The Relational Model

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Remember that the category **Rel** has sets as objects and binary relations as morphisms: $\mathbf{Rel}(A,B) = \mathscr{P}(A \times B)$.

1. Prove that **Rel** is isomorphic to the Kleisli category of **Set** for the monad $\mathscr{P}$ (remark page 3 of handout of lecture 10).

2. Prove that **Rel** has finite co-products (remember that it is enough for that to have binary co-products and an initial element).

3. Prove that **Rel** has finite products.

4. Given a category $\mathscr{C}$ and a monad M in $\mathscr{C}$, assuming that $f : A \to B$ is an isomorphism in $\mathscr{C}$, prove that $\eta_B \circ f$ is an isomorphism in the Kleisli category $\mathscr{C}_M$.

5. Monoidal structure on **Rel**.

   (a) Prove that $\times$ defines a bifunctor on **Rel**.

   (b) Prove that $(\mathbf{Rel}, \times, \{\star\})$ has a structure of monoidal category (MC).

   (c) Prove that $(\mathbf{Rel}, \times, \{\star\})$ has a structure of symmetric monoidal category (SMC).

   (d) Prove that $(\mathbf{Rel}, \times, \{\star\})$ has a structure of symmetric monoidal closed category (SMCC).

6. Explain why there is an isomorphism $(A+B) \times C \simeq (A \times C) + (B \times C)$ in **Rel** (see handout of lecture 10).

7. Prove that $d_A = \{(a,(a,a)) \mid a \in A\} : A \to A \times A$ does not define a natural transformation from the identity functor Id to the diagonal functor Δ.

8. Compute the interpretation of the following IMLL proof in **Rel**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{\dfrac{}{B \vdash B}\,ax \quad \dfrac{}{\vdash 1}\,1R}{B \vdash B \otimes 1}\,\otimes R}{A \multimap B, A \vdash B \otimes 1}\,\multimap L}{A, A \multimap B \vdash B \otimes 1}\,ex}{A \otimes (A \multimap B) \vdash B \otimes 1}\,\otimes L}{\vdash (A \otimes (A \multimap B)) \multimap (B \otimes 1)}\,\multimap R$$