Classical isomorphisms of types

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Abstract

The study of isomorphisms of types has been mainly carried out in an intuitionistic setting. We extend some of these works to classical logic for both call-by-name and call-by-value computations by means of polarized linear logic and game semantics. This leads to equational characterizations of these isomorphisms for all the propositional connectives.

Introduction

The study of isomorphisms of types started from theoretical questions about the characterization of isomorphisms in free cartesian closed categories [27] and in the λ-calculus [13, 9]. The possibility of using these results as a basis for more practical tools (programming library search [25], gluing of software components, ...) has considerably increased the interest of people in the subject.

The underlying problem is the following: given the definition of isomorphisms of types,

Definition 1 (Isomorphims of types)
Let $\mathcal{S}$ be a logical system, with an equational theory $=_{\mathcal{S}}$ on proofs, two formulas A and B are *isomorphic*, denoted by $A \simeq_{\mathcal{S}} B$, if there exist two proofs π_A of $B \vdash A$ and π_B of $A \vdash B$ such that by cutting π_A and π_B on the conclusion B (resp. A) we obtain a proof equal, up to $=_{\mathcal{S}}$, to the axiom $A \vdash A$ (resp. $B \vdash B$).

is it possible to find an equational theory corresponding exactly to the isomorphisms of the system $\mathcal{S}$?

Much work has been done for characterizing these isomorphisms of types for fragments of the propositional λ-calculus, second order λ-calculus, extensions with recursive types (see for example [23]), with dependent types (see for example [12, 7]), ... However all these works have been carried out in an *intuitionistic* setting. Our main goal is to give first attempts in the direction of extending these results to a *classical* setting.

While the question of the study of intuitionistic isomorphisms of types seems clear to formulate, the classical case is not so simple. There are mainly three different possible questions:

- what are the isomorphisms in a non-deterministic classical setting (such as LK)?

- what are the isomorphisms in a call-by-name classical setting?
- what are the isomorphisms in a call-by-value classical setting?

We are going to address these three questions in a propositional setting with all the classical connectives. The first question is essentially degenerate and does not give an interesting notion of isomorphism. In order to solve the two other questions, we will study the isomorphisms of the polarized variant of linear logic LLP [18] because both call-by-name and call-by-value classical logics are embedded into LLP through syntactical reduction-preserving translations. This entails that, from the characterization of isomorphisms in LLP, we can deduce both call-by-name and call-by-value isomorphisms.

For the study of the isomorphisms of LLP, we use a semantical approach. Such an approach consists in finding a denotational model of the system (so that any syntactical isomorphism leads to an isomorphism in the model) in which first the isomorphisms can be computed and second there are no more isomorphisms in the model than in the syntax. For that purpose, we will use game semantics which is known to give precise models of the syntax (full abstraction and full completeness results) and makes computations easier than in the syntax. This gives an application of game semantics to a purely syntactical question, showing that even if games seem very near to the syntax they are however abstract enough to simplify computations. The main consequences are two different characterizations of the polarized isomorphisms: an equational theory and a graphical representation through polarized forests.

We have tried to organize the paper by putting in different sections the intermediate results that use different techniques in such a way that the reader only interested in a particular topic could understand it by only reading the statements of the final results of the previous sections.

Section 1 is a short section devoted to solve our first question about non-deterministic classical isomorphisms of types. The other sections only look at the deterministic case.

Section 2 contains the syntactical definition of polarized linear logic (LLP) which is the main logical system used in the following sections. Section 3 gives some elements of game semantics and proves the key result of the paper which is the characterization of the isomorphisms in the game model. Section 4 defines the interpretation of LLP in the game model showing that any isomorphism in LLP leads to an isomorphism in games. Section 5 relates isomorphisms in the game model and isomorphisms in LLP (without using games anymore except through the underlying forests) by computing the corresponding equational theory and by showing that all the obtained isomorphisms are syntactically correct. This ends the study of isomorphisms in LLP. Section 6 uses embeddings of the call-by-name $\lambda\mu$-calculus and of the call-by-value $\lambda\mu$-calculus into LLP to derive the corresponding equational theories of isomorphisms. As a consequence, we get back the intuitionistic case (this is a new proof of a known result for the call-by-name case, but as far as we know this is new for the call-by-value case).

Section 7 explains the relation between our classical results on isomorphisms with disjunction and the intuitionistic ones of Balat-Di Cosmo-Fiore [15] by showing that they are in fact not really comparable due to different underlying theories of the disjunction. Section 8 presents a relation between our classical theory of isomorphisms and the equations of the real exponential field $(\mathbb{R}, +, 0, \cdot, 1, e)$ in the spirit of Tarski's problems for natural numbers studied in the usual theory of isomorphisms of types.

1 Non-deterministic isomorphisms

In this very short section, we just want to close quickly the question of isomorphisms in the non-deterministic classical setting by proving that it is degenerate. We prove that this non-determinism entails that any two equiprovable formulas are isomorphic due to the too strong equational theory on proofs generated by cut elimination (any two proofs of a given sequent are equal).

We consider proofs in Gentzen's sequent calculus LK up to cut elimination and:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash A, \Delta}}{\Gamma \vdash A, A, \Delta}\,\mathrm{wk}_r}{\Gamma \vdash A, \Delta}\,\mathrm{ctr}_r \quad = \quad \overset{\vdots}{\Gamma \vdash A, \Delta} \qquad \text{and} \qquad \dfrac{\dfrac{\overset{\vdots}{\Gamma, A \vdash \Delta}}{\Gamma, A, A \vdash \Delta}\,\mathrm{wk}_l}{\Gamma, A \vdash \Delta}\,\mathrm{ctr}_l \quad = \quad \overset{\vdots}{\Gamma, A \vdash \Delta}$$

Proposition 1 (Isomorphisms in LK)
$A \dashv\vdash_{\mathsf{LK}} B$ if and only if $A \simeq_{\mathsf{LK}} B$.

Proof: The second direction is immediate. For the first one, from $A \vdash_{\mathsf{LK}} B$ and $B \vdash_{\mathsf{LK}} A$ we can deduce a proof π_A of $A \vdash_{\mathsf{LK}} A$ and a proof π_B of $B \vdash_{\mathsf{LK}} B$, but using Lafont's critical pair, these two proofs are equal to the identity:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_A}{A \vdash A}}{A \vdash A, C}\,\mathrm{wk}_r \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax}}{C, A \vdash A}\,\mathrm{wk}_l}{A, A \vdash A, A}\,\mathrm{cut}}{A \vdash A, A}\,\mathrm{ctr}_l}{A \vdash A}\,\mathrm{ctr}_r$$

reduces to both π_A and the identity axiom. □

This explains why we have to go to some more constrained systems to get interesting theories of isomorphisms. That is a notion of isomorphisms that respects more structure of the objects.

From a semantics point of view, this corresponds to moving from Boolean algebras to control categories [26], according to the fact that Boolean algebras arise as control categories with at most one morphism between two objects.

2 Polarized Linear Logic

Polarized linear logic (LLP) is based on a restriction of the formulas of linear logic in order to get a simpler system rich enough to interpret the λ-calculus and classical logic. Polarized formulas are given by the following grammar:

$$\begin{array}{lcccccccccccc} P & ::= & X & \mid & 1 & \mid & 0 & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \\ N & ::= & X^{\perp} & \mid & \perp & \mid & \top & \mid & N \parr N & \mid & N \,\&\, N & \mid & ?P \end{array}$$

The orthogonal of a formula is defined by the De Morgan's laws:

$$
\begin{array}{ccccccc}
(X)^\perp & = & X^\perp & \qquad & (X^\perp)^\perp & = & X \\
1^\perp & = & \perp & & \perp^\perp & = & 1 \\
0^\perp & = & \top & & \top^\perp & = & 0 \\
(P \otimes Q)^\perp & = & P^\perp \parr Q^\perp & & (N \parr M)^\perp & = & N^\perp \otimes M^\perp \\
(P \oplus Q)^\perp & = & P^\perp \,\&\, Q^\perp & & (N \,\&\, M)^\perp & = & N^\perp \oplus M^\perp \\
(!N)^\perp & = & ?N^\perp & & (?P)^\perp & = & !P^\perp
\end{array}
$$

and for any polarized formula A, $(A^\perp)^\perp = A$.

The rules of LLP are not just the rules of LL restricted to polarized formulas, otherwise classical logic would not be so easy to embed. This is why we also generalize structural rules ($?w$, $?c$ and context of $!$) to any negative formula and not only $?$-formulas.

$$
\frac{}{\vdash N, N^\perp}\,ax \qquad \frac{\vdash \Gamma, N \qquad \vdash N^\perp, \Delta}{\vdash \Gamma, \Delta}\,cut
$$

$$
\frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\,\parr \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\,\otimes
$$

$$
\frac{\vdash \Gamma, N \qquad \vdash \Gamma, M}{\vdash \Gamma, N \,\&\, M}\,\& \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\,\oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\,\oplus_2
$$

$$
\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N}\,?w \qquad \frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\,?c
$$

$$
\frac{}{\vdash \Gamma, \top}\,\top \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash 1}\,1
$$

The context of the $\top$-rule must contain at most one positive formula and $\mathcal{N}$ is a context of negative formulas.

Lemma 1 (Positive formula)
If $\vdash \Gamma$ is provable in LLP, Γ contains at most one positive formula.

The equational theory we consider on LLP proofs, denoted by $\pi_1 =_{\beta\eta} \pi_2$, is given by cut-elimination (see appendix A), η-expansion of axioms (see appendix B) and the following equation (Rétoré's reduction):

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma, N\end{array}}{\vdash \Gamma, N, N}\,?w}{\vdash \Gamma, N}\,?c \quad = \quad \begin{array}{c}\vdots\\ \vdash \Gamma, N\end{array}
$$

We use the notation $=_{\beta\eta}$ with the idea that β means "up to cut elimination" and η means "up to expansion of axioms", even if this relation is stronger than the usual $\beta\eta$-equivalence of the λ-calculus.

Definition 2 (Polarized isomorphims)
Let A and B be two polarized formulas, A and B are *isomorphic*, denoted by $A \simeq_{\mathsf{LLP}} B$, if there

$$\begin{array}{rcl@{\qquad}rcl}
N \parr M & = & M \parr N & P \otimes Q & = & Q \otimes P \\
(N \parr M) \parr L & = & N \parr (M \parr L) & (P \otimes Q) \otimes R & = & P \otimes (Q \otimes R) \\
N \parr \bot & = & N & P \otimes 1 & = & P \\[1ex]
N \,\&\, M & = & M \,\&\, N & P \oplus Q & = & Q \oplus P \\
(N \,\&\, M) \,\&\, L & = & N \,\&\, (M \,\&\, L) & (P \oplus Q) \oplus R & = & P \oplus (Q \oplus R) \\
N \,\&\, \top & = & N & P \oplus 0 & = & P \\[1ex]
N \parr (M \,\&\, L) & = & (N \parr M) \,\&\, (N \parr L) & P \otimes (Q \oplus R) & = & (P \otimes Q) \oplus (P \otimes R) \\
N \parr \top & = & \top & P \otimes 0 & = & 0 \\[1ex]
!(N \,\&\, M) & = & !N \otimes !M & ?(P \oplus Q) & = & ?P \parr ?Q \\
!\top & = & 1 & ?0 & = & \top
\end{array}$$

for all negative formulas N, M and L and all positive formulas P, Q and R.

Figure 1: Equational theory $\mathcal{E}$

exist two proofs π_A of $\vdash B^\perp, A$ and π_B of $\vdash A^\perp, B$ such that by cutting π_A and π_B on the conclusion B (resp. A) we obtain a proof equal to the axiom $\vdash A^\perp, A$ (resp. $\vdash B^\perp, B$), up to $=_{\beta\eta}$.

Let N and M be two negative formulas, N and M are *intuitionistically isomorphic*, denoted by $N \simeq^i_{\mathsf{LLP}} M$, if there exist two proofs π_N of $\vdash ?M^\perp, N$ and π_M of $\vdash ?N^\perp, M$ such that:

- By cutting, on the conclusion $!M$, π_N and π_M with a promotion on M, we obtain a proof equal to the proof $\dfrac{\overline{\vdash N^\perp, N}\ ax}{\vdash ?N^\perp, N}\ ?d$ up to $=_{\beta\eta}$.
- By cutting, on the conclusion $!N$, π_M and π_N with a promotion on N, we obtain a proof equal to the proof $\dfrac{\overline{\vdash M^\perp, M}\ ax}{\vdash ?M^\perp, M}\ ?d$ up to $=_{\beta\eta}$.

If P and Q are two positive formulas, $P \simeq^i_{\mathsf{LLP}} Q$ if $P^\perp \simeq^i_{\mathsf{LLP}} Q^\perp$.

We immediately have $A \simeq_{\mathsf{LLP}} B \Rightarrow A \simeq^i_{\mathsf{LLP}} B$ by adding dereliction rules to π_A and π_B.

Definition 3 ($\mathcal{E}$-isomorphism)
$\mathcal{E}$ is the equational theory given in figure 1.

Let A and B be two polarized formulas, A and B are *$\mathcal{E}$-isomorphic*, denoted by $A \simeq_{\mathcal{E}} B$, if they are equal in the theory $\mathcal{E}$.

Lemma 2 (Correctness of $\mathcal{E}$)
If $A \simeq_{\mathcal{E}} B$ then $A \simeq_{\mathsf{LLP}} B$ (and thus $A \simeq^i_{\mathsf{LLP}} B$).

Proof: The proofs corresponding to the equations are:

$$\dfrac{\dfrac{\overline{\vdash N^\perp, N}\ ax \qquad \overline{\vdash M^\perp, M}\ ax}{\vdash N^\perp \otimes M^\perp, N, M}\ \otimes}{\vdash N^\perp \otimes M^\perp, M \parr N}\ \parr$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax \qquad \dfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash N^{\perp} \otimes M^{\perp}, N, M}\,\otimes \qquad \dfrac{}{\vdash L^{\perp}, L}\,ax}{\vdash (N^{\perp} \otimes M^{\perp}) \otimes L^{\perp}, N, M, L}\,\otimes}{\vdash (N^{\perp} \otimes M^{\perp}) \otimes L^{\perp}, N, M \parr L}\,\parr}{\vdash (N^{\perp} \otimes M^{\perp}) \otimes L^{\perp}, N \parr (M \parr L)}\,\parr
$$

$$
\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax \qquad \dfrac{}{\vdash 1}\,1}{\vdash N^{\perp} \otimes 1, N}\,\otimes
\qquad\qquad
\dfrac{\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax}{\vdash N^{\perp}, N, \perp}\,\perp}{\vdash N^{\perp}, N \parr \perp}\,\parr
$$

$$
\dfrac{\dfrac{}{\vdash N, \top, 0}\,\top}{\vdash N \parr \top, 0}\,\parr
\qquad\qquad
\dfrac{}{\vdash N^{\perp} \otimes 0, \top}\,\top
$$

$$
\dfrac{\dfrac{\dfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, M}\,\oplus_2 \qquad \dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, N}\,\oplus_1}{\vdash N^{\perp} \oplus M^{\perp}, M \,\&\, N}\,\&
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, N}\,\oplus_1}{\vdash (N^{\perp} \oplus M^{\perp}) \oplus L^{\perp}, N}\,\oplus_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, M}\,\oplus_2}{\vdash (N^{\perp} \oplus M^{\perp}) \oplus L^{\perp}, M}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash L^{\perp}, L}\,ax}{\vdash (N^{\perp} \oplus M^{\perp}) \oplus L^{\perp}, L}\,\oplus_2}{\vdash (N^{\perp} \oplus M^{\perp}) \oplus L^{\perp}, M \,\&\, L}\,\&}{\vdash (N^{\perp} \oplus M^{\perp}) \oplus L^{\perp}, N \,\&\, (M \,\&\, L)}\,\&
$$

$$
\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax \qquad \dfrac{}{\vdash N^{\perp}, \top}\,\top}{\vdash N^{\perp}, N \,\&\, \top}\,\&
\qquad\qquad
\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax}{\vdash N^{\perp} \oplus 0, N}\,\oplus_1
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax \qquad \dfrac{\dfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash M^{\perp} \oplus L^{\perp}, M}\,\oplus_1}{\vdash N^{\perp} \otimes (M^{\perp} \oplus L^{\perp}), N, M}\,\otimes}{\vdash N^{\perp} \otimes (M^{\perp} \oplus L^{\perp}), N \parr M}\,\parr \qquad \dfrac{\dfrac{\dfrac{}{\vdash N^{\perp}, N}\,ax \qquad \dfrac{\dfrac{}{\vdash L^{\perp}, L}\,ax}{\vdash M^{\perp} \oplus L^{\perp}, L}\,\oplus_2}{\vdash N^{\perp} \otimes (M^{\perp} \oplus L^{\perp}), N, L}\,\otimes}{\vdash N^{\perp} \otimes (M^{\perp} \oplus L^{\perp}), N \parr L}\,\parr}{\vdash N^{\perp} \otimes (M^{\perp} \oplus L^{\perp}), (N \parr M) \,\&\, (N \parr L)}\,\&
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N, N^{\perp}}\,ax \qquad \dfrac{}{\vdash M, M^{\perp}}\,ax}{\vdash N, M, N^{\perp} \otimes M^{\perp}}\,\otimes}{\vdash N, M, (N^{\perp} \otimes M^{\perp}) \oplus (N^{\perp} \otimes L^{\perp})}\,\oplus_1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash N, N^{\perp}}\,ax \qquad \dfrac{}{\vdash L, L^{\perp}}\,ax}{\vdash N, L, N^{\perp} \otimes L^{\perp}}\,\otimes}{\vdash N, L, (N^{\perp} \otimes M^{\perp}) \oplus (N^{\perp} \otimes L^{\perp})}\,\oplus_2}{\vdash N, M \,\&\, L, (N^{\perp} \otimes M^{\perp}) \oplus (N^{\perp} \otimes L^{\perp})}\,\&}{\vdash N \parr (M \,\&\, L), (N^{\perp} \otimes M^{\perp}) \oplus (N^{\perp} \otimes L^{\perp})}\,\parr
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash N^\perp, N$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, N$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), N$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), !N$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash M^\perp, M$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, M$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), !M$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), !N \otimes !M$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), !N \otimes !M$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash N, N^\perp$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash N, ?N^\perp$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash N, ?N^\perp, ?M^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash M, M^\perp$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash M, ?M^\perp$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash M, ?N^\perp, ?M^\perp$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash N \,\&\, M, ?N^\perp, ?M^\perp$}
\RightLabel{$!$}
\UnaryInfC{$\vdash !(N \,\&\, M), ?N^\perp, ?M^\perp$}
\RightLabel{$⅋$}
\UnaryInfC{$\vdash !(N \,\&\, M), ?N^\perp ⅋ ?M^\perp$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\top$}
\UnaryInfC{$\vdash \top$}
\RightLabel{$!$}
\UnaryInfC{$\vdash !\top$}
\RightLabel{$\perp$}
\UnaryInfC{$\vdash !\top, \perp$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\vdash 1$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?0, 1$}
\end{prooftree}
$$

We then have to verify that, up to $=_{\beta\eta}$, cutting the two corresponding proofs gives an axiom. Let us, for example, have a quick look at the proof of $\vdash ?(N^\perp \oplus M^\perp), !(N \,\&\, M)$ thus obtained:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash N^\perp, N$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, N$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), N$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), N$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash M^\perp, M$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, M$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), N \,\&\, M$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), !(N \,\&\, M)$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), !(N \,\&\, M)$}
\end{prooftree}
$$

if we introduce cuts with the appropriate η-expansions of axioms and if we eliminate them, we obtain:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash N^\perp, N$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, N$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), N$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), N$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), N$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash M^\perp, M$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash N^\perp \oplus M^\perp, M$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), M$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash ?(N^\perp \oplus M^\perp), N \,\&\, M$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(N^\perp \oplus M^\perp), !(N \,\&\, M)$}
\end{prooftree}
$$

and then, with Rétoré's reduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash N^\perp \oplus M^\perp, N}\,\oplus_1}{\vdash ?(N^\perp \oplus M^\perp), N}\,?d \qquad \dfrac{\dfrac{\dfrac{}{\vdash M^\perp, M}\,ax}{\vdash N^\perp \oplus M^\perp, M}\,\oplus_2}{\vdash ?(N^\perp \oplus M^\perp), M}\,?d}{\vdash ?(N^\perp \oplus M^\perp), N \mathbin{\&} M}\,\&}{\vdash ?(N^\perp \oplus M^\perp), !(N \mathbin{\&} M)}\,!$$

which is a normal form of the following proof, obtained by cutting two expansions of axioms:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash N^\perp \oplus M^\perp, N}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash M^\perp, M}\,ax}{\vdash N^\perp \oplus M^\perp, M}\,\oplus_2}{\vdash N^\perp \oplus M^\perp, N \mathbin{\&} M}\,\&}{\vdash ?(N^\perp \oplus M^\perp), N \mathbin{\&} M}\,?d}{\vdash ?(N^\perp \oplus M^\perp), !(N \mathbin{\&} M)}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash N^\perp \oplus M^\perp, N}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash M^\perp, M}\,ax}{\vdash N^\perp \oplus M^\perp, M}\,\oplus_2}{\vdash N^\perp \oplus M^\perp, N \mathbin{\&} M}\,\&}{\vdash ?(N^\perp \oplus M^\perp), N \mathbin{\&} M}\,?d}{\vdash ?(N^\perp \oplus M^\perp), !(N \mathbin{\&} M)}\,!}{\vdash ?(N^\perp \oplus M^\perp), !(N \mathbin{\&} M)}\,cut$$

$\square$

The goal of sections 3, 4 and 5 is to prove the converse of lemma 2 which is the difficult direction (theorem 2) by means of game semantics.

3 Game semantics

Game semantics gives denotational models for various systems. We will use it because it is far enough from the syntax to simplify computations and not too much to betray the syntax. Types are interpreted by arenas (forests) and proofs or programs by strategies on these arenas.

3.1 Arenas and strategies

We just recall the key definitions of usual game semantics. All these definitions with more details, explanations and justifications appear in [16].

Definition 4 (Forest)
A *forest* is a partial order $(E, \leq)$ such that for any $x \in E$, $(\{y \in E \mid y \leq x\}, \leq)$ is a finite total order.

The *nodes* of a forest $(E, \leq)$ are the elements of E and the forest is *finite* if E is finite. The *roots* are the minimal elements. If x is the maximum element under y, we say that y is a *son* of x, denoted by $x \vdash y$. If $(E, \leq)$ is a forest, the relation $\leq$ is the transitive reflexive closure of $\vdash$ and we will often represent $(E, \leq)$ by $(E, \vdash)$.

Definition 5 (Morphism)
A morphism between two forests $(E, \leq)$ and $(E', \leq')$ is a function from E to E' which respects the order (if $x \leq y$ then $f(x) \leq' f(y)$).

If there exists a forest isomorphism between E and E', we use the notation $E \simeq_f E'$.

Definition 6 (Arena)
An *arena* A is a finite forest whose nodes are called *moves*.

The polarity $\lambda_A(a)$ of a move a is O (resp. P) if the length of the path (*i.e.* its number of edges) going from a root of A to a is even (resp. odd).

A move a of A is *initial*, denoted by $\vdash_A a$, if it is a root of A. If b is a son of a in A, we say that a enables b, denoted by $a \vdash_A b$. The set of the initial moves of A is denoted by A^i.

Definition 7 (Arrow)
Let $(A, \leq_A)$ and $(B, \leq_B)$ be two arenas, the arena $(A \to B, \leq_{A\to B})$ is defined by:

- the set $A \to B$ is the disjoint union of $A \times B^i$ and B;
- if $(a, b_0) \in A \times B^i$ and $(a', b_0) \in A \times B^i$ with $a \vdash_A a'$, then $(a, b_0) \vdash_{A\to B} (a', b_0)$;
- if $b \in B$ and $b' \in B$ with $b \vdash_B b'$, then $b \vdash_{A\to B} b'$;
- if $b_0 \in B^i$ and $(a_0, b_0) \in A^i \times B^i$, then $b_0 \vdash_{A\to B} (a_0, b_0)$.

Definition 8 (Justified sequence)
Let A be an arena, a *justified sequence* s on A is a sequence of moves of A with, for each non-initial move b of s, a pointer to an earlier occurrence of move a of s, called the *justifier* of b, such that $a \vdash_A b$.

Definition 9 (Projections in $A \to B$)
If s is a justified sequence on $A \to B$, the *projection* $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sequence on A (resp. B) containing only the moves a such that (a, b_0) is a move of s for some b_0 (resp. the moves b such that b is a move of s).

In this spirit, given a justified sequence s on $A \to B$, we will often say that a move of s is in A when it is in $A \times B^i$.

Definition 10 (Play)
Let A be an arena, a *play* s on A is a justified sequence on A with moves of alternated polarity.

The set of plays of A is denoted by $\mathcal{P}_A$. We use the notation $t \leq^P s$ if t is a prefix of s ending with a P-move. We say that t is a *P-prefix* of s.

Definition 11 (View)
Let A be an arena and s be a play on A, the *view* $\ulcorner s \urcorner$ of s is the sub-play of s defined by:

- $\ulcorner sa \urcorner = \ulcorner s \urcorner a$ if a is a P-move;
- $\ulcorner sa \urcorner = a$ if a is an initial O-move;
- $\ulcorner satb \urcorner = \ulcorner s \urcorner ab$ is b is an O-move justified by a.

Definition 12 (Strategy)
A *strategy* σ on A, denoted by $\sigma : A$, is a non-empty P-prefix closed set of even length plays of A such that:

- *determinism*: if $sab \in \sigma$ and $sac \in \sigma$, then $sab = sac$.
- *visibility*: if $sab \in \sigma$, the justifier of b is in $\ulcorner sa \urcorner$.
- *innocence*: if $sab \in \sigma$, $t \in \sigma$, $ta \in \mathcal{P}_A$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$ then $tab \in \sigma$.

Remark: Due to the innocence condition, a strategy is completely characterized by its views.

Definition 13 (Composition)
Let A, B and C be three arenas, an *interaction sequence* u on A, B and C is a justified sequence on $(A \to B) \to C$ such that $u \upharpoonright_{A \to B} \in \mathcal{P}_{A \to B}$, $u \upharpoonright_{B \to C} \in \mathcal{P}_{B \to C}$ and $u \upharpoonright_{A \to C} \in \mathcal{P}_{A \to C}$. A move of u in A pointing to a move in B is an initial move of A and its justifier is an initial move of B, the play $u \upharpoonright_{A \to C}$ is obtained by choosing as a pointer for these initial moves of A the justifier of their justifier which is an initial move of C (the other moves in A are pointing in $u \upharpoonright_A$ and the moves in C are pointing in $u \upharpoonright_C$). The set of the interaction sequences on A, B and C is denoted by $\mathrm{int}(A, B, C)$.

Let $\sigma : A \to B$ and $\tau : B \to C$ be two strategies, the *composition* of σ and τ is the strategy $\sigma;\tau = \{u \upharpoonright_{A \to C} \mid u \in \mathrm{int}(A, B, C) \wedge u \upharpoonright_{A \to B} \in \sigma \wedge u \upharpoonright_{B \to C} \in \tau\} : A \to C$.

Lemma 3 (Zipping)
Let $\sigma : A \to B$ and $\tau : B \to C$ be two strategies, if s is a play in $\sigma;\tau$, there exists exactly one interaction sequence u ending with a move in A or C such that $u \upharpoonright_{A \to B} \in \sigma$, $u \upharpoonright_{B \to C} \in \tau$ and $u \upharpoonright_{A \to C} = s$.

Proof: The existence of u is immediately given by the definition of composition.

Let u and v be two such interaction sequences on A, B and C. We prove the result by induction on the length of s. If $s = \varepsilon$, we have $u = v = \varepsilon$. If $s = tmn$, we decompose u (resp. v) into $u_1 u_2 m u_3 n$ (resp. $v_1 v_2 m v_3 n$) where u_1 ends with the last move of t, and u_2 and u_3 contain only moves in B. By induction hypothesis applied to t, we have $u_1 = v_1$. We first show that $u_2 = v_2 = \varepsilon$: the last move of u_1 is a P-move in $A \to C$ thus by the alternation conditions (see the state diagram in [16]) it cannot be followed by a move in B. We show that $u_3 = v_3$:

- If u_3 is a strict prefix of v_3, let $b \in B$ be the first move in v_3 not in u_3. We have both $u_1 m u_3 n \upharpoonright_{A \to B} \in \sigma$ and $v_1 m u_3 b \upharpoonright_{A \to B} \in \sigma$ which is not possible with $n \in A$ by determinism of σ and not possible with $n \in C$ by determinism of τ.
- If v_3 is a strict prefix of u_3, this is the same.
- Otherwise, let w be the maximal common prefix of u_3 and v_3. If $u_3 \neq v_3$, we have $u_3 = wb$ and $v_3 = wb'$ with $b \neq b'$. If b and b' are P-moves (resp. O-moves) in B, we have $u_1 m w b \upharpoonright_{A \to B} \in \sigma$ (resp. τ) and $v_1 m w b' \upharpoonright_{A \to B} \in \sigma$ (resp. τ) contradicting the determinism of σ (resp. τ).

This entails $u_3 = v_3$ and finally $u = v$. □

Definition 14 (Identity)
Let A be an arena, the *identity* strategy id_A is $id_A = \{s \in \mathcal{P}_{A_1 \to A_2} \mid \forall t \leq^P s, t \upharpoonright_{A_1} = t \upharpoonright_{A_2}\} : A \to A$ (the indexes are only used to distinguish occurrences).

Lemma 4
If s is a justified sequence on A, there exists a unique play $s' \in \mathcal{P}_{A_1 \to A_2}$ of id_A such that $s' \upharpoonright_{A_1} = s' \upharpoonright_{A_2} = s$.

PROOF: By induction on the length of s. If $s = \varepsilon$, we have $s' = \varepsilon$. If $s = ta$, by induction hypothesis, there exists a play t' of id_A such that $t' \upharpoonright_{A_1} = t' \upharpoonright_{A_2} = t$. If a is an O-move, we have $s' = t'a_2a_1$ and if a is a P-move, we have $s' = t'a_1a_2$ where a_i is the copy of a in A_i. □

Definition 15 (Total strategy)
Let $\sigma : A$ be a strategy, σ is *total* if whenever $s \in \sigma$ and $sa \in \mathcal{P}_A$, there exists some b such that $sab \in \sigma$.

3.2 Isomorphisms

We next give a characterization of isomorphisms in the game model, which is the key result of the paper.

Proposition 2
If f is a forest isomorphism from A to B, then $\{s \in \mathcal{P}_{A \to B} \mid \forall t \leq^P s, f(t \upharpoonright_A) = t \upharpoonright_B\}$ is a strategy, called the strategy generated *by f (where $f(s)$ is the justified sequence obtained from s by replacing each move a by $f(a)$ and by preserving the pointers).*

PROOF: We denote this set of plays by σ. By definition, σ is a P-prefix closed set of even length plays of $A \to B$. We have to prove the three properties of strategies:

- If $sab \in \sigma$ and $sac \in \sigma$, and assuming a in A, the condition in the definition of σ entails that b and c are in B, so that $sab \upharpoonright_B = f(sab \upharpoonright_A) = f(sa \upharpoonright_A) = f(sac \upharpoonright_A) = sac \upharpoonright_B$ and thus $sab = sac$.
- If $sab \in \sigma$, and assuming a in A, the condition in the definition of σ entails that b is in B and, since $sab \upharpoonright_B = f(sab \upharpoonright_A)$, the justifier of b is the move just before the justifier of a in sab so that it is in the view $\ulcorner sa \urcorner$.
- If $sab \in \sigma$, $t \in \sigma$, $ta \in \mathcal{P}_{A \to B}$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$, and assuming a in A, the condition in the definition of σ entails that b is in B and, we have $sab \upharpoonright_B = f(sab \upharpoonright_A)$ thus $b = f(a)$ and we conclude $tab \upharpoonright_B = f(tab \upharpoonright_A)$ that is $tab \in \sigma$. □

Definition 16 (Game isomorphism)
Let A and B be two negative arenas, a *game isomorphism* between A and B is a pair of strategies $\sigma : A \to B$ and $\tau : B \to A$ such that $\sigma;\tau = id_A$ and $\tau;\sigma = id_B$.

If there exists a game isomorphism between A and B, we use the notation $A \simeq_g B$.

Definition 17 (Zig-zag play)
A play s in $A \to B$ is *zig-zag* if:

- each P-move following an O-move in A (resp. B) is in B (resp. A);
- each P-move in A following an initial O-move in B is justified by it;
- $s \upharpoonright_A$ and $s \upharpoonright_B$ have the same pointers.

If only the first two conditions are verified, s is a *pre-zig-zag* play.

Lemma 5 (Dual pre-zig-zag play)
If s is a pre-zig-zag play of even length in $A \to B$, there exists a unique pre-zig-zag play $\bar{s}$ in $B \to A$ such that $\bar{s} \upharpoonright_A = s \upharpoonright_A$ and $\bar{s} \upharpoonright_B = s \upharpoonright_B$.

Proof: We define $\bar{s}$ by induction on the length of s: if $s = \varepsilon$, $\bar{s} = \varepsilon$; if $s = tab$ then $\bar{s} = \bar{t}ba$; if $s = tba$ then $\bar{s} = \bar{t}ab$.

It is easy to check that $\bar{s}$ is a pre-zig-zag play. Moreover due to the alternation condition of pre-zig-zag plays and due to the condition $\bar{s} \upharpoonright_A = s \upharpoonright_A$ and $\bar{s} \upharpoonright_B = s \upharpoonright_B$, $\bar{s}$ is unique: the moves in A and B and their pointers are given by the projections and their interleaving is given by alternation. □

Lemma 6 (Composition of zig-zag plays)
If $s \in \mathcal{P}_{A \to B}$ is a pre-zig-zag play of even length, there exists an interaction sequence $\tilde{s} \in \mathrm{int}(B, A, B)$ such that $\tilde{s} \upharpoonright_{B \to A} = \bar{s}$ and $\tilde{s} \upharpoonright_{A \to B} = s$.

Proof: We prove the result by induction on the length of s. If $s = \varepsilon$, we have $\bar{s} = \varepsilon$ and $\tilde{s} = \varepsilon$. If $s = tab$, by induction hypothesis, there exists an interaction sequence $\tilde{t}$ such that $\tilde{t} \upharpoonright_{B \to A} = \bar{t}$ and $\tilde{t} \upharpoonright_{A \to B} = t$. If a is in A thus b is in B (resp. a in B and b in A), we consider $\tilde{s} = \tilde{t}b_1ab_2$ where b_1 is the copy of b in the leftmost B and b_2 is the copy of b in the rightmost B (resp. $\tilde{s} = \tilde{t}a_2ba_1$ where a_2 is the copy of a in the rightmost B and a_1 in the leftmost one). □

Theorem 1 (Strict isomorphisms)
If there exists a game isomorphism (σ, τ) between A and B, then there exists a forest isomorphism f from A to B, such that moreover σ (resp. τ) is the strategy generated by f (resp. f^{-1}).

Proof: We first show by induction on the even number k that if $s \in \sigma$ with length k then s is a zig-zag play and $\{t \mid t \in \tau \wedge |t| = k\} = \{\bar{s} \mid s \in \sigma \wedge |s| = k\}$:

- If $k = 0$ then $s = \varepsilon$ and the result is immediate.
- If $k = k'+2$ and s is a play in σ of length k, we have $s = tmn$ with $t \in \sigma$ and by induction hypothesis t is zig-zag and $\bar{t} \in \tau$. We assume $m \in B$ (the case $m \in A$ is similar). If $n \in B$ then, using lemma 6, $\tilde{t}mn \in \mathrm{int}(B, A, B)$ is such that $\tilde{t}mn \upharpoonright_{B \to A} = \tilde{t} \upharpoonright_{B \to A} = \bar{t} \in \tau$ and $\tilde{t}mn \upharpoonright_{A \to B} = \tilde{t} \upharpoonright_{A \to B} mn = tmn \in \sigma$, so that $\tilde{t}mn \upharpoonright_{B \to B} \in \tau; \sigma = id_B$ but this is impossible because a play in id_B cannot contain two successive moves in the same B. This entails $n \in A$.

  According to lemma 4, we consider the play $s' \in \mathcal{P}_{B_1 \to B_2}$ of id_B such that $s' \upharpoonright_{B_1} = s' \upharpoonright_{B_2} = s \upharpoonright_B = t \upharpoonright_B m$. By definition of the composition $\tau; \sigma$, there exists $u \in \mathrm{int}(B, A, B)$ such that $u \upharpoonright_{B \to B} = s'$, $u \upharpoonright_{B \to A} \in \tau$ and $u \upharpoonright_{A \to B} \in \sigma$. By lemma 3, $u = \tilde{t}mnm$, which entails that n is justified by m in s if m is initial (otherwise $u \upharpoonright_{B \to B} \notin id_B$), so that s is a pre-zig-zag play. Moreover $\tilde{t}mnm \upharpoonright_{B \to A} = \tilde{s} \upharpoonright_{B \to A} = \bar{s} \in \tau$ so that $\{\bar{s} \mid s \in \sigma \wedge |s| = k\} \subset \{t \mid t \in \tau \wedge |t| = k\}$. We obtain the converse in the same way.

  We still have to show that s is a zig-zag play. The justifier of n in $s \upharpoonright_A$ must be a move before the justifier of m in $s \upharpoonright_B$ by visibility of σ, and it must be the move just before it, otherwise the justifier of m in $\bar{t}nm \in \tau$ is not in $\ulcorner \bar{t}n \urcorner$, contradicting the visibility condition for τ.

We have shown that $\tau = \{\bar{s} \mid s \in \sigma\}$. We have almost shown that σ is total: if $s \in \sigma$ and $sm \in \mathcal{P}_{A \to B}$, we assume $m \in B$ (the case $m \in A$ is similar), we consider the play $s' \in \mathcal{P}_{B_1 \to B_2}$

of id_B such that $s' \upharpoonright_{B_1} = s' \upharpoonright_{B_2} = sm \upharpoonright_B$ (lemma 4). By definition of the composition $\tau;\sigma$, there exists $u \in \mathrm{int}(B, A, B)$ such that $u \upharpoonright_{B \to B} = s'$, $u \upharpoonright_{B \to A} \in \tau$ and $u \upharpoonright_{A \to B} \in \sigma$. We can write $u = u'mnm$ for some n since σ and τ contain only zig-zag plays, and by lemma 3, we have $u' = \tilde{s}$ so that $\tilde{s}mnm \upharpoonright_{A \to B} = smn \in \sigma$. This immediately gives τ also total.

Let a be a move of A and $a_1 \dots a_p$ be the sequence of moves of A such that a_1 is initial, a_i enables a_{i+1} for each $1 \leq i \leq p-1$ and $a_p = a$, we are going to define a function f from the moves of A to the moves of B by induction on p. In fact we enrich the induction hypothesis by requiring that $f(a_1)a_1a_2f(a_2)f(a_3)a_3 \ldots \in \sigma$. If $p = 1$, we look at the unique play of the shape ab in τ (which exists by totality) and we define $f(a) = b$. If $p = p' + 1$ with p odd, we have by induction hypothesis $a_1f(a_1)f(a_2)a_2a_3f(a_3) \ldots f(a_{p'})a_{p'} \in \tau$, let $f(a_p)$ be the unique move such that $a_1f(a_1)f(a_2)a_2a_3f(a_3) \ldots f(a_{p'})a_{p'}a_pf(a_p) \in \tau$ which exists by totality of τ. If $p = p' + 1$ with p even, we have by induction hypothesis $f(a_1)a_1a_2f(a_2)f(a_3)a_3 \ldots f(a_{p'})a_{p'} \in \sigma$, let $f(a_p)$ be the unique move such that $f(a_1)a_1a_2f(a_2)f(a_3)a_3 \ldots f(a_{p'})a_{p'}a_pf(a_p) \in \sigma$ which exists by totality of σ.

In the same way, we can associate a function g with τ and we easily verify that $f \circ g$ is the identity on the moves of B and $g \circ f$ is the identity on the moves of A so that f is a bijection. Moreover, by construction, if $a \leq a'$ in A we have $f(a) \leq f(a')$ in B, so that f is an isomorphism between the forests A and B.

Finally, we show that if σ is innocent, σ is the strategy generated by f. Let σ_f be this strategy, we just have to show that if s is a view then $s \in \sigma \iff s \in \sigma_f$. Let s be a zig-zag view in $A \to B$, we prove by induction on the length of s that $s \upharpoonright_A$ is a sequence $a_1 \dots a_p$ such that a_i justifies a_{i+1} for $1 \leq i \leq p-1$ and $s \upharpoonright_B$ is a sequence $b_1 \dots b_q$ such that b_j justifies b_{j+1} for $1 \leq j \leq q-1$. If $s = \varepsilon$ the result is straightforward. If $s = mn$, $m \in B$ and $n \in A$ are initial. If $s = tn'm'mn$, m is justified by m' since s is a view and n is justified by n' since s is zig-zag (and we apply the induction to t). If s is a view in σ_f, it is a zig-zag view and we have already shown that plays of the shape $f(a_1)a_1a_2f(a_2)\ldots$ where a_i justifies a_{i+1} are in σ. If s is a view in σ, it is zig-zag and a straightforward induction shows that s is of the shape $f(a_1)a_1a_2f(a_2)\ldots$ thus $s \in \sigma_f$. □

Remark: In the setting of sequential algorithms [8], Berry and Curien have given a similar characterization of isomorphisms based on a decomposition of algorithms into a purely functional part (giving the underlying order-theoretic isomorphism) and an index choice part.

4 The game model of LLP

In order to characterize the isomorphisms of types in LLP, we give the game interpretation of LLP and we show that it gives a denotational model. In fact the relation between LLP and its game model is very much stronger (completeness, ...) but we just need soundness here. A precise comparison between LLP and the same game model is given in [19].

Definition 18 (Polarized arena)
A *polarized arena* A is an arena with a polarity π_A which is P or O (also denoted by $+$ or $-$).

The polarity $\lambda_A(a)$ of a move a is π_A (resp. $\overline{\pi_A}$) if the length of the path going from a root of A to a is even (resp. odd).

Definition 19 (Constructions of arenas)
We consider the following constructions on polarized arenas:

Dual. If A is an arena, its dual $A^\perp$ is obtained by changing its polarity.

Empty. There are two empty polarized arenas: the positive one and the negative one.

Unit. The unit arenas are the forests reduced to one node $\circ$ and with polarity O or P.

Sum. If A and B are two arenas of the same polarity, $A+B$ is the disjoint union of the two forests.

Product. If A and B are two arenas of the same polarity, the trees of $A \times B$ are obtained by taking a tree in A and a tree in B and by identifying their roots. More formally:

- the underlying set of $A \times B$ is $(A^i \times B^i) + (A \setminus A^i) \times B^i + A^i \times (B \setminus B^i)$;
- if $(a_0, b_0) \in A^i \times B^i$, $(a, b_0) \in (A \setminus A^i) \times B^i$ and $a_0 \vdash_A a$ then $(a_0, b_0) \vdash_{A \times B} (a, b_0)$;
- if $(a_0, b_0) \in A^i \times B^i$, $(a_0, b) \in A^i \times (B \setminus B^i)$ and $b_0 \vdash_B b$ then $(a_0, b_0) \vdash_{A \times B} (a_0, b)$;
- if $(a, b_0) \in (A \setminus A^i) \times B^i$, $(a', b_0) \in (A \setminus A^i) \times B^i$ and $a \vdash_A a'$ then $(a, b_0) \vdash_{A \times B} (a', b_0)$;
- if $(a_0, b) \in A^i \times (B \setminus B^i)$, $(a_0, b') \in A^i \times (B \setminus B^i)$ and $b \vdash_B b'$ then $(a_0, b) \vdash_{A \times B} (a_0, b')$;
- $\pi_{A \times B} = \pi_A = \pi_B$.

Lift. If A is an arena of polarity π_A, $\updownarrow A$ is obtained by adding a unique new root $\circ$ under all the trees of A. It is an arena of polarity $\overline{\pi_A}$.

Remark: If N and M are two negative arenas, $N \to M \simeq_f \updownarrow N^\perp \times M$, and this corresponds to an encoding of implication in Linear Logic: $A \to B = ?A^\perp \parr B$.

Definition 20 (Projections)
If s is a justified sequence on $A + B$, the *projection* $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sequence containing only the moves of s in A (resp. in B).

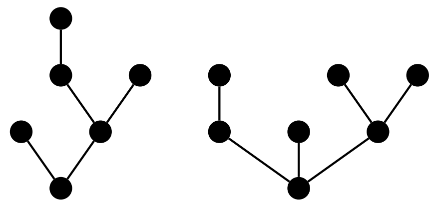
If s is a justified sequence on $A \times B$, the *projection* $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sequence containing only the moves a (resp. b) such that (a, b_0) (resp. (a_0, b)) is a move of s for some initial move b_0 (resp. a_0). In this spirit, we will say that a move of the shape (a, b_0) with a non-initial (resp. (a_0, b) with b non-initial) is a move in A (resp. in B).

A polarized formula A is interpreted by a polarized arena $A^\star$ of the same polarity. Given such an interpretation for the variables, we then have:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
0^\star & = & (\emptyset, P) & \top^\star & = & (\emptyset, O) \\
1^\star & = & (\circ, P) & \perp^\star & = & (\circ, O) \\
(P \oplus Q)^\star & = & P^\star + Q^\star & (N \mathbin{\&} M)^\star & = & N^\star + M^\star \\
(P \otimes Q)^\star & = & P^\star \times Q^\star & (N \parr M)^\star & = & N^\star \times M^\star \\
(!N)^\star & = & \updownarrow N^\star & (?P)^\star & = & \updownarrow P^\star
\end{array}
$$

with the property $A^{\star\perp} = A^{\perp\star}$.

Example 1
The polarized arena associated with $?(1 \oplus !(?!\top \mathbin{\&} \perp)) \mathbin{\&} (?(!\perp \oplus 1) \parr ?!(\perp \mathbin{\&} \perp))$ is:



with polarity O.

We now move to the constructions on strategies.

The product of strategies is not definable for any two strategies (as given by the structure of control categories [26]) but only for a particular class of strategies: *central strategies.*

Definition 21 (Central strategy)
Let $\sigma : A \to B$ be a strategy,

- σ is *linear* if in each play of σ, each initial move in B has exactly one move in A justified by it.
- σ is *strict* if, for each initial move b in B, it contains a play ba with a in A.
- σ is *central*, denoted by $\sigma : A \overset{\bullet}{\to} B$, if it is strict and linear.

Definition 22 (Product)
If $\sigma : A \overset{\bullet}{\to} C$ and $\tau : B \overset{\bullet}{\to} D$ are two central strategies, $\sigma \times \tau$ is the central strategy on $A \times B \overset{\bullet}{\to} C \times D$ defined by:

$$\sigma \times \tau = \{s \in \mathcal{P}_{A\times B \to C\times D} \mid s \upharpoonright_{A \to C} \in \sigma \wedge s \upharpoonright_{B \to D} \in \tau\}$$

Definition 23 (Weakening)
The set of views of the *weakening strategy* w_N is $\{\varepsilon\} \cup \{n\circ \mid n \text{ initial in } N\}$ which gives a central strategy on $\perp \overset{\bullet}{\to} N$.

Definition 24 (Contraction)
If s is a play on $N_1 \times N_2 \to N_0$ we define s_i $(i = 1, 2)$ to be the sub-sequence of s defined by:

- the initial moves in N_0 are in s_i;
- the moves in N_i are in s_i;
- a P-move in N_0 following a move of s_i is in s_i;
- a O-move in N_0 pointing to a move of s_i is in s_i.

The *contraction strategy* is $c_N = \{s \in \mathcal{P}_{N_1\times N_2 \to N_0} \mid \forall t \leq^P s, t_1 \in id_N \wedge t_2 \in id_N\} : N \times N \overset{\bullet}{\to} N$.

By lemma 1, a provable sequent of LLP is $\vdash \mathcal{N}, \Pi$ where Π is empty or is a positive formula. The interpretation of a proof π of $\vdash \mathcal{N}, \Pi$ is a strategy $\pi^\star$ on $\Pi^\perp \overset{\bullet}{\to} \mathcal{N}$ (this is a notation for $\mathcal{N}^\star$ if Π is empty and $P^{\star\perp} \overset{\bullet}{\to} \mathcal{N}^\star$ if $\Pi = P$) central if Π is not empty.

The strategy $\pi^\star$ is defined by induction on π:

(ax) $ax^\star = id_N$.

(cut) if $\Gamma = \Gamma', \Pi$, $\pi_1^\star : \Pi^\perp \overset{\bullet}{\to} \Gamma' \parr N$ and $\pi_2^\star : N \overset{\bullet}{\to} \Delta$, the strategy $\pi^\star$ is $\pi_1^\star; (id_{\Gamma'} \times \pi_2^\star) : \Pi^\perp \overset{\bullet}{\to} \Gamma' \parr \Delta$.

($\mathbin{⅋}$) this rule does not modify the interpretation.

($\otimes$) if $\pi_1^\star : P^\perp \xrightarrow{\bullet} \Gamma$ and $\pi_2^\star : Q^\perp \xrightarrow{\bullet} \Delta$, the strategy $\pi^\star$ is $\pi_1^\star \times \pi_2^\star : (P \otimes Q)^\perp \xrightarrow{\bullet} \Gamma \mathbin{⅋} \Delta$.

($\&$) if $\Gamma = \Gamma', \Pi$, $\pi_1^\star : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} N$ and $\pi_2^\star : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} M$, the views of $\pi^\star$ are the views of $\pi_1^\star$ and the views of $\pi_2^\star$ which give a strategy on $\Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} (M \mathbin{\&} N)$.

($\oplus_i$) if $\pi_1^\star : P_i^\perp \xrightarrow{\bullet} \Gamma$, $\pi^\star$ is $\pi_1^\star$ seen as a strategy on $(P_1 \oplus P_2)^\perp \xrightarrow{\bullet} \Gamma$.

($!$) if $\pi_1^\star : \mathcal{N} \mathbin{⅋} N$, a view of $\pi^\star : (!N)^\perp \xrightarrow{\bullet} \mathcal{N}$ is ε, or $m\circ$ with m initial in $\mathcal{N}$, or $m\circ ns$ where $(m,n)s$ is a view of $\pi_1^\star$.

($?d$) if $\pi_1^\star : P^\perp \xrightarrow{\bullet} \Gamma$, we have $\pi^\star = \{\varepsilon\} \cup \{s[{}^{(a,\circ)}/_a] \mid s \in \pi_1^\star\} : \Gamma \mathbin{⅋} ?P$ where $s[{}^{(a,\circ)}/_a]$ is obtained from s by replacing any initial move a by $(a,\circ)$.

($?w$) if $\Gamma = \Gamma', \Pi$ and $\pi_1^\star : \Pi^\perp \xrightarrow{\bullet} \Gamma'$, we can define a strategy σ on $\Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} \perp$ as in the case of a $\perp$-rule, we have $\pi^\star = \sigma; (id_{\Gamma'} \times w_N) : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} N$.

($?c$) if $\Gamma = \Gamma', \Pi$ and $\pi_1^\star : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} N \mathbin{⅋} N$, we have $\pi^\star = \pi_1^\star; (id_{\Gamma'} \times c_N) : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} N$.

($\top$) if $\Gamma = \Gamma', \Pi$, we have $\top^\star = \{\varepsilon\} : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} \top$.

($\perp$) if $\Gamma = \Gamma', \Pi$ and $\pi_1^\star : \Pi^\perp \xrightarrow{\bullet} \Gamma'$, we have $\pi^\star = \{\varepsilon\} \cup \{s[{}^{(a,\circ)}/_a] \mid s \in \pi_1^\star\} : \Pi^\perp \xrightarrow{\bullet} \Gamma' \mathbin{⅋} \perp$.

(1) The views of $1^\star$ are ε and $\circ_\perp \circ_1$ which give a strategy on $1^\perp \xrightarrow{\bullet} \perp$.

Proposition 3 (Soundness)
If π and π' are two proofs of $\vdash \Gamma$ in LLP, *$\pi =_{\beta\eta} \pi'$ entails $\pi^\star = \pi'^\star$.*

Proof: The proof of this result is given in [19]. □

Corollary 3.1 (Model of isomorphisms)
If $A \simeq^i_{\mathsf{LLP}} B$ then $A^\star \simeq_g B^\star$.

5 Polarized type isomorphisms

Using the previous two sections, we can move from the study of isomorphisms in LLP to the study of isomorphic forests: $A \simeq_{\mathsf{LLP}} B \Rightarrow A \simeq^i_{\mathsf{LLP}} B \Rightarrow A^\star \simeq_g B^\star \Rightarrow A^\star \simeq_f B^\star$.

5.1 Polarized isomorphisms without variables

The game interpretation of variables is not precise enough for us to directly obtain a characterization of types isomorphisms with variables. This is why we first study the variable free case.

Definition 25 (Additive form)
A positive formula P is an *additive form* if it can be written:

$$P \equiv \bigoplus_{i=1}^{n} !P_i^\perp$$

up to associativity of $\oplus$, with the P_is in additive form. The case $n = 0$ corresponds to the formula 0.

A negative formula N is an *additive form* if $N^\perp$ is.

Remark: A polarized formula is an additive form if and only if it does not contain any of the connectives $\otimes$, $\parr$, 1 and $\bot$ and it does not contain any sub-formula of the shape $\_ \oplus 0$ or $\_ \mathbin{\&} \top$ (or $0 \oplus \_$ or $\top \mathbin{\&} \_$).

Lemma 7 (Additive translation)
Let A be a polarized formula without variable, there exists an additive form A^a such that $A \simeq_{\mathcal{E}} A^a$.

Proof: We consider the following rewriting system (up to commutativity of the binary connectives) which is derived from the theory $\mathcal{E}$:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
N \parr \bot & \rightsquigarrow & N & P \otimes 1 & \rightsquigarrow & P \\
N \mathbin{\&} \top & \rightsquigarrow & N & P \oplus 0 & \rightsquigarrow & P \\
N \parr \top & \rightsquigarrow & \top & P \otimes 0 & \rightsquigarrow & 0 \\
N \parr (M \mathbin{\&} L) & \rightsquigarrow & (N \parr M) \mathbin{\&} (N \parr L) & P \otimes (Q \oplus R) & \rightsquigarrow & (P \otimes Q) \oplus (P \otimes R) \\
?P \parr ?Q & \rightsquigarrow & ?(P \oplus Q) & !N \otimes !M & \rightsquigarrow & !(N \mathbin{\&} M) \\
\bot & \rightsquigarrow & ?0 & 1 & \rightsquigarrow & !\top
\end{array}$$

To show the termination of this system, we define a function φ which associates with each polarized formula A without variable an integer $\varphi(A) \geq 2$ by:

$$\begin{array}{ccccc}
\varphi(A \parr B) & = & \varphi(A \otimes B) & = & \varphi(A)\varphi(B) \\
\varphi(A \mathbin{\&} B) & = & \varphi(A \oplus B) & = & \varphi(A) + \varphi(B) + 1 \\
\varphi(!A) & = & \varphi(?A) & = & \varphi(A) + 1 \\
\varphi(\bot) & = & \varphi(1) & = & 4 \\
\varphi(\top) & = & \varphi(0) & = & 2
\end{array}$$

For any rewriting rule $A \rightsquigarrow B$, we have $\varphi(A) > \varphi(B)$.

According to the remark after definition 25, we have to show that a normal form does not contain any $\parr$, $\otimes$, $\bot$ and 1. By $\bot \rightsquigarrow ?0$ and $1 \rightsquigarrow !\top$, the case of $\bot$ and 1 is immediate. We consider the case of $\otimes$ ($\parr$ is the same). If $A \otimes B$ appears in the formula with A and B which are additive forms, the main connective of A and B must be $\oplus$, 0 or $!$. If one of these two main connectives is $\oplus$ or 0, we can apply a rule $P \otimes (Q \oplus R) \rightsquigarrow (P \otimes Q) \oplus (P \otimes R)$ or $P \otimes 0 \rightsquigarrow 0$ and if both of them are $!$, we can apply $!N \otimes !M \rightsquigarrow !(N \mathbin{\&} M)$. $\square$

Remark: If P is an additive form, the number of trees of $P^\star$ is exactly the natural number n of definition 25 and each $!P_i^\perp$ corresponds to one of these trees.

Lemma 8 (Additive arenas)
Let A and B be two polarized formulas without variable which are additive forms, and such that $A^\star \simeq_f B^\star$, we have $A \simeq_{\mathcal{E}} B$.

Proof: By induction on the size of $A^\star$, with A and B positive:

- If $A^\star$ is empty, $B^\star$ is also empty and $A = B = 0$.
- If $A^\star$ is a tree, $B^\star$ is also a tree and $A = !A'$ and $B = !B'$. The forest $A'^\star$ (resp. $B'^\star$) is obtained by removing the root of $A^\star$ (resp. $B^\star$) so that $A'^\star \simeq_f B'^\star$ and by induction hypothesis, $A' \simeq_{\mathcal{E}} B'$ and finally $A \simeq_{\mathcal{E}} B$.

- If $A^\star$ contains $n \geq 2$ trees, $B^\star$ also contains n trees and A is obtained from n formulas A_1, ..., A_n by adding the correct $\oplus$ connectives, this entails by associativity: $A \simeq_{\mathcal{E}} ((A_1 \oplus A_2) \cdots \oplus A_{n-1}) \oplus A_n$ where each $A_i^\star$ is one of the trees of $A^\star$. In the same way, we get $B \simeq_{\mathcal{E}} ((B_1 \oplus B_2) \cdots \oplus B_{n-1}) \oplus B_n$ and each $B_i^\star$ is one of the trees of $B^\star$. From $A^\star \simeq_f B^\star$, we can find a permutation θ of the trees of $A^\star$ such that for each $1 \leq i \leq n$, $A^\star_{\theta(i)} \simeq_f B_i^\star$. By induction hypothesis, $A_{\theta(i)} \simeq_{\mathcal{E}} B_i$ so that using commutativity of $\oplus$ in $\mathcal{E}$ we have $A \simeq_{\mathcal{E}} B$. □

Proposition 4 (Variable free polarized isomorphisms)
Let A and B be two polarized formulas without variable*,*

$$A \simeq_{\mathsf{LLP}} B \iff A^\star \simeq_g B^\star \iff A^\star \simeq_f B^\star \iff A \simeq_{\mathcal{E}} B$$

Proof: We prove the following implications:

- $A \simeq_{\mathcal{E}} B \Rightarrow A \simeq_{\mathsf{LLP}} B$, by lemma 2.
- $A \simeq_{\mathsf{LLP}} B \Rightarrow A \simeq^i_{\mathsf{LLP}} B \Rightarrow A^\star \simeq_g B^\star$, by soundness of the game model (corollary 3.1).
- $A^\star \simeq_g B^\star \Rightarrow A^\star \simeq_f B^\star$, by theorem 1.
- $A^\star \simeq_f B^\star \Rightarrow A \simeq_{\mathcal{E}} B$, we have $A \simeq_{\mathcal{E}} A^a$ and $B \simeq_{\mathcal{E}} B^a$ by lemma 7, and this entails with the previous implications: $A^{a\star} \simeq_f A^\star \simeq_f B^\star \simeq_f B^{a\star}$. By lemma 8 we can deduce $A^a \simeq_{\mathcal{E}} B^a$ and finally $A \simeq_{\mathcal{E}} B$. □

5.2 Recovering variables

The last result allows us to forget everything about games but their underlying forests. We show that the correspondence between (isomorphic) polarized formulas and forests can be extended with variables.

Definition 26 (Multiplicative forms)
A positive formula P is a *multiplicative form* if it can be written:

$$P \equiv \bigotimes_{i=1}^{n} !P_i^\perp \otimes \bigotimes_{j=1}^{m} X_j$$

up to associativity of $\otimes$, with the P_is in multiplicative form. The case $n = 0$ and $m = 0$ corresponds to the formula 1. A negative formula N is a multiplicative form if $N^\perp$ is.

A positive formula P is a *quasi-multiplicative form* if it can be written:

$$P \equiv \bigoplus_{k=1}^{p} P_k$$

up to associativity of $\oplus$, with the P_ks in multiplicative form. The case $p = 0$ corresponds to the formula 0. A negative formula N is a quasi-multiplicative form if $N^\perp$ is.

Remark: A polarized formula is a multiplicative form if and only if it does not contain any of the connectives $\oplus$, $\&$, 0 and $\top$ and it does not contain any sub-formula of the shape $\_ \otimes 1$ or $\_ \parr \perp$ (or $1 \otimes \_$ or $\perp \parr \_$).

Lemma 9 (Quasi-multiplicative translation)
Let A be a polarized formula (possibly with variables), there exists a quasi-multiplicative form A^m such that $A \simeq_{\mathcal{E}} A^m$.

PROOF: We use the same proof as for lemma 7 but we reverse the following rewriting rules:

$$\begin{array}{rcl@{\qquad\qquad}rcl} ?(P \oplus Q) & \rightsquigarrow & ?P \parr ?Q & !(N \,\&\, M) & \rightsquigarrow & !N \otimes !M \\ ?0 & \rightsquigarrow & \bot & !\top & \rightsquigarrow & 1 \end{array}$$

and we use:

$$\begin{array}{ccccc} \varphi(A \parr B) & = & \varphi(A \otimes B) & = & \varphi(A)\varphi(B) \\ \varphi(A \,\&\, B) & = & \varphi(A \oplus B) & = & \varphi(A) + \varphi(B) + 1 \\ \varphi(!A) & = & \varphi(?A) & = & \varphi(A)! \\ \varphi(\bot) & = & \varphi(1) & = & 2 \\ \varphi(\top) & = & \varphi(0) & = & 3 \\ \varphi(X^{\bot}) & = & \varphi(X) & = & 2 \end{array}$$

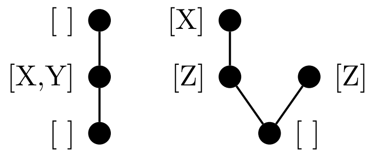
According to the remark after definition 26, we have to show that a normal form does not contain any $\top$, 0, $\&$, $\oplus$ except in head position. If there is such a connective in a normal form which is not in head position, we consider one such connective with a different connective just above it. We first consider the case of 0 ($\top$ is the same), if the connective above it is a $\otimes$ we can apply $A \otimes 0 \rightsquigarrow 0$, if it is a $\oplus$ we can apply $A \oplus 0 \rightsquigarrow A$ and if it is a $?$ we can apply $?0 \rightsquigarrow \bot$. For a $\oplus$ connective ($\&$ is the same), if the connective above it is a $\otimes$ we can apply $A \otimes (B \oplus C) \rightsquigarrow (A \otimes B) \oplus (A \otimes C)$ and if it is a $?$ we can apply $?(A \oplus B) \rightsquigarrow ?A \parr ?B$. $\square$

Definition 27 (Atomized forests)
An *atomized forest* is a forest with nodes labelled by finite (possibly empty) multi-sets of positive variables.

We can generalize the interpretation of LLP formulas by forests to an interpretation of LLP formulas by atomized forests, still denoted by $(.)^\star$:

- The atomized forest $X^\star$ (resp. $(X^{\bot})^\star$) is the one point positive (resp. negative) forest labelled by $[X]$.
- The atomized forest $1^\star$ (resp. $\bot^\star$) as an empty associated multi-set.
- The atomized forest $(P \oplus Q)^\star$ (resp. $(N \,\&\, M)^\star$) has the same multi-sets as the ones associated with P and Q (resp. N and M).
- The atomized forest $(P \otimes Q)^\star$ (resp. $(N \parr M)^\star$) has the same multi-sets as the ones associated with P and Q (resp. N and M) for the non-root nodes, and the union of the labelling multi-sets for the identified roots.
- The atomized forest $(!N)^\star$ (resp. $(?P)^\star$) is obtained by adding the new root with an empty associated multi-set.

Example 2
The interpretation of $?(X^{\bot} \otimes Y^{\bot} \otimes !1) \,\&\, ?(Z^{\bot} \otimes (!X \oplus 1))$ is the negative atomized forest:



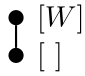
Definition 28 (Atomized forest morphism)
Let A and B be two atomized forests, an *atomized forest morphism* from A to B is a forest morphism f from A to B such that, if n is a node of A, the multi-set of variables of n is included in the multi-set of $f(n)$.

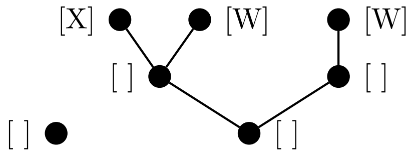
Definition 29 (Atomized forest substitution)
A *forest substitution* θ is a partial function from positive variables to atomized forests.

If θ is a forest substitution and A is a forest, the forest $A\theta$ is obtained by replacing each node with associated multi-set $[X_1, \ldots, X_n, Y_1, \ldots, Y_p]$ (where the X_js are in the domain of θ and the Y_ks are not) by the forest $\theta(X_1) \times \cdots \times \theta(X_n) \times Y_1 \times \cdots \times Y_p$.

More formally, by induction on the size of A:

- If A is empty, $A\theta$ is empty.
- Otherwise, for each root r_i of A with the forest F_i above it and with labelling multi-set $[X_1^i, \ldots, X_{n_i}^i, Y_1^i, \ldots, Y_{p_i}^i]$ (where the X_j^is are in the domain of θ and the Y_k^is are not), we build the forest $F_i' = \updownarrow(F_i\theta) \times \theta(X_1^i) \times \cdots \times \theta(X_{n_i}^i) \times Y_1^i \times \cdots \times Y_{p_i}^i$ (which is empty if one of the $\theta(X_j^i)$s is empty). The forest $A\theta$ is the union (or sum) of the F_i's.

Example 3
The application of the substitution $Y \mapsto \emptyset$ and $Z \mapsto$  to the forest of example 2 leads to:



Lemma 10
If A is a polarized formula and θ is a substitution, we can associate with θ a forest substitution $\theta^\star$ by $\theta^\star(X) = (\theta(X))^\star$ for any X. We have $(A\theta)^\star = A^\star\theta^\star$.

Remark: If P is a multiplicative form, $P^\star$ is a tree with a root labelled with $[X_1, \ldots, X_m]$ and it has n sons where the X_js and n are given as in definition 26. Moreover each $P_i^\star$ is one of the immediate sub-trees of $P^\star$

If P is a quasi-multiplicative form, $P^\star$ is a forest of p trees where p is given as in definition 26 and each P_k corresponds to one of these trees.

Lemma 11 (Quasi-multiplicative arenas)
Let A and B be two quasi-multiplicative forms such that $A^\star \simeq_f B^\star$, we have $A \simeq_{\mathcal{E}} B$.

Proof: By induction on the size of $A^\star$ (that is the number of nodes of $A^\star$ plus the sum of the sizes of its multi-sets), with A and B positive.

- If $A^\star$ and $B^\star$ are empty, $A = B = 0$.

- If $A^\star$ and $B^\star$ are two trees with roots labelled with a multi-set containing X, according to the previous remark, A (resp. B) can be written $A = A' \otimes X$ (resp. $B = B' \otimes X$) where $A'^\star$ (resp. $B'^\star$) is obtained by removing an occurrence of X in the multi-set associated with the root of $A^\star$ (resp. $B^\star$). By induction hypothesis applied to A' and B', we have $A' \simeq_{\mathcal{E}} B'$ and thus $A \simeq_{\mathcal{E}} B$.
- If $A^\star$ and $B^\star$ are two trees with an empty associated multi-set, let n be the number of sons of their root (which is the same for $A^\star$ and $B^\star$). Moreover, according to the previous remark, we have $A = \bigotimes_{i=1}^{n} !A_i$ and $B = \bigotimes_{i=1}^{n} !B_i$ where each A_i (resp. B_i) corresponds to an immediate sub-tree of $A^\star$ (resp. $B^\star$). From $A^\star \simeq_f B^\star$ we can deduce a permutation θ of the A_i's such that $A^\star_{\theta(i)} \simeq_f B^\star_i$. By induction hypothesis, $A_{\theta(i)} \simeq_{\mathcal{E}} B_i$ and thus, up to commutativity, $A \simeq_{\mathcal{E}} B$.
- If $A^\star$ and $B^\star$ are two forests containing $p \geq 2$ trees, according to the previous remark, we have $A = \bigoplus_{k=1}^{p} A_k$ and $B = \bigoplus_{k=1}^{p} B_k$ where each A_k (resp. B_k) corresponds to a tree of $A^\star$ (resp. $B^\star$). As in the previous case, up to a permutation θ, we can apply the induction hypothesis to the A_ks and B_ks and we conclude $A \simeq_{\mathcal{E}} B$. $\square$

Remark: It would also be possible to use an adaptation of the additive translation in the previous lemma but the quasi-multiplicative one makes things simpler.

The main result of the paper follows, it gives two characterizations of polarized isomorphisms: a "geometrical" one (two types are isomorphic if the associated forests are isomorphic) and an equational one.

Theorem 2 (Polarized isomorphisms)
Let A and B be two polarized formulas,

$$A \simeq_{\mathsf{LLP}} B \iff A^\star \simeq_f B^\star \iff A \simeq_{\mathcal{E}} B$$

PROOF: We prove the following implications:

- $A \simeq_{\mathcal{E}} B \Rightarrow A \simeq_{\mathsf{LLP}} B$, by lemma 2.
- $A \simeq_{\mathsf{LLP}} B \Rightarrow A^\star \simeq_f B^\star$. First note that, for any substitution θ, we have $A\theta \simeq_{\mathsf{LLP}} B\theta$. In particular, assuming that the set of variables is indexed by natural numbers: X_1, X_2, ... we can define the substitution θ_h given by $X_i \mapsto !?!\cdots?1$ or $!?!\cdots!\bot$ with exactly $i(h+1)-1$ exponential connectives, where h is the height of $A^\star$ (which is also the height of $B^\star$). We have $A\theta_h \simeq_{\mathsf{LLP}} B\theta_h$ and $A\theta_h$ and $B\theta_h$ are closed formulas, so that by proposition 4 we have $(A\theta_h)^\star \simeq_f (B\theta_h)^\star$ and by lemma 10 $A^\star\theta_h^\star \simeq_f B^\star\theta_h^\star$. We still have to deduce $A^\star \simeq_f B^\star$.

  The forest $A^\star\theta_h^\star$ is $A^\star[{}^{c_i}/_{X_i}]$ where c_i is the branch of length $i(h+1)$. We prove the result by induction on the number of elements of all the multi-sets associated with the nodes of $A^\star$. If all these multi-sets are empty, we have $A^\star\theta_h^\star = A^\star$ and we can conclude. If there are $k+1$ occurrences of variables, let n be a node of $A^\star\theta_h^\star$ of maximal height, its height is of the shape $k + i_0(h+1)$ with $0 \leq k \leq h$. Let m be the node which is $i_0(h+1)$ levels under n, this node has an occurrence X of X_{i_0} associated with it in $A^\star$. It is also the case for $f(m)$ in $B^\star$, otherwise $f(m)$ cannot have a branch of length $i_0(h+1)$ above it in $B^\star\theta_h^\star$. By induction hypothesis applied to $A^\star[{}^{\bullet}/_X]$ and $B^\star[{}^{\bullet}/_X]$, we have $A^\star[{}^{\bullet}/_X] \simeq_f B^\star[{}^{\bullet}/_X]$ and we can conclude $A^\star \simeq_f B^\star$ (where $\bullet$ is the one node forest with an empty associated multi-set, that is $1^\star$).

- $A^\star \simeq_f B^\star \Rightarrow A \simeq_{\mathcal{E}} B$, we have $A \simeq_{\mathcal{E}} A^m$ and $B \simeq_{\mathcal{E}} B^m$ by lemma 9, and this entails with the previous implications: $A^{m\star} \simeq_f A^\star \simeq_f B^\star \simeq_f B^{m\star}$. By lemma 11 we can deduce $A^m \simeq_{\mathcal{E}} B^m$ and finally $A \simeq_{\mathcal{E}} B$. □

Corollary 2.1
Let A and B be two polarized formulas, $A \simeq_{\mathsf{LLP}} B \iff A \simeq^i_{\mathsf{LLP}} B \iff A \simeq_{\mathcal{E}} B$.

PROOF: We have $A \simeq_{\mathsf{LLP}} B \Rightarrow A \simeq^i_{\mathsf{LLP}} B \Rightarrow A^\star \simeq_f B^\star \Rightarrow A \simeq_{\mathcal{E}} B \Rightarrow A \simeq_{\mathsf{LLP}} B$. □

6 Deterministic classical isomorphisms

As shown by many works [10, 11, 18, 20] deterministic classical systems can be classified into two categories: call-by-name and call-by-value systems. In each category the different systems are essentially equivalent. To represent these two evaluation paradigms, we will use call-by-name and call-by-value $\lambda\mu$-calculi.

We consider Selinger's extension of the $\lambda\mu$-calculus with disjunction [26]. The $\lambda\mu$-terms are built from two disjoint sets of variables, the λ-variables x, y, ... and the μ-variables α, β, ...:

$$\begin{array}{rcccccccccccccc} t & ::= & x & \mid & \lambda x.t & \mid & (t)t & \mid & <t,t> & \mid & \pi_1 t & \mid & \pi_2 t & \mid & \star \\ & \mid & \mu\alpha.t & \mid & [\alpha]t & \mid & \mu(\alpha,\beta).t & \mid & [\alpha,\beta]t \end{array}$$

The associated simple types are:

$$A \ ::= \ X \mid A \rightarrow A \mid A \wedge A \mid \top \mid A \vee A \mid \bot$$

and a typing judgment has the shape $\Gamma \vdash t : A \mid \Delta$ where Γ contains typing declarations for the λ-variables and Δ contains typing declarations for the μ-variables. The typing rules are:

$$\frac{}{x : A \vdash x : A \mid} \qquad \frac{}{\Gamma \vdash \star : \top \mid \Delta}$$

$$\frac{\Gamma \vdash t : B \mid \Delta}{\Gamma \setminus \{x : A\} \vdash \lambda x.t : A \rightarrow B \mid \Delta} \qquad \frac{\Gamma \vdash t : A \rightarrow B \mid \Delta \qquad \Gamma' \vdash u : A \mid \Delta'}{\Gamma \cup \Gamma' \vdash (t)u : B \mid \Delta \cup \Delta'}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \vdash u : B \mid \Delta}{\Gamma \vdash <t,u> : A \wedge B \mid \Delta} \qquad \frac{\Gamma \vdash t : A \wedge B \mid \Delta}{\Gamma \vdash \pi_1 t : A \mid \Delta} \qquad \frac{\Gamma \vdash t : A \wedge B \mid \Delta}{\Gamma \vdash \pi_2 t : B \mid \Delta}$$

$$\frac{\Gamma \vdash t : \bot \mid \Delta}{\Gamma \vdash \mu\alpha.t : A \mid \Delta \setminus \{\alpha : A\}} \qquad \frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash [\alpha]t : \bot \mid \Delta \cup \{\alpha : A\}}$$

$$\frac{\Gamma \vdash t : \bot \mid \Delta}{\Gamma \vdash \mu(\alpha,\beta).t : A \vee B \mid \Delta \setminus \{\alpha : A, \beta : B\}} \qquad \frac{\Gamma \vdash t : A \vee B \mid \Delta}{\Gamma \vdash [\alpha,\beta]t : \bot \mid \Delta \cup \{\alpha : A, \beta : B\}}$$

If we consider the call-by-name $=^{\mathrm{cbn}}_{\beta\eta\mu\rho\theta}$ and the call-by-value $=^{\mathrm{cbv}}_{\beta\eta\mu\rho\theta}$ equational theories (defined in the next sections), we obtain the following notions of isomorphisms:

Definition 30 (Isomorphims of types)
Two types A and B are *isomorphic* in the call-by-name (resp. call-by-value) $\lambda\mu$-calculus, denoted by $A \simeq^{\mathrm{cbn}}_{\lambda\mu} B$ (resp. $A \simeq^{\mathrm{cbv}}_{\lambda\mu} B$), if there exist two $\lambda\mu$-terms t such that $x : B \vdash t : A \mid$ and u such that $y : A \vdash u : B \mid$ with $(\lambda x.t)u =^{\mathrm{cbn}}_{\beta\eta\mu\rho\theta} y$ (resp. $(\lambda x.t)u =^{\mathrm{cbv}}_{\beta\eta\mu\rho\theta} y$) and $(\lambda y.u)t =^{\mathrm{cbn}}_{\beta\eta\mu\rho\theta} x$ (resp. $(\lambda y.u)t =^{\mathrm{cbv}}_{\beta\eta\mu\rho\theta} x$).

$$
\begin{array}{rlll}
(\lambda x.t)u & =_\beta & t[{}^{u}/_{x}] & : A \\
\lambda x.(t)x & =_\eta & t & : A \to B \quad x \notin t \\
\pi_1{<}t,u{>} & =_\beta & t & : A \\
\pi_2{<}t,u{>} & =_\beta & u & : A \\
{<}\pi_1 t, \pi_2 t{>} & =_\eta & t & : A \wedge B \\
\star & =_\eta & t & : \top \\
(\mu\alpha.t)u & =_\mu & \mu\alpha.t[{}^{[\alpha](v)u}/_{[\alpha]v}] & : A \\
\pi_1\mu\alpha.t & =_\mu & \mu\alpha.t[{}^{[\alpha]\pi_1 v}/_{[\alpha]v}] & : A \\
\pi_2\mu\alpha.t & =_\mu & \mu\alpha.t[{}^{[\alpha]\pi_2 v}/_{[\alpha]v}] & : A \\
[\beta]\mu\alpha.t & =_\rho & t[{}^{\beta}/_{\alpha}] & : \bot \\
\mu\alpha[\alpha]t & =_\theta & t & : A \quad \alpha \notin t \\
[\alpha,\beta]\mu\gamma.t & =_\rho & t[{}^{[\alpha,\beta]v}/_{[\gamma]v}] & : \bot \\
[\alpha',\beta']\mu(\alpha,\beta).t & =_\rho & t[{}^{\alpha'}/_{\alpha}, {}^{\beta'}/_{\beta}] & : \bot \\
\mu(\alpha,\beta)[\alpha,\beta]t & =_\theta & t & : A \vee B \quad \alpha,\beta \notin t \\
[\alpha]t & =_\rho & t & : \bot
\end{array}
$$

where $t[{}^{C\{v\}}/_{[\alpha]v}]$ is obtained by substituting any subterm of t of the shape $[\alpha]v$ by $C\{v\}$ and any subterm of t of the shape $[\alpha,\beta]v$ by $C\{\mu\alpha[\alpha,\beta]v\}$.

Figure 2: Call-by-name typed equational theory of the $\lambda\mu$-calculus

6.1 Call-by-name isomorphisms

The call-by-name typed equational theory $=^{\text{cbn}}_{\beta\eta\mu\rho\theta}$ of the $\lambda\mu$-calculus [26] is given in figure 2.

In order to apply our result about polarized isomorphisms of types to the $\lambda\mu$-calculus, we use a translation into LLP.

The translation of the call-by-name $\lambda\mu$-calculus into LLP is obtained by translating types as negative formulas:

$$
\begin{aligned}
X^- &= X^\perp \\
(A \to B)^- &= ?A^{-\perp} \parr B^- \\
(A \wedge B)^- &= A^- \mathbin{\&} B^- \\
\top^- &= \top \\
(A \vee B)^- &= A^- \parr B^- \\
\bot^- &= \bot
\end{aligned}
$$

the judgment $\Gamma \vdash t : A \mid \Delta$ is translated as $\vdash ?(\Gamma^-)^\perp, A^-, \Delta^-$. The translation of terms is then easy to derive and is given in [18].

Proposition 5 (Simulation)
If t and u are two $\lambda\mu$-terms such that $t =^{\text{cbn}}_{\beta\eta\mu\rho\theta} u$ then $t^- =_{\beta\eta} u^-$.

Definition 31 ($\mathcal{E}_n$-isomorphism)
$\mathcal{E}_n$ is the equational theory generated by the equations of figure 3, and $\mathcal{E}_n(\to, \wedge, \top)$ is the one given by figure 4.

$$\begin{aligned}
A \wedge B &= B \wedge A \\
A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\
A \wedge \top &= A
\end{aligned}$$

$$\begin{aligned}
A \vee B &= B \vee A \\
A \vee (B \vee C) &= (A \vee B) \vee C \\
A \vee \bot &= A
\end{aligned}$$

$$\begin{aligned}
A \vee (B \wedge C) &= (A \vee B) \wedge (A \vee C) \\
A \vee \top &= \top
\end{aligned}$$

$$\begin{aligned}
(A \wedge B) \to C &= A \to (B \to C) \\
\top \to A &= A
\end{aligned}$$

$$A \to (B \vee C) = (A \to B) \vee C$$

Figure 3: Equational theory $\mathcal{E}_n$

$$\begin{aligned}
A \wedge B &= B \wedge A \\
A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\
A \wedge \top &= A
\end{aligned}$$

$$\begin{aligned}
(A \wedge B) \to C &= A \to (B \to C) \\
\top \to A &= A
\end{aligned}$$

$$\begin{aligned}
A \to (B \wedge C) &= (A \to B) \wedge (A \to C) \\
A \to \top &= \top
\end{aligned}$$

Figure 4: Equational theory $\mathcal{E}_n(\to, \wedge, \top)$

Two types A and B are $\mathcal{E}_n$-*isomorphic*, denoted by $A \simeq_{\mathcal{E}_n} B$, if they are equal in the theory $\mathcal{E}_n$. Two types A and B are $\mathcal{E}_n(\to, \wedge, \top)$-*isomorphic*, denoted by $A \simeq_{\mathcal{E}_n(\to,\wedge,\top)} B$, if they are equal in the theory $\mathcal{E}_n(\to, \wedge, \top)$.

The theory $\mathcal{E}_n(\to, \wedge, \top)$ corresponds to the correct restriction of $\mathcal{E}_n$ to the connectives ($\to$, $\wedge$, $\top$) and it will be used in corollaries 6.1 and 6.2.

Lemma 12
The theory $\mathcal{E}_n(\to, \wedge, \top)$ is included in $\mathcal{E}_n$.

Proof: We have two equations to prove:

$$\begin{aligned} A \to (B \wedge C) &= A \to (\bot \vee (B \wedge C)) \\ &= (A \to \bot) \vee (B \wedge C) \\ &= ((A \to \bot) \vee B) \wedge ((A \to \bot) \vee C) \\ &= (A \to (\bot \vee B)) \wedge ((A \to \bot) \vee C) \\ &= (A \to B) \wedge ((A \to \bot) \vee C) \\ &= (A \to B) \wedge (A \to (\bot \vee C)) \\ &= (A \to B) \wedge (A \to C) \end{aligned}$$

and

$$\begin{aligned} A \to \top &= A \to (\bot \vee \top) \\ &= (A \to \bot) \vee \top \\ &= \top \end{aligned}$$

□

Proposition 6
If A and B are two types, $A \simeq^{\mathrm{cbn}}_{\lambda\mu} B \iff A^- \simeq_{\mathsf{LLP}} B^- \iff A \simeq_{\mathcal{E}_n} B$.

Proof: The first implication $A \simeq^{\mathrm{cbn}}_{\lambda\mu} B \Rightarrow A^- \simeq_{\mathsf{LLP}} B^-$ is a consequence of proposition 5: if $x : B \vdash t : A \mid$ and $y : A \vdash u : B \mid$ with compositions equal to identity, we obtain two proofs t^- of $\vdash ?A^{-\bot}, B^-$ and u^- of $\vdash ?B^{-\bot}, A^-$ which are isomorphisms in LLP (see definition 2 and corollary 2.1).

The second implication $A^- \simeq_{\mathsf{LLP}} B^- \Rightarrow A \simeq_{\mathcal{E}_n} B$ is proved with theorem 2 which implies $A^{-\star} \simeq_f B^{-\star}$. In the spirit of lemma 9, we can show that any type is equal in $\mathcal{E}_n$ to a type with the connectives $\wedge$ and $\top$ appearing only in head position and with only arrow types of the shape $A \to \bot$. We then show, as for lemma 11, that two such types corresponding to isomorphic forests are equal in $\mathcal{E}_n$.

The third implication $A \simeq_{\mathcal{E}_n} B \Rightarrow A \simeq^{\mathrm{cbn}}_{\lambda\mu} B$ is just a syntactical verification. □

Corollary 6.1
If A and B are two types using only the connectives $\to$, $\wedge$ and $\top$, $A \simeq^{\mathrm{cbn}}_{\lambda\mu} B \iff A \simeq_{\mathcal{E}_n(\to,\wedge,\top)} B$.

PROOF: By proposition 6, we have $A \simeq_{\mathcal{E}_n} B$. Up to the equations of $\mathcal{E}_n(\to, \wedge, \top)$, any formula with only $\to$, $\wedge$ and $\top$ is equal to a formula which is either $\top$ or a conjunction of formulas with only $\to$ (this corresponds to lemma 9). We can then verify that two such formulas A and B equal in $\mathcal{E}_n$ (thus such that $A^{-\star} \simeq_f B^{-\star}$) are equal in $\mathcal{E}_n(\to, \wedge, \top)$ (following the idea of lemma 11).

The second direction is immediate with proposition 6 and lemma 12. □

If isomorphisms in the λ-calculus are considered up to usual $\beta\eta$-equivalence, we have:

Corollary 6.2
If A and B are two types using only the connectives $\to$, $\wedge$ and $\top$ then $A \simeq_\lambda B \iff A \simeq_{\mathcal{E}_n(\to,\wedge,\top)} B$.

PROOF: Since the λ-calculus is a subsystem of the $\lambda\mu$-calculus, any type isomorphism of the λ-calculus is an isomorphism of the $\lambda\mu$-calculus. We conclude with corollary 6.1. In the other direction we just verify that the theory $\mathcal{E}_n(\to, \wedge, \top)$ is validated by the λ-calculus. □

This gives a new proof of the equational theory of the isomorphisms of types for the λ-calculus [27, 14].

6.2 Call-by-value isomorphisms

Values are particular $\lambda\mu$-terms defined by the following grammar:

$$\begin{array}{lllllllll} V & ::= & x & \mid & \lambda x.t & \mid\ <V,V> & \mid\ \pi_1 V & \mid\ \pi_2 V & \mid\ \star \\ & \mid & \mu(\alpha,\beta)[\alpha]V & \mid & \mu(\alpha,\beta)[\beta]V & & & & \end{array}$$

with $\alpha, \beta \notin V$.

The call-by-value typed equational theory $=^{\mathrm{cbv}}_{\beta\eta\mu\rho\theta}$ of the $\lambda\mu$-calculus [26] is given in figure 5.

The translation of the call-by-value $\lambda\mu$-calculus into LLP is obtained by translating types as positive formulas:

$$\begin{aligned} X^+ &= X \\ (A \to B)^+ &= !(A^+ \multimap ?B^+) = !(A^{+\perp} \invamp ?B^+) \\ (A \wedge B)^+ &= A^+ \otimes B^+ \\ \top^+ &= 1 \\ (A \vee B)^+ &= A^+ \oplus B^+ \\ \perp^+ &= 0 \end{aligned}$$

the judgment $\Gamma \vdash t : A \mid \Delta$ is translated as $\vdash (\Gamma^+)^\perp, ?A^+, ?\Delta^+$. The translation of terms is then easy to derive and is given in [18].

Proposition 7 (Simulation)
If t and u are two $\lambda\mu$-terms such that $t =^{\mathrm{cbv}}_{\beta\eta\mu\rho\theta} u$ then $t^+ =_{\beta\eta} u^+$.

Definition 32 ($\mathcal{E}_v$-isomorphism)
$\mathcal{E}_v$ is the equational theory generated by the equations of figure 6, and $\mathcal{E}_v(\to, \wedge, \top)$ is the one given by figure 7.

Two types A and B are $\mathcal{E}_v$*-isomorphic*, denoted by $A \simeq_{\mathcal{E}_v} B$, if they are equal in the theory $\mathcal{E}_v$. Two types A and B are $\mathcal{E}_v(\to, \wedge, \top)$*-isomorphic*, denoted by $A \simeq_{\mathcal{E}_v(\to,\wedge,\top)} B$, if they are equal in the theory $\mathcal{E}_v(\to, \wedge, \top)$.

$$
\begin{array}{rcll}
(\lambda x.t)V & =_\beta & t[^V/_x] & : A \\
\lambda x.(V)x & =_\eta & V & : A \to B \quad x \notin V \\
\pi_1 {<}V, W{>} & =_\beta & V & : A \\
\pi_2 {<}V, W{>} & =_\beta & W & : A \\
{<}\pi_1 V, \pi_2 V{>} & =_\eta & V & : A \wedge B \\
\star & =_\eta & V & : \top \\
(\lambda x.x)t & =_\beta & t & : A \\
(\lambda y.u)(\lambda x.t)v & =_\beta & (\lambda x.(\lambda y.u)t)v & : A \quad x \notin u \\
(\lambda x.(\lambda y.(x)y)t)u & =_\beta & (u)t & : A \quad x \notin t \\
(\lambda x.(\lambda y.{<}x, y{>})t)u & =_\beta & {<}u, t{>} & : A \wedge B \quad x \notin t \\
(\lambda x.\pi_1 x)t & =_\beta & \pi_1 t & : A \\
(\lambda x.\pi_2 x)t & =_\beta & \pi_2 t & : A \\
(\lambda x.t)\mu\alpha.u & =_\mu & \mu\alpha.u[^{(\lambda x.[\alpha]t)v}/_{[\alpha]v}] & : A \quad \alpha \notin t \\
[\beta]\mu\alpha.t & =_\rho & t[^\beta/_\alpha] & : \bot \\
\mu\alpha[\alpha]t & =_\theta & t & : A \quad \alpha \notin t \\
[\alpha', \beta']\mu(\alpha, \beta).t & =_\rho & t[^{\alpha'}/_\alpha, {}^{\beta'}/_\beta] & : \bot \\
\mu(\alpha, \beta)[\alpha, \beta]t & =_\theta & t & : A \vee B \quad \alpha, \beta \notin t \\
[\alpha]t & =_\rho & t & : \bot \\
(\lambda x.[\alpha]x)t & =_\beta & [\alpha]t & : \bot \\
(\lambda x.[\alpha, \beta]x)t & =_\beta & [\alpha, \beta]t & : \bot
\end{array}
$$

where V and W are values.

Figure 5: Call-by-value typed equational theory of the $\lambda\mu$-calculus

$$
\begin{aligned}
A \wedge B &= B \wedge A \\
A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\
A \wedge \top &= A \\[1ex]
A \vee B &= B \vee A \\
A \vee (B \vee C) &= (A \vee B) \vee C \\
A \vee \bot &= A \\[1ex]
A \wedge (B \vee C) &= (A \wedge B) \vee (A \wedge C) \\
A \wedge \bot &= \bot \\[1ex]
(A \vee B) \to C &= (A \to C) \wedge (B \to C) \\
\bot \to A &= \top
\end{aligned}
$$

Figure 6: Equational theory $\mathcal{E}_v$

$$
\begin{aligned}
A \wedge B &= B \wedge A \\
A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\
A \wedge \top &= A
\end{aligned}
$$

Figure 7: Equational theory $\mathcal{E}_v(\to, \wedge, \top)$

The theory $\mathcal{E}_v(\to, \wedge, \top)$ corresponds to the correct restriction of $\mathcal{E}_v$ to the connectives ($\to$, $\wedge$, $\top$) and it will be used in corollaries 8.1 and 8.2.

Remark: As given by figure 7, $A \to (B \to C) \not\simeq B \to (A \to C)$ in call-by-value because $\lambda x.\lambda y.(f)xy \neq f$.

Proposition 8
If A and B are two types, $A \simeq^{\mathrm{cbv}}_{\lambda\mu} B \iff A^+ \simeq_{\mathsf{LLP}} B^+ \iff A \simeq_{\mathcal{E}_v} B$.

Proof: The first implication $A \simeq^{\mathrm{cbv}}_{\lambda\mu} B \Rightarrow A^+ \simeq_{\mathsf{LLP}} B^+$ is a consequence of proposition 7: if $x : B \vdash t : A \mid$ and $y : A \vdash u : B \mid$ with compositions equal to identity, we obtain two proofs t^+ of $\vdash {A^+}^\perp, ?B^+$ and u^+ of $\vdash {B^+}^\perp, ?A^+$ which are isomorphisms in LLP (see definition 2 and corollary 2.1).

The second implication $A^+ \simeq_{\mathsf{LLP}} B^+ \Rightarrow A \simeq_{\mathcal{E}_v} B$ is proved with theorem 2 which implies ${A^+}^\star \simeq_f {B^+}^\star$. In the spirit of lemmas 9 and 11, we can show that any type is equal in $\mathcal{E}_v$ to a type with the connectives $\vee$ and $\perp$ appearing only in head position, and two such types corresponding to isomorphic forests are equal in $\mathcal{E}_v$.

The third implication $A \simeq_{\mathcal{E}_v} B \Rightarrow A \simeq^{\mathrm{cbv}}_{\lambda\mu} B$ is just a syntactical verification. □

Corollary 8.1
If A and B are two types using only the connectives $\to$, $\wedge$ and $\top$, $A \simeq^{\mathrm{cbv}}_{\lambda\mu} B \iff A \simeq_{\mathcal{E}_v(\to,\wedge,\top)} B$.

Proof: By theorem 2, we have ${A^+}^\star \simeq_f {B^+}^\star$ and we can show that two types based only on $\to$, $\wedge$ and $\top$ with the same associated forest are equal in $\mathcal{E}_v(\to, \wedge, \top)$ as for lemma 11.

The other direction is immediate with proposition 8. □

The equational theory for the call-by-value λ-calculus [24] (or λ_v-calculus) is given in figure 8.
We obtain a characterization of the isomorphisms of types for the λ_v-calculus. The associated theory is very weak but has not been identified before.

Corollary 8.2
If A and B are two types using only the connectives $\to$, $\wedge$ and $\top$, $A \simeq_{\lambda_v} B \iff A \simeq_{\mathcal{E}_v(\to,\wedge,\top)} B$.

Proof: The call-by-value λ-calculus is a subsystem of the call-by-value $\lambda\mu$-calculus and that gives the first implication by corollary 8.1. In the other direction we just verify that the theory $\mathcal{E}_v(\to, \wedge, \top)$ is validated by the λ_v-calculus. □

$$
\begin{array}{rcll}
(\lambda x.t)V & =_\beta & t[^V/_x] & : A \\
\lambda x.(V)x & =_\eta & V & : A \to B \qquad\qquad x \notin V \\
\pi_1{<}V, W{>} & =_\beta & V & : A \\
\pi_2{<}V, W{>} & =_\beta & W & : A \\
{<}\pi_1 V, \pi_2 V{>} & =_\eta & V & : A \wedge B \\
\star & =_\eta & V & : \top
\end{array}
$$

Figure 8: Call-by-value typed equational theory of the λ-calculus

7 Classical and intuitionistic disjunctions

In the previous sections, we have been able to give a finite axiomatization of the theory of isomorphisms of types for the call-by-name $\lambda\mu$-calculus with disjunction ($\vee$) and contradiction ($\bot$) types. However, Balat, Di Cosmo and Fiore [15] have shown that, in the λ-calculus case, this theory is *not* finitely axiomatizable. We are going to discuss this seeming contradiction.

The equational theory of the call-by-name $\lambda\mu$-calculus for the language corresponding to types $\to$, $\wedge$ and $\top$ is a conservative extension of the equational theory of the λ-calculus for the same types. When we go to the disjunction case, this is not true anymore. The equational theory of the λ-calculus with disjunction corresponding to the axioms of bicartesian closed categories contains:

$$
\begin{array}{rcl}
\texttt{case}\ t\ \texttt{with}\ x \mapsto u[^{\texttt{inl}\ x}/_z] & & \\
y \mapsto u[^{\texttt{inr}\ y}/_z] & = & u[^t/_z]
\end{array}
\tag{1}
$$

which is used to prove the isomorphism $(A \vee B) \to C \simeq (A \to C) \wedge (B \to C)$ (wrong in the call-by-name $\lambda\mu$-calculus). If we consider the two main particular cases of (1): $u = z$ and $t = z$, we obtain for $u = z$:

$$
\begin{array}{rcl}
\texttt{case}\ t\ \texttt{with}\ x \mapsto \texttt{inl}\ x & & \\
y \mapsto \texttt{inr}\ y & = & t
\end{array}
\tag{2}
$$

This equation is validated in the $\lambda\mu$-calculus and corresponds to:

$$
\mu\gamma[\gamma](\lambda y.\mu(\alpha,\beta)[\beta]y)\mu\beta[\gamma](\lambda x.\mu(\alpha,\beta)[\alpha]x)\mu\alpha[\alpha,\beta]t = t
$$

with $\alpha, \beta \notin t$. But with $t = z$ in (1):

$$
\begin{array}{rcl}
\texttt{case}\ z\ \texttt{with}\ x \mapsto u[^{\texttt{inl}\ x}/_z] & & \\
y \mapsto u[^{\texttt{inr}\ y}/_z] & = & u
\end{array}
\tag{3}
$$

and this equation is not realized in the $\lambda\mu$-calculus.

From a categorical point of view, the notion of *control category* [26] which gives the models of the call-by-name $\lambda\mu$-calculus is based on a disjunction which is a binoidal functor and not a bifunctor. Selinger proved that a control category with a bifunctorial disjunction is a Boolean algebra and we are back to section 1. This shows that, in some sense, the theories of the λ-calculus with disjunction and of the $\lambda\mu$-calculus with disjunction are incompatible.

This explains how our result is not comparable with Balat-Di Cosmo-Fiore's one, in particular the disjunction distributes over the conjunction here and this is the converse in their setting. The only result we can deduce about the λ-calculus from ours are the equations for the isomorphisms of types of the λ-calculus with a constrained disjunction which verifies equation (2) but not equation (1).

8 Tarski's problem

The question of isomorphisms of types consists in finding some equational characterization of the isomorphisms of a given logical system, whereas Tarski's problem consists in proving that some equational theory characterizes the equality in some number structure (natural numbers with selected operations: $+$, $\times$, ..., real numbers, ...).

Composing these two questions, we can try to find number structures and logical systems such that equality in the first one corresponds to isomorphisms of types in the second one. The key result of this kind is given by theorem 3 and an interesting use of such a correspondence is given in [15].

We are going to describe the relations between polarized isomorphisms and the associated number models. As a consequence, these results give also relations with call-by-name and call-by-value classical isomorphisms.

Definition 33 (Tarski model of isomorphisms)
Let $\mathcal{S}$ be a logical system, a *Tarski model of the isomorphisms of* $\mathcal{S}$ is a model of the equational theory corresponding to the isomorphisms of types of $\mathcal{S}$ which does not validate any other universally closed equation.

Theorem 3 (Soloviev [27])
$(\mathbb{N}, (-)^{-}, \cdot, 1)$ *is a Tarski model of the isomorphisms of the simply typed* λ*-calculus with the connectives (*$\rightarrow$, $\wedge$, $\top$*).*

Corollary 3.1
$(\mathbb{N}, (-)^{-}, \cdot, 1)$ *is a Tarski model of the isomorphisms of the simply typed call-by-name* $\lambda\mu$*-calculus with the connectives (*$\rightarrow$, $\wedge$, $\top$*).*

Proof: We have shown in corollary 6.2 that the equational theory for the isomorphisms of types of the call-by-name $\lambda\mu$-calculus and of the λ-calculus are the same for the language ($\rightarrow$, $\wedge$, $\top$). $\square$

We are going to use Macintyre's result on Schanuel's conjecture and exponential rings [21] to give a sufficient condition to be a Tarski model of the positive isomorphisms of LLP. By duality, this gives also models of the negative isomorphisms and, as a consequence, models of call-by-name and call-by-value classical isomorphisms.

We recall some definitions about exponential rings [28] with the slightly more general case of exponential semi-rings. For the basic notions of algebra, see [17].

Definition 34 (Semi-ring)
A *semi-ring* is a tuple $(R, +, 0, \cdot, 1)$ such that $(R, +, 0)$ is a commutative monoid and $(R, \cdot, 1)$ is a monoid (if it is commutative, the semi-ring is commutative), and moreover:

$$\begin{aligned} x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ (y + z) \cdot x &= (y \cdot x) + (z \cdot x) \\ x \cdot 0 &= 0 \\ 0 \cdot x &= 0 \end{aligned}$$

$$
\begin{array}{ccc}
A \otimes B & = & B \otimes A \\
(A \otimes B) \otimes C & = & A \otimes (B \otimes C) \\
A \otimes 1 & = & A \\
A \oplus B & = & B \oplus A \\
(A \oplus B) \oplus C & = & A \oplus (B \oplus C) \\
A \oplus 0 & = & A \\
A \otimes (B \oplus C) & = & (A \otimes B) \oplus (A \otimes C) \\
A \otimes 0 & = & 0 \\
\neg(A \oplus B) & = & \neg A \otimes \neg B \\
\neg 0 & = & 1
\end{array}
$$

where $\neg P$ stands for $!P^{\perp}$.

Figure 9: Positive equational theory for LLP

Definition 35 (Exponential semi-ring)
An *exponential semi-ring* $(R, +, 0, \cdot, 1, E)$ is a commutative semi-ring $(R, +, 0, \cdot, 1)$ with a map E from R to R such that:

$$
\begin{aligned}
E(x + y) &= E(x) \cdot E(y) \\
E(0) &= 1
\end{aligned}
$$

so that E is a monoid morphism from $(R, +, 0)$ to $(R, \cdot, 1)$.

The theory of exponential semi-rings is equational and coincides with the theory of polarized isomorphisms written in a purely positive language, as in figure 9. We just interpret $\oplus$ by $+$, 0 by 0, $\otimes$ by $\cdot$, 1 by 1 and $\neg$ by E.

Definition 36 (Exponential ring)
An exponential semi-ring R is an *exponential ring* if the underlying semi-ring is a ring.

Definition 37 (Schanuel's condition)
Let R be an exponential ring which is an integral domain and has characteristic 0, R satisfies *Schanuel's condition* if for any α_1, ..., α_n in R linearly independent over $\mathbb{Q}$, the ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_n, E(\alpha_1), \ldots, E(\alpha$
has transcendence degree at least n over $\mathbb{Z}$.

Theorem 4 (Macintyre [21])
Let R be an exponential ring which satisfies Schanuel's condition, if $e_1 = e_2$ is a closed equation in the language $(+, 0, \cdot, 1, E)$ true in R, it is also true in the free exponential ring on 0 generators.

Corollary 4.1
If R is an exponential ring which satisfies Schanuel's condition, R is a Tarski model of the positive isomorphisms of LLP.

The fact that the exponential ring $(\mathbb{R}, +, 0, \cdot, 1, e)$ satisfies Schanuel's condition is called *Schanuel's conjecture.*

Corollary 4.2
If Schanuel's conjecture is true, $(\mathbb{R}, +, 0, \cdot, 1, e)$ is a Tarski model of the positive isomorphisms of LLP.

The terminology for the linear connectives given by Girard is the following: $\otimes$, $\parr$, 1 and $\perp$ are the *multiplicative* connectives, $\oplus$, &, 0 and $\top$ are the *additive* connectives and ! and ? are the *exponential* connectives. This originally came from linear algebra and from the key isomorphism of LL: $!(A \mathbin{\&} B) = !A \otimes !B$. We somehow give here a concrete interpretation of these ideas by really interpreting the linear connectives by the corresponding operations on real numbers: additives by $+$, multiplicatives by $\cdot$ and exponentials by e.

Macintyre's result gives a sufficient condition for having a Tarski model. Concerning a necessary condition we can just make a few remarks. A Tarski model of the positive isomorphisms of LLP must be an exponential semi-ring S and its characteristic must be 0. Otherwise there exists some p such that $\underbrace{1 + \cdots + 1}_{p} = 0$. This gives an equation which is not valid in LLP. We can always move to the case of rings by introducing the free exponential ring R generated by S. If this ring does not satisfy Schanuel's condition, it must at least verify that $E(1)$, $E(E(1))$, $E(E(E(1)))$, ... are transcendental, otherwise from a polynomial P such that $P(E^n(1)) = 0$ we can derive an equation on the language $(+, 0, \cdot, 1, E)$ by decomposing $P = P_1 - P_2$ with P_1 and P_2 in this language: $P_1(E^n(1)) = P_2(E^n(1))$ which is not valid in LLP. In particular if $(\mathbb{R}, +, 0, \cdot, 1, e)$ is a Tarski model of LLP, e^e must be transcendental and this is an open problem in number theory.

Remark: If we look at the translation of the call-by-value implication $A \to B$ into LLP, we obtain $!(A \multimap ?B) = !(A^\perp \parr ?B)$. A naive matching between this expression and the definition of exponentiation: $B^A = e^{A \ln B}$ for the reals would lead to match ? with ln. However ? transforms $\oplus$ into $\parr$ while ln transforms $\cdot$ into $+$, we have a mismatch!

Assuming Schanuel's conjecture, $\mathbb{R}$ is a Tarski model of the isomorphisms of LLP, this entails that we can associate with any equation on real numbers in the language $(+, 0, \cdot, 1, e)$, an algorithmic interpretation by computing the proofs in LLP corresponding to this isomorphism (or the corresponding $\lambda\mu$-terms).

Further work

We have shown how a game semantics approach allows us to characterize classical isomorphisms of types (and as a consequence to give a new proof for the intuitionistic call-by-name ones and a new result for the intuitionistic call-by-value ones), in a propositional setting. Previous syntactical works [14] offered results on polymorphic systems as well. We wonder if it is possible to extend our game approach to the second order setting.

An advantage of game semantics is the possibility of modeling not only the λ-calculus and other logical systems but also programming primitives of various kinds: non-determinism [16], ground type references [3], general references [1], ... The key result of our approach is theorem 1 and if we look at the proof we can see that we only use determinism and visibility of strategies and from a computational viewpoint this corresponds to the game model of Idealized Algol (IA) [3]. As a consequence this theorem gives a characterization of the isomorphisms of types for IA by the theory $\mathcal{E}_n(\to, \wedge, \top)$ (figure 4). However we use the visibility condition so that our result cannot be directly applied to general references and we wonder if it is possible to remove this constraint in our proof and to extend our results in that direction.

A more general question than isomorphisms of types is the characterization of retractions in a given logical system. This is known to be a difficult problem for the λ-calculus [22]. We can try to apply game semantics ideas to address this problem.

Another related question is the study of isomorphisms of types in usual Linear Logic. Balat and Di Cosmo have given a syntactical characterization of the isomorphisms of types in MLL [6] (the multiplicative fragment of Linear Logic). The question of the extension of this result to richer fragments of LL remains open. Our complete characterization of polarized isomorphisms cannot be directly of help for the LL problem which is a strongly more general one. However we could hope to apply game semantics, as we have done here, in a non-polarized setting. For example MLL game semantics [2] might be used to give an alternative proof of Balat and Di Cosmo's result. The extension to richer fragments is once again problematic since we still do not know if game models are precise enough for these fragments. Two possible fragments to look at are the MALL (resp. MELL) case for which Abramsky and Melliès' model [4] (resp. Baillot-Danos-Ehrhard-Regnier's model [5]) might be used.

A Cut elimination in LLP (β-rules)

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash \Gamma, A}\,cut \quad\rightsquigarrow\quad \vdash \Gamma, A$$

$$\dfrac{\dfrac{\vdash \Gamma, M^{\perp} \qquad \vdash \Delta, N^{\perp}}{\vdash \Gamma, \Delta, M^{\perp} \otimes N^{\perp}}\,\otimes \qquad \dfrac{\vdash M, N, \Xi}{\vdash M \parr N, \Xi}\,\parr}{\vdash \Gamma, \Delta, \Xi}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, M^{\perp} \qquad \dfrac{\vdash \Delta, N^{\perp} \qquad \vdash M, N, \Xi}{\vdash M, \Delta, \Xi}\,cut}{\vdash \Gamma, \Delta, \Xi}\,cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N_i^{\perp}}{\vdash \Gamma, N_1^{\perp} \oplus N_2^{\perp}}\,\oplus_i \qquad \dfrac{\vdash N_1, \Delta \qquad \vdash N_2, \Delta}{\vdash N_1 \mathbin{\&} N_2, \Delta}\,\&}{\vdash \Gamma, \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, N_i^{\perp} \qquad \vdash N_i, \Delta}{\vdash \Gamma, \Delta}\,cut$$

$$\dfrac{\dfrac{}{\vdash 1}\,1 \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot}{\vdash \Gamma}\,cut \quad\rightsquigarrow\quad \vdash \Gamma$$

$$\dfrac{\dfrac{}{\vdash \top, \Gamma, A}\,\top \qquad \vdash A^{\perp}, \Delta}{\vdash \top, \Gamma, \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{}{\vdash \top, \Gamma, \Delta}\,\top$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \dfrac{\vdash N^{\perp}, \Gamma}{\vdash ?N^{\perp}, \Gamma}\,?d}{\vdash \mathcal{N}, \Gamma}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash \mathcal{N}, \Gamma}\,cut$$

$$\dfrac{\dfrac{\vdash M, \mathcal{N}, N}{\vdash !M, \mathcal{N}, N}\,! \qquad \vdash N^{\perp}, \Gamma}{\vdash !M, \mathcal{N}, \Gamma}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash M, \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash M, \mathcal{N}, \Gamma}\,cut}{\vdash !M, \mathcal{N}, \Gamma}\,!$$

$$\dfrac{\dfrac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\,?c \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, N, N \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta, N}\,cut \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta, \Delta}\,cut}{\vdash \Gamma, \Delta}\,?c$$

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, N}\,?w \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma}{\vdash \Gamma, \Delta}\,?w$$

The commutative steps are left to the reader (see [18] for a precise description in proof-nets).

B Expansion of axioms (η-rules)

$$\dfrac{}{\vdash 1, \bot}\,ax \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1, \bot}\,\bot$$

$$\frac{}{\vdash 0, \top}\,ax \qquad \rightsquigarrow \qquad \frac{}{\vdash 0, \top}\,\top$$

$$\frac{}{\vdash N^{\perp} \otimes M^{\perp}, N \parr M}\,ax \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\cfrac{}{\vdash N^{\perp}, N}\,ax \qquad \cfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash N^{\perp} \otimes M^{\perp}, N, M}\,\otimes}{\vdash N^{\perp} \otimes M^{\perp}, N \parr M}\,\parr$$

$$\frac{}{\vdash N^{\perp} \oplus M^{\perp}, N \,\&\, M}\,ax \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\cfrac{}{\vdash N^{\perp}, N}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, N}\,\oplus_1 \qquad \cfrac{\cfrac{}{\vdash M^{\perp}, M}\,ax}{\vdash N^{\perp} \oplus M^{\perp}, M}\,\oplus_2}{\vdash N^{\perp} \oplus M^{\perp}, N \,\&\, M}\,\&$$

$$\frac{}{\vdash !N, ?N^{\perp}}\,ax \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\cfrac{}{\vdash N, N^{\perp}}\,ax}{\vdash N, ?N^{\perp}}\,?d}{\vdash !N, ?N^{\perp}}\,!$$

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