Homework 6: Light linear logic
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to be returned by dec 11, 2015

Notations: Here application in λ-calculus will be denoted as $(t\ u)$. We will also write $\lambda x_1 x_2.t$ for $\lambda x_1.\lambda x_2.t$ (please indicate in case you use other notations).

We write $\underline{n}$ the Church (unary) integer $\lambda f x.(f\ (f\ \ldots (f\ x)\ldots)$ (with n occurrences of f).

Exercise 1:

In homework 5 we typed the following λ-term in IELL:

$$t = \lambda n f x.(n\ f\ (n\ f\ (f\ x)))$$

In this exercise we will study it in Intuitionistic Light Linear Logic (ILLL). We consider the following type for Church unary integers in ILLL:

$$\mathbf{N}^{\mathbf{L}}_{\alpha} = !(\alpha \multimap \alpha) \multimap \S(\alpha \multimap \alpha)$$

1. Show that there is an ILLL derivation $\mathcal{D}'$ (decorated with terms) of the following judgement:

$$\vdash t : !\mathbf{N}^{\mathbf{L}}_{\alpha} \multimap \S\mathbf{N}^{\mathbf{L}}_{\alpha}.$$

2. Translate the derivation $\mathcal{D}'$ into an LLL proof-net S. Using results of the course, what can we say about a time complexity bound for the reduction of a proof-net obtained by applying S to a proof-net representing a Church integer $\underline{n}$ (and thus representing the term $(t\ \underline{n})$)?

   Write down the reduction of this proof-net in the particular case of $n = 1$.

Exercise 2 [Polynomials in IELL and ILLL]:

In this exercise we use the two following types for Church integers respectively in IELL and ILLL:

$$\begin{aligned} \mathbf{N}^{\mathbf{E}} &= \forall\alpha.!(\alpha \multimap \alpha) \multimap !(\alpha \multimap \alpha) \\ \mathbf{N}^{\mathbf{L}} &= \forall\alpha.!(\alpha \multimap \alpha) \multimap \S(\alpha \multimap \alpha). \end{aligned}$$

1. Show that any polynomial function with one variable and coefficients in $\mathbb{N}$, so of the shape $f(x) = \Sigma_{i=1}^{n} a_i x^i$, can be represented in IELL by a proof of conclusion $!\mathbf{N}^{\mathbf{E}} \multimap !\mathbf{N}^{\mathbf{E}}$.

2. Can we hope for a similar representation in ILLL of all polynomial functions with a single type, say for instance with type $!\mathbf{N}^{\mathbf{L}} \multimap \S\mathbf{N}^{\mathbf{L}}$?

3. Consider the polynomial function $f(x) = x^{2^k}$, for $k \in \mathbb{N}$. Show that it can be represented in ILLL by a proof of conclusion $\mathbf{N}^{\mathbf{L}} \multimap \S^{4k}\mathbf{N}^{\mathbf{L}}$.