Subject 4: IMAL, IMEAL and the naive exponential

to be returned on the lundi 10 novembre

The questions marked ($\star$) are possibly more difficult.

1 Simple games and IMAL

We recall that IMAL (for Intuitionistic Multiplicative Affine Logic) is obtained from IMLL by adding the unrestricted left weakening rule:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} w_A$$

Let X be a propositional variable, and set $\mathbf{B} = (X \otimes X) \multimap X$. We write $\mathbb{B}$ for the set $\{\textsc{True}, \textsc{False}\}$.

Q. 1. Give all the cut-free IMAL proofs of $\mathbf{B}$.

Among those, we name one true and another one false *(the rest of the subject should work whatever your choice is... But one choice makes more sense than the other!)*. We write $\overline{\textsc{True}} = \mathsf{true}$ and $\overline{\textsc{False}} = \mathsf{false}$.

As in Subject 3, if A is a formula we write $\mathbf{B}[A] = \mathbf{B}[A/X] = A \otimes A \multimap A$. Obviously we have $\mathbf{B}[X] = \mathbf{B}$. If ϖ is a proof of sequent $B_1, \ldots, B_n \vdash C$ and A is a formula, we write $\varpi[A/X]$ for the same proof where occurrences of X have been replaced by A – it is now a proof of the sequent $B_1[A/X], \ldots, B_n[A/X] \vdash C[A/X]$. We say that an IMAL proof ϖ of $\mathbf{B}[A_1], \ldots, \mathbf{B}[A_n] \vdash \mathbf{B}$ $(A_1, \ldots, A_n)$*-represents* a boolean function $f : \mathbb{B}^n \to \mathbb{B}$ if, given $(b_i)_{1 \leq i \leq n} \in \mathbb{B}^n$, cutting appropriately $\overline{b_i}[A_i/X]$ with ϖ *(you should be able to infer what this means)* yields $\overline{f(b_1, \ldots, b_n)}$ by cut elimination.

If ϖ $(X, \ldots, X)$-represents a function $f : \mathbb{B}^n \to \mathbb{B}$, we simply say that it *represents* it.

Q. 2. Give an IMAL proof of the sequent $\mathbf{B}[\mathbf{B}], \mathbf{B}, \mathbf{B} \vdash \mathbf{B}$ that (B, X, X)-represents the function $\mathsf{if} : \mathbb{B}^3 \to \mathbb{B}$.

We now consider the game semantics model of IMAL. Propositional variables are interpreted by the one-move game $\bot = (\circ, \{\circ \mapsto O\}, \{\epsilon, \circ\})$. We will often omit the semantic brackets $[\![-]\!]$ and denote the game $[\![A]\!]$ simply by A.

Q. 3. Describe the game obtained as the interpretation of $\mathbf{B}$.

Q. 4. Compute the interpretations of the proofs of Q1, detailing the process. We keep the notations true and false for respectively $[\![\mathsf{true}]\!]$ and $[\![\mathsf{false}]\!]$.

Are there other strategies on $\mathbf{B}$? If yes, describe them.

Q. 5. Describe a strategy neg on $\mathbf{B} \multimap \mathbf{B}$ representing the negation function. Detailing the process, compute the compositions $\mathsf{true}; \mathsf{neg}$ and $\mathsf{neg}; \mathsf{neg}$.

Q. 6. You have seen in the Subject 3 that there are some functions from $\mathbb{B}$ to $\mathbb{B}$ that cannot be represented by a MLL proof net. Show that any function from $\mathbb{B}$ to $\mathbb{B}$ can be represented by some strategy on $\mathbf{B} \multimap \mathbf{B}$. Comment.

Q. 7. Compute the interpretation of your proof of Q2 *(as a – quite lengthy – set of plays, or in the language of strategies along with an informal description).*

Q. 8. Define what it means for a strategy $\sigma : \mathbf{B} \otimes \ldots \otimes \mathbf{B} \multimap \mathbf{B}$ to represent a boolean function $f : \mathbb{B}^n \to \mathbb{B}$ in such a way that if a proof ϖ of $\mathbf{B}, \ldots, \mathbf{B} \vdash \mathbf{B}$ represents f, then $[\![\varpi]\!]$ represents f as well (prove it!).

Q. 9. Give a strategy $\sigma : \mathbf{B} \otimes \mathbf{B} \otimes \mathbf{B} \multimap \mathbf{B}$ that represents if.

Q. 10. ($\star$) Show that any boolean function $f : \mathbb{B}^n \to \mathbb{B}$ is represented by a strategy on $\mathbf{B} \otimes \ldots \otimes \mathbf{B} \multimap \mathbf{B}$.

Q. 11. ($\star\star$) Show that the strategy of Q9 is not IMAL-definable. In fact, show that there is no IMAL proof of $\mathbf{B}, \mathbf{B}, \mathbf{B} \vdash \mathbf{B}$ that represents if.

2 The naive exponential

In the lecture we have seen the *Hyland exponential*, giving a model of IMELL. It is a bit complicated. Here we investigate what seems to be a simpler option, until it breaks down.

A strategy $\sigma : A$ is *history-free* iff: for all $s, t \in \sigma$, if $smn \in \sigma$ and $tm \in P_A$, then $tmn \in \sigma$. In other words, the behaviour of an history-free strategy only depends on the last Opponent move.

Q. 12. Show that each history-free strategy on a game A induces a partial function

$$f_\sigma : \lambda_A^{-1}(\{O\}) \to \lambda_A^{-1}(\{P\})$$

such that the mapping $\sigma \mapsto f_\sigma$ is injective. Recall the two-moves game Σ. For $A = \Sigma \multimap \Sigma \otimes \Sigma$, show that the mapping above is not surjective.

So when we compare history-free strategies it suffices to compare the corresponding partial functions. We also sometimes define history-free strategies by their partial functions, but then because of the issue above we need to carefully check that they indeed correspond to history-free strategies.

We write $!^H A$ for the *Hyland exponential* seen during the lecture. Its corresponding structural strategies are written:

$$\begin{array}{rclcl} d_A^H & : & !^H A & \multimap & A \\ c_A^H & : & !^H A & \multimap & !^H A \otimes !^H A \\ \delta_A^H & : & !^H A & \multimap & !^H !^H A \end{array} \qquad \begin{array}{rcrcl} m_{A,B}^H & : & !^H A \otimes !^H B & \multimap & !^H (A \otimes B) \\ m_1^H & : & 1 & \multimap & !^H 1 \\ w_A^H & : & !^H A & \multimap & 1 \end{array}$$

And the functorial promotion is written $!^H \sigma : !^H A \multimap !^H B$.

Q. 13. Show that $c_A^H, \delta_A^H, m_{A,B}^H$ are not history-free.

We now define, for a game A, its *naive exponential* $!^N A$. It has:

$$\begin{array}{rcl} M_{!^N A} & = & \mathbb{N} \times M_A \\ \lambda_{!^N A} & = & \{(n, a) \mapsto \lambda_A(a) \mid (n, a) \in \mathbb{N} \times M_A\} \\ P_{!^N A} & = & \{s \in M_{!^N A}^* \mid s \text{ alternates}, \forall n \in \mathbb{N}, s \upharpoonright_n \in P_A\} \end{array}$$

So the difference with $!^H A$ is that new copies of A can be opened in an arbitrary order.

Q. 14. Describe a history-free strategy $\sigma : !^N A \multimap !^H A$. Describe a strategy $\tau : !^H A \multimap !^N A$. Is it history-free?

Q. 15. Give history-free structural strategies $d^N_A, c^N_A, \delta^N_A, m^N_{A,B}, m^N_1$ and w^N_A, and a functorial promotion $!^N\sigma : !^N A \multimap !^N B$ that preserves history-freeness.

Recall that for the translation of the λ-calculus to be a simulation, we needed to consider proof nets up-to (among others) the *Rétoré equivalence*. This observation also holds in the sequent calculus, where the Rétoré equivalence states in particular:

$$\dfrac{\dfrac{\dfrac{\varpi}{\Gamma, !A_1 \vdash B}}{\Gamma, !A_1, !A_2 \vdash B}}{\Gamma, !A \vdash B} \quad \sim \quad \dfrac{\varpi}{\Gamma, !A \vdash B} \quad \sim \quad \dfrac{\dfrac{\dfrac{\varpi}{\Gamma, !A_2 \vdash B}}{\Gamma, !A_1, !A_2 \vdash B}}{\Gamma, !A \vdash B}$$

where the annotations $1, 2$ serve to disambiguate the weakened hypothesis. Call ϖ_l (resp. ϖ_r) the left hand side proof (resp. right hand side).

Q. 16. Compute the interpretation of ϖ_l and ϖ_r using the naive exponential. Show that there is an instantiation of Γ, A, B and ϖ such that at least one of the two equivalences is not validated by the games model.

Because of that, $!^N$ might still be invariant under cut elimination in IMEAL but it seems unlikely that it yields a model of the simply-typed λ-calculus, since that typically requires Rétoré to hold. Oh well, let us try to fix it. It seems that the strategies for ϖ_l, ϖ_r and ϖ only differ with respect to their choice of copy indices, let us formalize that.

If $\pi : \mathbb{N} \to \mathbb{N}$ is a partial injection and $s \in P_{!^N A \multimap B}$, we define:

$$\begin{array}{rcl} \pi \cdot \epsilon & = & \epsilon \\ \pi \cdot sb & = & (\pi \cdot s)b \\ \pi \cdot s(n, a) & = & (\pi \cdot s)(\pi(n), a) \end{array}$$

and undefined where $\pi(n)$ is undefined, where $b \in M_B$ and $(n, a) \in M_{!^N A}$.

Q. 17. Show that $\cdot$ is a group action of the group of permutations on $\mathbb{N}$ on $P_{!^N A \multimap B}$. *(that means that for all $s \in P_{!^N A \multimap B}$, $\pi \cdot s \in P_{!^N A \multimap B}$ as well, and that $\mathrm{id}_{\mathbb{N}} \cdot s = s$ and $(\pi_1 \circ \pi_2) \cdot s = \pi_1 \cdot (\pi_2 \cdot s)$).*

Recall that if $\cdot$ is a group action of G on a set S and $s \in S$, the orbit of s is $\{\pi \cdot s \mid \pi \in G\}$. For $\sigma, \tau : !^N A \multimap B$, we write $\sigma \simeq \tau$ if their plays reach the same orbits.

Q. 18. Show that $\simeq$ is an equivalence relation. Show that the strategies for ϖ_l, ϖ_r and ϖ are pairwise equivalent for $\simeq$.

So maybe we can consider strategies up to $\simeq$ and save the naive exponential. For that of course we need all operations on strategies to respect $\simeq$. Unfortunately:

Q. 19. Give a proof ϖ of $!X \vdash X$ and a history-free strategy $\tau : (!^N X \multimap X) \multimap \mathbf{B}$ such that $\varpi_l ; \tau = \mathsf{true}$ and $\varpi_r ; \tau = \mathsf{false}$. Comment.

Q. 20. ($\star\star\star\star$) Show that in fact τ cannot be defined in IMEAL, and that composition with strategies obtained from the interpretation of IMEAL proofs respect $\simeq$.