Commutative Diagrams

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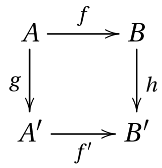
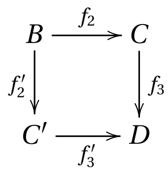
Exercise 1: **Reasoning with Commutative Diagrams**

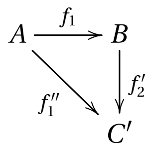
Given a category $\mathscr{C}$, a *diagram* on $\mathscr{C}$ is a directed graph with vertices labeled with objects of $\mathscr{C}$ and edges labeled with morphisms of $\mathscr{C}$ (with, if the source of edge e is labeled A and the target of e is labeled B, then the label of e is a morphism from A to B). In other words, it is a directed graph equipped with a directed graph morphism into the underlying directed graph of $\mathscr{C}$.
A diagram is *commutative* if given any two vertices x and y and any two *non-empty* directed paths p and q from x to y, the two morphisms in $\mathscr{C}$ obtained by composing the morphisms labeling the edges of p on one side, and those labeling the edges of q on the other side, are equal.
Building commutative diagrams gives a way to reason about equality of morphisms in categories which can be substituted to equational reasoning. As an example, here is a commutative square and the corresponding equation it describes on morphisms:



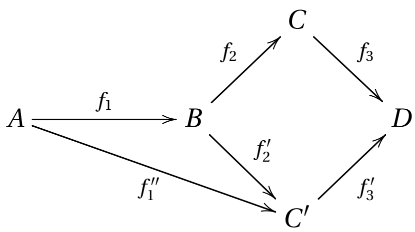
$$\begin{aligned} h \circ f &= f' \circ g \\ f\,;h &= g\,;f' \end{aligned}$$

As another example, assuming we have morphisms $f_1 : A \to B$, $f_2 : B \to C$, $f_3 : C \to D$, $f_2' : B \to C'$, $f_3' : C' \to D$, $f_1'' : A \to C'$ satisfying: $f_2\,;f_3 = f_2'\,;f_3'$ and $f_1\,;f_2' = f_1''$. These two equations can be represented with the following two commutative diagrams:



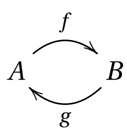


By equational reasoning, we can deduce: $f_1\,;f_2\,;f_3 = f_1\,;f_2'\,;f_3' = f_1''\,;f_3'$. The same result can be obtained by diagrammatic reasoning by considering the following commutative diagram:

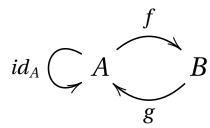


All this relies heavily on the associativity of composition in categories.

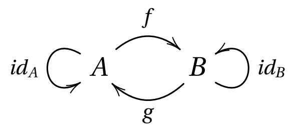
1. What are the equations induced by the following commutative diagram?



2. Same question with:



3. Same question with:



What property is it defining about f and g?

4. Prove that, in any commutative diagram $\mathbb{D}$, adding a *new* vertex v and a *new* edge e with source v and target a vertex of $\mathbb{D}$ which is not in a cycle (resp. with target v and source a vertex of $\mathbb{D}$ which is not in a cycle) gives another commutative diagram.

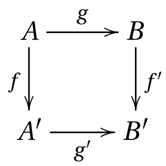
5. Prove that, in any commutative diagram with an edge e labeled $f\,;g$ (and with source and target not belonging to cycles), adding a new vertex v and splitting e into an edge toward v labeled f and an edge from v labeled g gives another commutative diagram.

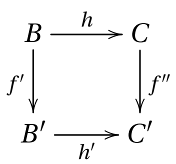
6. Prove that, in any commutative diagram, removing an edge gives another commutative diagram.

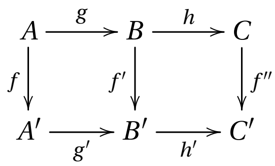
7. Prove that, in any commutative diagram, removing a node and all its adjacent edges gives another commutative diagram.

8. Prove that, in any commutative diagram containing a non-empty directed path p whose associated morphism is f (composition of the labels of the edges of p), adding a new edge with same source as p, same target as p and label f gives another commutative diagram.

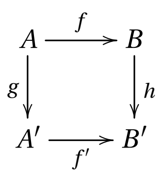
9. Prove that if the first two diagrams are commutative, then the third one is commutative as well:

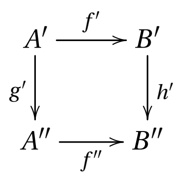


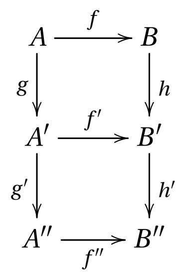




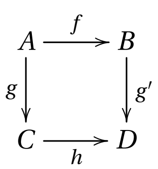
10. Prove that if the first two diagrams are commutative, then the third one is commutative as well:

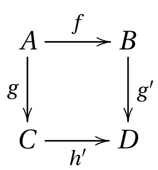


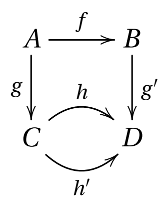




11. By choosing objects and morphisms in **Set**, prove that it is possible to have the first two diagrams commutative but not the third one:

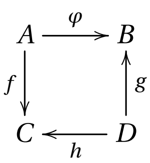


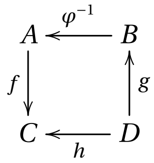




12. Given a commutative diagram in which all the labels of edges are isomorphisms, prove that the diagram obtained by reversing all edges and by turning each edge label φ into φ^{-1} is also commutative.

13. By using morphisms from $\{0,1\}$ to $\{0,1\}$ in **Set**, prove that given an isomorphism φ, the commutation of the first diagram does not imply the commutation of the second one:

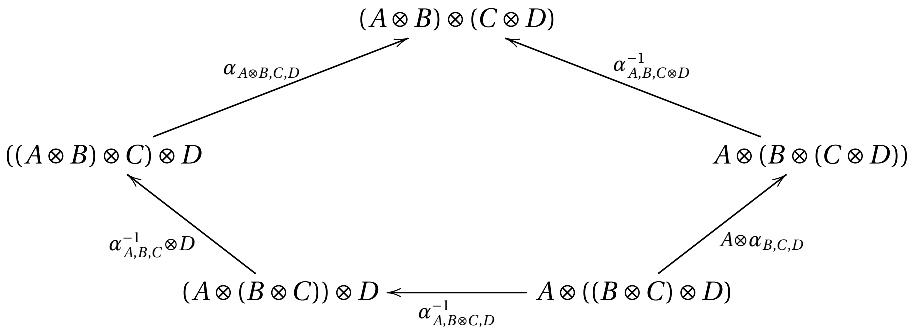




14. Given a commutative diagram obtained exactly by identifying the two sources and the two targets of two disjoint non-empty directed paths (no other vertex or edge), and an edge e which is labeled by an isomorphism φ, prove that replacing e with the reverse edge labeled φ^{-1} gives a commutative diagram.

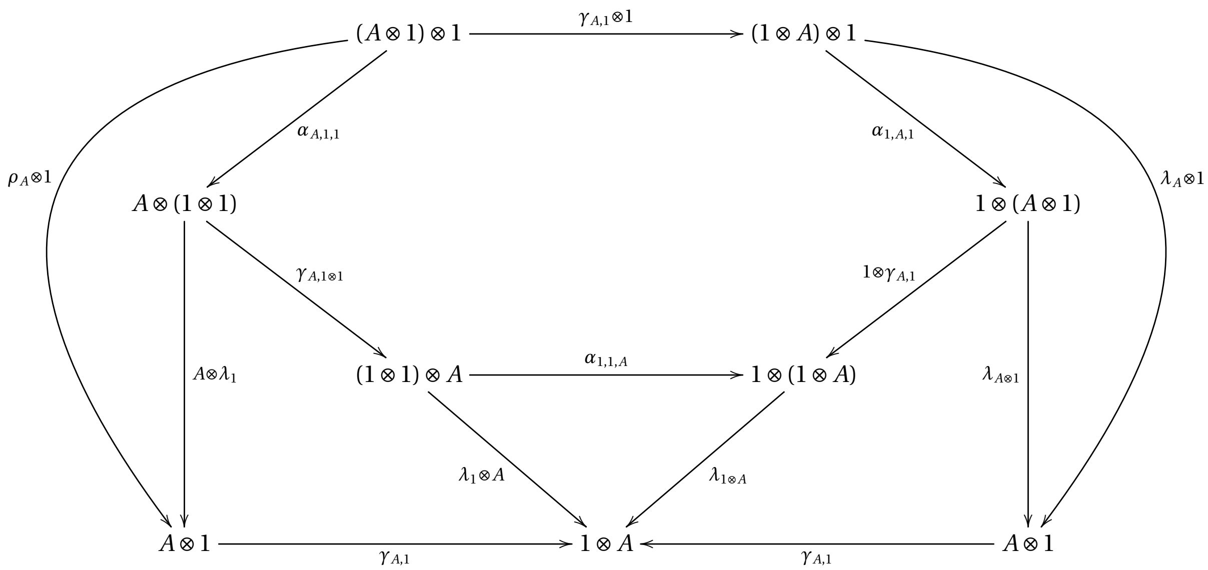
    Note that it is in particular the case for any commutive acyclic triangle.

15. Given a monoidal category $(\mathscr{C},\lambda,\rho,\alpha)$, prove the following diagram is commutative:



    And give also an equational proof.

16. Justify the commutativity of the following diagram in any symmetric monoidal category $(\mathscr{C},\lambda,\rho,\alpha,\gamma)$:



17. Deduce $\rho_A = \gamma_{A,1}\,;\lambda_A$ for any object A in any symmetric monoidal category $(\mathscr{C},\lambda,\rho,\alpha,\gamma)$.

Exercise 2: **Preservation of Commutative Diagrams through Functors**

1. Prove the image of a commutative diagram under a functor is also a commutative diagram.

2. A morphism f is called an *epimorphism* if for any g and h, $f\,;g = f\,;h$ implies $g = h$.

   (a) Prove that any isomorphism is an epimorphism.

   (b) What are the epimorphisms of **Set**?

3. If F is an endofunctor and there exists a natural transformation from F to Id whose elements are epimorphisms, prove that the converse of Question 1 also holds: if the image under F of a diagram is commutative then the original diagram is commutative as well.

4. In any *monoidal* category, prove that $f \otimes 1 = g \otimes 1$ implies $f = g$.