Subject 1: Classical Sequent Calculus

to be returned on Monday, September 22nd

Questions marked with (*) might be a bit more involved. (Part of) some questions have already been addressed in the lecture, but please redo them properly anyway.
In the whole subject, exchange rules can be left implicit.

Two Sequent Calculi

We consider and compare here two one-sided sequent calculi LK_m and LK_r.

Formulas are given by:

$$A ::= X \mid \neg X \mid A \wedge A \mid A \vee A \mid \top \mid \bot$$

X ranges over the elements of a given set of variables $\mathcal{V}$. X and $\neg X$ are called *atomic formulas*. For A not in $\mathcal{V}$, $\neg A$ is defined in the usual way by induction on A using De Morgan's laws.

Rules of LK_m are:

$$\frac{}{\vdash A, \neg A}\ ax^m \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex(\sigma) \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ ctr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ wk$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\ \wedge^m \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad \frac{}{\vdash \top}\ \top^m \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot$$

Rules of LK_r are:

$$\frac{}{\vdash \Gamma, X, \neg X}\ ax^a \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex(\sigma) \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ ctr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ wk$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge^a \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad \frac{}{\vdash \Gamma, \top}\ \top^a \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot$$

Question 1. Prove $(\neg(X \vee X) \vee Y) \vee X$ in LK_r.

Question 2. Prove $(\neg(X \vee X) \vee Y) \vee X$ in LK_m.

Question 3. Prove that a sequent $\vdash \Gamma$ provable in LK_r is also provable in LK_m.

Question 4. Prove the $(\wedge^m)$ rule is derivable in LK_r.

Question 5. Prove that $\vdash A, \neg A$ is provable in LK_r for any formula A.

Question 6. Prove that, in any proof of LK_r, if a rule R is just above a (wk) rule, one can transform the proof by making the (wk) rule go up or disappear.

Question 7. (*) Deduce that a sequent $\vdash \Gamma$ provable in LK_r is also provable in LK_r without using the (wk) rule.

Question 8. Prove that a sequent $\vdash \Gamma$ provable in LK_m is also provable in LK_r without using the (wk) rule.

Soundness and Completeness

Question 9. Prove all the rules of LK_r are valid: for any Boolean valuation φ, if all the premisses of the rule are mapped to *true* by φ, then the conclusion as well.

Question 10. *Soundness:* Prove that any provable formula A of LK_r is valid (*i.e.* mapped to *true* by any Boolean valuation).

A rule is *semantically reversible* if whenever its conclusion is mapped to *true* by all Boolean valuations then all the premisses of the rule as well.

Question 11. Prove the (wk) rule is not semantically reversible.

The system LK_r^- is obtained from the system LK_r by removing the (ctr) and (wk) rules.

Question 12. Prove all the rules of LK_r^- are semantically reversible.

Question 13. Prove that, by applying the rules of LK_r^- in a bottom-up way, it is possible to associate with any formula A a set of *atomic* sequents (*i.e.* containing atomic formulas only), such that the formula A is valid if and only if all these atomic sequents are.

Question 14. Give a syntactic necessary and sufficient condition for atomic sequents to be provable in LK_r^-.

Question 15. Give a syntactic necessary and sufficient condition for atomic sequents to be valid.

Question 16. *Completeness:* Prove that any valid formula is provable in LK_r^-.

Question 17. Deduce that any valid formula is provable in LK_r.

Question 18. Deduce that any valid formula is provable in LK_m.

Semantic Cut Elimination

The system LK_m^+ is obtained from LK_m by adding the following (cut) rule:

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\ cut$$

Question 19. Prove that any formula A provable in LK_m is also provable in LK_m^+.

We are now going to prove the converse.

Question 20. Prove the (cut) rule is valid.

Question 21. Prove that any provable formula A of LK_m^+ is valid.

Question 22. *Cut Elimination:* Conclude that any formula A provable in LK_m^+ is also provable in LK_m.