Proofs and Categories

CR15 CTCS — Homework 6 — return date: January 16, 2024

Olivier . Laurent @ens-lyon.fr

Exercise 1: **The Logic of a Category**

In the whole exercise, we fix a category $\mathscr{C}$.

The sequent calculus $\mathscr{L}(\mathscr{C})$ is defined by:

- *formulas* are objects of $\mathscr{C}$
- *sequents* are of the shape $A \vdash B$ where A and B are formulas
- *rules* are:

$$\frac{}{A \vdash B}\ mph(f) \qquad \text{for each } f \in \mathscr{C}(A,B)$$

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ cut$$

It is important here to see the difference between $\mathscr{L}(\mathscr{C})$ and $\mathsf{L}(ob(\mathscr{C}))$. The second one, used in the course, does not depend on morphisms of $\mathscr{C}$. One can see $\mathscr{L}(\mathscr{C})$ as an extension of $\mathsf{L}(ob(\mathscr{C}))$ by:

$$\frac{}{A \vdash A}\ ax \qquad \mapsto \qquad \frac{}{A \vdash A}\ mph(id_A)$$

1. Since $\mathscr{L}(\mathscr{C})$ extends $\mathsf{L}(ob(\mathscr{C}))$, describe how to extend $[\![\_]\!]$ to $\mathscr{L}(\mathscr{C})$ (we use here $[\![A]\!] = A$ for formulas since they are already objects of $\mathscr{C}$).

2. As we want to try to eliminate cuts in $\mathscr{L}(\mathscr{C})$, give a cut-free right-hand side for reducing the following pattern containing a (cut) rule:
$$\frac{\dfrac{}{A \vdash B}\ mph(f) \qquad \dfrac{}{B \vdash C}\ mph(g)}{A \vdash C}\ cut \qquad \rightsquigarrow \qquad ?$$
in such a way that both sides have the same image under $[\![\_]\!]$.

3. We extend $\rightsquigarrow$ to contexts, by allowing it to be applied anywhere inside a proof. From now on, we use the notation $\rightsquigarrow$ for this extension.

   Prove that $\rightsquigarrow$ is terminating.

4. Prove cut-elimination for $\mathscr{L}(\mathscr{C})$, *i.e.* for any proof π of $A \vdash B$, there exists a cut-free proof π' of $A \vdash B$ such that $[\![\pi]\!] = [\![\pi']\!]$.

5. Given a proof π, provide a direct way of computing a normal form π_0 of π for $\rightsquigarrow$ (*i.e.* a reduct of π, after possibly many steps, which cannot be reduced anymore: $\pi \rightsquigarrow^* \pi_0 \not\rightsquigarrow$).

6. Conclude that the normal form is unique.

Exercise 2: **Preorder Semantics of Logic**

Given a category $\mathscr{C}$, we want to build a category $\mathsf{th}(\mathscr{C})$ such that:

- $ob(\mathsf{th}(\mathscr{C})) = ob(\mathscr{C})$
- Given two objects A and B of $\mathscr{C}$,

$$\mathsf{th}(\mathscr{C})(A,B) = \begin{cases} \emptyset & \text{if } \mathscr{C}(A,B) = \emptyset \\ \{\star\} & \text{otherwise} \end{cases}$$

  (note that $\mathsf{th}(\mathscr{C})(A,B)$ has at most one element).

1. Define identities and composition so that $\mathsf{th}(\mathscr{C})$ becomes a category.
2. We consider $W(A) = A$ on objects of $\mathscr{C}$, and $W(f) = \star$ on morphisms of $\mathscr{C}$.
   (a) Prove that W defines a functor from $\mathscr{C}$ to $\mathsf{th}(\mathscr{C})$.
   (b) Is it a full functor? if not, under which hypotheses on $\mathscr{C}$ would it be a full functor?
   (c) Is it a faithful functor? if not, under which hypotheses on $\mathscr{C}$ would it be a faithful functor?
3. Given a proof π of $A \vdash B$ in $\mathsf{L}(ob(\mathsf{th}(\mathscr{C})))$, compute $[\![\pi]\!]$ in $\mathsf{th}(\mathscr{C})$.
4. Let $(\mathscr{X}, \leq)$ be a preorder, we know that it can be seen as a category whose objects are elements of $\mathscr{X}$ and with at most one morphism between two objects.

   By taking inspiration from $W([\![\_]\!])$, explain how to interpret proofs of $\mathsf{L}(\mathscr{X})$ in a preorder $(\mathscr{X}, \leq)$ over $\mathscr{X}$.
5. Explain why this kind of semantic interpretation in a preorder is called "semantics of provability" while interpretations in more general categories are called "semantics of proofs".