b(x) + c(x) - g(x))$. When the context is clear, the argument x will be omitted. Clearly, parsing properly configuration x gives the following useful relationships.

Fact 6 *For all configurations $x \in Q^U$, the following equalities hold:*

$$\begin{aligned} |x|_{\mathtt{01}} &= b + d = e + f = c + d = e + g = |x|_{\mathtt{10}}, \\ |x|_{\mathtt{001}} &= b = c = |x|_{\mathtt{100}}, \\ |x|_{\mathtt{011}} &= f = g = |x|_{\mathtt{110}}. \end{aligned}$$

Let us now analyze the worst expected convergence time for DQECAs. Due to space constraints, most of the proofs are omitted and can be found in [6].

5.1 "Coupon collector" DQECAs

The behavior of the DQECAs in this class (see Fig. 1(a)) is similar to the classic Coupon Collector random process (e.g., [7]).

Theorem 7. *Under fully asynchronous dynamics, DQECAs E and DE converge a.s. to a fixed point on any initial configuration. Their worst expected convergence time is $\Theta(n \ln n)$. The fixed points for E and DE respectively are the configurations without isolated $\mathtt{1}$ and the configurations without isolated $\mathtt{0}$ and $\mathtt{1}$.*

Proof. These rules simply erase either isolated $\mathtt{0}$s, isolated $\mathtt{1}$s or both. They never create any of them (by Fact 2), and reach a fixed point as soon as no more $\mathtt{0}$ or $\mathtt{1}$ are isolated (by Fact 3). These processes are then similar to a coupon collector process that has to collect all the isolated $\mathtt{0}$s or $\mathtt{1}$s, by drawing at each time step a random location uniformly in $\{1, \ldots, n\}$ (see e.g., [7]). If the number of remaining isolated $\mathtt{0}$s and $\mathtt{1}$s is i, the probability to draw one of them is i/n, and then, one of them is drawn on expectation after n/i steps. The expected convergence time is then bounded by $n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) = O(n \ln n)$.

Finally, configuration $(\mathtt{010})^{\lfloor n/3 \rfloor} \mathtt{0}^{n \bmod 3}$, which is a proper coupon collector process, provides a lower bound of $\Omega(n \ln n)$ for both rules.

5.2 Quadratic DQECAs

Figure 1(b) illustrates the typical space-time diagram in this class. All the results of this section are obtained by finding a proper *variant* whose convergence implies the convergence of the DQECA, and which decreases by more than a given constant on expectation.

Lemma 8 *Given an initial configuration x, for each DQECA B, BC, BDE, BCDE, BCDEG, BE, EF, BCE, EFG, BCEFG, and BEFG, there exists a sequence (X_t) of random variables with values in $\{0, \ldots, n\}$ (the* variant*), such that:*

(a) if $X_t = 0$, then x^t is a fixed point.
(b) for all t such that x^t is not a fixed point, $\mathbb{E}[\Delta X_{t+1}|X_t] \leqslant -p$.

Proof. **Rules B and BC.** Set $X_t = |x^t|_{\mathtt{0}}$ the number of $\mathtt{0}$s in x^t. (a) is clear since $X_t = 0$ implies that $x^t = \mathtt{1}^n$. We obtain (b) by noticing that each application of transitions B or C decreases X_t by one, and that for any non fixed-point configuration, an active transition is performed with probability greater or equal to p. Similarly, $X_t = |x^t|_{\mathtt{1}}$ is suitable for rules EF and EFG.
Remaining rules. We need to take into account the presence of isolated $\mathtt{0}$s and $\mathtt{1}$s. We set $X_t = |x^t|_{\mathtt{0}} + Z(x^t)$ for rules BDE, BCDE, BE, BCE, and BCDEG; and $X_t = |x^t|_{\mathtt{1}} + Z(x^t)$ for rule BEFG. Consider automaton BEFG. Clearly, $X_t \in \{0, \ldots, n\}$, and we have (a) $X_t = 0$ implies that $x^t = \mathtt{0}^n$. For this rule,

$$\mathbb{E}[\Delta X_{t+1}|x^t] = p \cdot (b - e - f - g)(x^t) - p \cdot e(x^t),$$

since only transition E acts on $Z(x^t)$. By Fact 6, one can rewrite

$$\mathbb{E}[\Delta X_{t+1}|x^t] = -p \cdot (d + e + g)(x^t).$$

Second, if x is not a fixed point, then $(b + e + f + g)(x) > 0$. But by Fact 6, if $d + e = 0$, then $b = f = g$. Thus, $b + e + f + g > 0$ implies $d + e + g > 0$. We conclude that if x^t is not a fixed point, we have (b). The proof is similar for all the remaining automata. We can now state the theorem.

Theorem 9. *Under fully asynchronous dynamics, DQECAs B, BC, BDE, BCDE, BCDEG, BE, EF, BCE, EFG, BCEFG, and BEFG converge almost surely to a fixed point on any initial configuration. Their worst expected convergence time is $\Theta(n^2)$. Only the DQECAs B, BC, BE, and BCE have non-trivial fixed points, which are the configurations where all the $\mathtt{0}$s are isolated.*

Proof. The property on the fixed points is a direct application of Fact 3. Consider now one of the rules. Let X_t be the variant given by Lemma 8. X_t does not exactly verify the hypotheses of Lemma 4: X_t needs to be extended beyond a fixed point if it is reached before $X_t = 0$. We consider the random sequence X'_t defined as follow: $X'_t = X_t$ if x^t is not a fixed point, and $X'_t = 0$ otherwise. Thus, $X'_t = 0$ if and only if x^t is a fixed point, and we can now apply Lemma 4 with $m = n$ and $\epsilon = p$ and we obtain $\mathbb{E}[T] \leqslant X_0/p = O(n^2)$.

The lower bound $\Omega(n^2)$ on the convergence time is simply given by considering the following initial configuration $x = \mathtt{0}^{\lceil n/2 \rceil}\mathtt{1}^{\lfloor n/2 \rfloor}$. Note that $X_t = |x^t|_{\mathtt{1}}$ works for all the rules on initial configuration x and its exact expected convergence time is straightforward to compute by first step analysis (see [2]).

Observe that we can divide this class into two subcategories: the automata that are monotonic, for which the variant is a non-increasing function of time, and the non-monotonic, for which the variant follows a biased random walk (see Tab. 1). Interestingly enough, this distinction is observed on the space-time diagrams.

5.3 Cubic DQECAs

Figure 1(c) and 1(d) illustrate the typical behaviors in this class: one can observe that the dynamics of the regions in the space-time diagram are similar to unbiased random walks. Furthermore, one can observe that the process of the frontiers between regions is similar to annihilating random walks (e.g.,[11]): each frontier follow a random walk and two frontiers vanish when they meet.

All the results of this section are obtained by coupling the process with a suitable unbiased bounded random walk, such that the DQECA is guaranteed to reach a fixed point before the walk reaches a (or one distinguished) boundary.

The upperbounds in Theorem 11 are straightforward applications of the following lemma 10 in combination with the probabilistic lemma 5. The lower bounds are again obtained by analyzing the expected convergence time on the initial configuration $x = \mathsf{0}^{\lceil n/2 \rceil}\mathsf{1}^{\lfloor n/2 \rfloor}$ with variant $X_t = |x^t|_{\mathsf{1}}$.

Lemma 10 *Given an initial configuration x,*

- *for each DQECA* BDEF, BDEG, *and* BCDEFG, *there exists an integer $m \leqslant 2n$ and a random integer sequence (X_t) of type I (see section 4) with values in $\{0, \ldots, m\}$, such that: for all t, if $X_t = 0$ or $X_t = m$, then x^t is a fixed point.*
- *for each DQECA* BEF, BEG, *and* BCEFG, *there exists an integer $m \leqslant 2n$ and a random integer sequence (X_t) of type II (see section 4) with values in $\{0, \ldots, m\}$, such that for all t, if $X_t = 0$, then x^t is a fixed point.*

Theorem 11. *Under fully asynchronous dynamics, DQECAs* BDEF, BDEG, BCDEFG, BEF, BEG, *and* BCEFG *converge almost surely to a fixed point on any initial configuration. Their worst expected convergence time is $\Theta(n^3)$. All of them admit only $\mathsf{0}^n$ and $\mathsf{1}^n$ as fixed point.*
For DQECAs BDEF, BDEG, *and* BCDEFG, *the fixed points $\mathsf{0}^n$ and $\mathsf{1}^n$ can be reached from any configuration (respectively distinct from $\mathsf{1}^n$ and $\mathsf{0}^n$). For DQECAs* BEF, BEG, *and* BCEFG, *any configuration distinct from $\mathsf{1}^n$ converges almost surely to $\mathsf{0}^n$.*

5.4 Exponential DQECA

Figure 1(e) illustrates the typical behavior of this class. The illustrated process will eventually converge to $\mathsf{0}^n$. The trajectory of the 0-regions is similar to a coalescing random walk : the 0-regions follow a kind of coalescing random walk and merge when they meet, until only one 0-region remains. The size of the remaining 0-region then follows a random walk, biased towards 1, that will eventually converge to n after an exponential time (note that a 0-region cannot disappear for rule BCEF). This result is obtained by coupling the process with a process applying the same rule on a suitable single 0-region configuration. The following lemma analyzes the latter process first, from which we deduce the theorem. Note that the expected convergence time is independent of the initial (non-fixed point) configuration, up to a multiplicative constant.

Theorem 12. *The fixed points of DQECA* BCEF *are $\mathsf{0}^n$ and $\mathsf{1}^n$. From any non-fixed point initial configuration, DQECA* BCEF *converges almost surely to $\mathsf{0}^n$ and its expected convergence time is exactly $\Theta(n2^n)$.*

5.5 Diverging DQECAs

Figure 1(f) illustrates the typical behavior of a divergent DQECA: the number of regions is conserved, and all reachable configurations from a given initial configuration are accessed an infinite number of times almost surely. The proof of the following result relies essentially on applying Fact 3.

Theorem 13. *Under fully asynchronous dynamics, the DQECAs* BF*,* BG*,* BCF*, and* BCFG *diverge almost surely on any initial configuration that is not one of the three following fixed points* $\mathtt{0}^n$*,* $\mathtt{1}^n$ *and, if* n *is even,* $(\mathtt{01})^{n/2}$*. Furthermore, given an initial configuration, all reachable configurations are accessed an infinite number of times almost surely.*

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