Proof Systems and n-ary Functors

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Olivier . Laurent @ens-lyon.fr

Exercise 1: **Natural Deduction and Sequent Calculus**

We consider propositional intuitionistic logic with two connectives $\rightarrow$ and $\wedge$. Given a set of atoms denoted X, Y, etc, its formulas are:

$$A ::= X \mid A \rightarrow A \mid A \wedge A$$

Sequents have the shape $\Gamma \vdash A$ where Γ is a list of formulas. The rules of natural deduction NJ are:

$$\frac{}{\Gamma, A, \Delta \vdash A}\ ax \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I \qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow E$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge_1 E \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge_2 E$$

1. Prove that if $\Gamma, A, B, \Delta \vdash C$ is provable in NJ, then $\Gamma, B, A, \Delta \vdash C$ as well.
2. Prove that if $\Gamma, \Delta \vdash C$ is provable in NJ, then $\Gamma, A, \Delta \vdash C$ as well.
3. Prove that if $\Gamma, A, A, \Delta \vdash C$ is provable in NJ, then $\Gamma, A, \Delta \vdash C$ as well.

The sequent calculus LJ is another deduction system for propositional intuitionistic logic using the same formulas and the same sequents but not the same rules:

$$\frac{}{A \vdash A}\ ax_m \qquad \frac{\Delta \vdash A \qquad \Gamma, A, \Sigma \vdash C}{\Gamma, \Delta, \Sigma \vdash C}\ cut \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ ex \qquad \frac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ wk \qquad \frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ co$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow R \qquad \frac{\Delta \vdash A \qquad \Gamma, B, \Sigma \vdash C}{\Gamma, A \rightarrow B, \Delta, \Sigma \vdash C} \rightarrow L$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge R \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \wedge L$$

4. Prove that if $\Gamma \vdash A$ is provable in LJ then $\Gamma \vdash A$ is provable in NJ.
5. Prove that if $\Gamma \vdash A$ is provable in NJ then $\Gamma \vdash A$ is provable in LJ.

Exercise 2: **0-ary Functors**

Let $\mathscr{C}$ be an arbitrary category. Let $\mathbb{T}$ be the terminal category (one object, one morphism), and $\tau_{\mathscr{C}}$ the unique functor from $\mathscr{C}$ to $\mathbb{T}$.

1. Describe what is a functor from $\mathbb{T}$ to $\mathscr{C}$.
2. Given an object C of $\mathscr{C}$, prove that mapping any object of $\mathscr{C}$ to C and any morphism to the identity on C, one gets a functor (called a *constant functor*).
3. Prove that constant functors from $\mathscr{C}$ to $\mathscr{C}$ are exactly compositions of $\tau_{\mathscr{C}}$ with a functor from $\mathbb{T}$ to $\mathscr{C}$ (and moreover this decomposition is unique).

Exercise 3: **2-ary Functors**

Let $\mathscr{C}$ be an arbitrary category. A *binoidal functor* on $\mathscr{C}$ is a triple (o, φ, ψ) where:

- o is a function from pairs of objects of $\mathscr{C}$ to objects of $\mathscr{C}$;
- φ is a family, indexed by objects C of $\mathscr{C}$, of functions $\varphi_C : \mathscr{C}(A,B) \to \mathscr{C}(o(A,C), o(B,C))$;
- ψ is a family, indexed by objects C of $\mathscr{C}$, of functions $\psi_C : \mathscr{C}(A,B) \to \mathscr{C}(o(C,A), o(C,B))$.

such that:

- for each object C of $\mathscr{C}$, Φ_C which maps object A to $o(A,C)$ and morphism $f \in \mathscr{C}(A,B)$ to $\varphi_C(f) \in \mathscr{C}(o(A,C), o(B,C))$ defines a functor from $\mathscr{C}$ to $\mathscr{C}$;
- for each object C of $\mathscr{C}$, Ψ_C which maps object A to $o(C,A)$ and morphism $f \in \mathscr{C}(A,B)$ to $\psi_C(f) \in \mathscr{C}(o(C,A), o(C,B))$ defines a functor from $\mathscr{C}$ to $\mathscr{C}$.

1. Prove that any bifunctor F on $\mathscr{C}$ (*i.e.* functor from $\mathscr{C} \times \mathscr{C}$ to $\mathscr{C}$) defines a binoidal functor on $\mathscr{C}$ with:

$$\begin{aligned} o(A,B) &:= F(A,B) \\ \varphi_C(f) &:= F(f, id_C) \\ \psi_C(f) &:= F(id_C, f) \end{aligned}$$

Given a binoidal functor $H = (o, \varphi, \psi)$, we say that $f \in \mathscr{C}(A_1, B_1)$ is *central for H* if the following two diagrams commute for all $g \in \mathscr{C}(A_2, B_2)$:

$$\begin{array}{ccc} o(A_1, A_2) & \xrightarrow{\psi_{A_1}(g)} & o(A_1, B_2) \\ {\scriptstyle \varphi_{A_2}(f)} \downarrow & & \downarrow {\scriptstyle \varphi_{B_2}(f)} \\ o(B_1, A_2) & \xrightarrow[\psi_{B_1}(g)]{} & o(B_1, B_2) \end{array} \qquad \begin{array}{ccc} o(A_2, A_1) & \xrightarrow{\psi_{A_2}(f)} & o(A_2, B_1) \\ {\scriptstyle \varphi_{A_1}(g)} \downarrow & & \downarrow {\scriptstyle \varphi_{B_1}(g)} \\ o(B_2, A_1) & \xrightarrow[\psi_{B_2}(f)]{} & o(B_2, B_1) \end{array}$$

2. Prove that central morphisms of H constitute a wide (*i.e.* with all objects) subcategory of $\mathscr{C}$, called the *center* of H.
3. Prove that, in the construction of Question 1, all morphisms are central for the defined binoidal functor.
4. Prove that a binoidal functor on $\mathscr{C}$ whose center is $\mathscr{C}$ induces a bifunctor on $\mathscr{C}$.