Subject 3: Multiplicative Proof-Nets

to be returned on Tuesday, October 10th

Formulas are given by:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

where X ranges over the elements of a given countable set $\mathcal{V}$ of variables.
We consider the following rules for the one-sided multiplicative linear sequent calculus MLL:

$$\frac{}{\vdash A, A^{\perp}}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ cut \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

We only use the terminology *proof-net* for a proof-structure which satisfies the Danos-Regnier correctness criterion. A proof-structure is *atomic* if the conclusions of all its axiom nodes are labelled with atomic formulas (that is of the shape X or $X^{\perp}$).

Proof Nets

Question 1. For each sequent below, give a proof in MLL and the associated proof-net.

a. $\vdash X^{\perp}, X \otimes (X^{\perp} \parr X)$

b. $\vdash X^{\perp} \parr (Y^{\perp} \otimes Z^{\perp}), (X \otimes Y) \parr Z$

c. $\vdash X^{\perp} \otimes X, (X \parr X^{\perp}) \otimes (X \parr X^{\perp})$

d. $\vdash X \parr X^{\perp}, (X^{\perp} \otimes X) \parr (X^{\perp} \otimes X)$

Question 2. For each formula below, give all possible atomic cut-free proof-structures with this formula as unique conclusion. For all the obtained proof-structures, check whether they satisfy the Danos-Regnier criterion or not. For each obtained proof-structure which satisfies the criterion, give a sequentialization in MLL.

a. $X \parr X^{\perp}$

b. $(X \otimes X^{\perp}) \parr (X \parr X^{\perp})$

c. $(X \otimes X^{\perp}) \otimes (X \parr X^{\perp})$

d. $(X \otimes X^{\perp}) \parr (X \otimes X^{\perp})$

e. $(X \parr X^{\perp}) \parr (X \parr X^{\perp})$

f. $(X \parr X^{\perp}) \otimes (X \parr X^{\perp})$

g. $((X \otimes X^{\perp}) \parr (X \otimes X^{\perp})) \parr (X \parr X^{\perp})$

h. $(X \otimes (Z \parr Y)) \parr (((Y^{\perp} \parr X^{\perp}) \otimes (U^{\perp} \parr V^{\perp})) \parr ((V \parr Z^{\perp}) \otimes U))$

Boolean Computation

We consider the formula $\mathbf{B} = (X^\perp \parr X^\perp) \parr (X \otimes X)$.

Question 3. Give all the cut-free proof-nets with a unique conclusion labelled $\mathbf{B}$.

Among the two atomic proof-nets of Question 3, only one can be obtained by axiom expansion. We call it TRUE. The other atomic one is called FALSE. The set of Booleans is $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$ and we define a translation of Booleans into proof-nets: $\overline{\mathsf{true}} = \textsc{true}$ and $\overline{\mathsf{false}} = \textsc{false}$.
A function f from $\mathbb{B}$ to $\mathbb{B}$ is said to be *represented* by the proof-net $\mathcal{R}$ with two conclusions $\mathbf{B}^\perp$ and $\mathbf{B}$ if the normal form of the proof-net $\mathcal{R}_b$ (obtained by putting a cut node between the conclusion $\mathbf{B}$ of $\overline{b}$ and the conclusion $\mathbf{B}^\perp$ of $\mathcal{R}$) is $\overline{f(b)}$, for any $b \in \mathbb{B}$.

Question 4. Give a proof-net representing the negation function $\mathbb{B} \to \mathbb{B}$:

$$\mathsf{true} \mapsto \mathsf{false}$$
$$\mathsf{false} \mapsto \mathsf{true}$$

Question 5. Give all the atomic cut-free proof-nets with two conclusions: $\mathbf{B}^\perp$ and $\mathbf{B}$.

Question 6. Give a function from $\mathbb{B}$ to $\mathbb{B}$ which cannot be represented by a proof-net.