Subject 5: Playing with innocent strategies and semantic normalization

to be returned on the 1st december

1 Playing and counting with innocent strategies

Here $\mathbf{B}$ denotes the arena for booleans, having one Opponent move q enabling two Player moves $\mathtt{tt}$ and $\mathtt{ff}$. Likewise, $\perp$ denotes the one-move arena.

In this first part we consider the *partial* λ*-calculus* with one base type α, and a constant $\Omega : \alpha$ interpreted as $[\![\Omega]\!] = \{\epsilon\}$. We will write $N = (\alpha \to \alpha) \to \alpha \to \alpha$ for the type of Church numerals, and $\underline{n} : N$ for the usual representation of $n \in \mathbb{N}$ as a term of type N.

Q. 1. Draw the arenas:

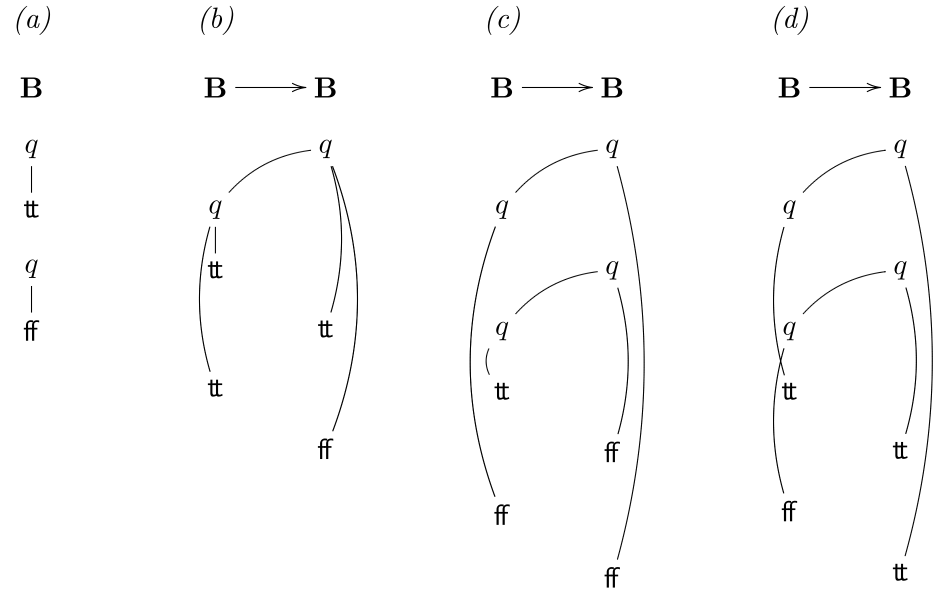
(a) $\mathbf{B} \Rightarrow \mathbf{B}$,

(b) $(\mathbf{B} \Rightarrow \perp) \Rightarrow \mathbf{B}$,

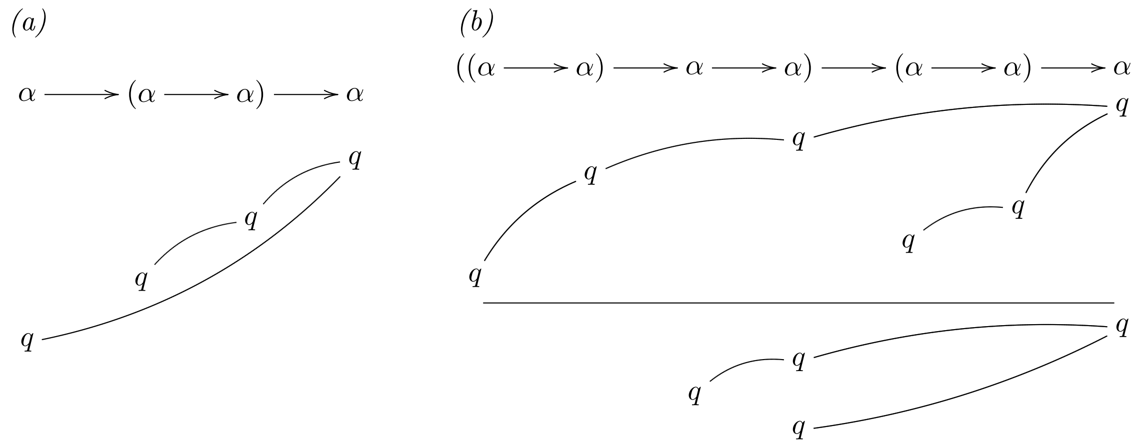
(c) $[\![N]\!]$,

(d) $[\![N \to N]\!]$.

Q. 2. For each of the following plays, can it belong to a single-threaded strategy *(stratégie filaire)*? A P-visible strategy? An innocent strategy? Justify your answer.



Q. 3. Using definability, give η-long normal forms for the finite innocent strategies described by the following sets of maximal P-views.



Q. 4. Give the maximal P-views of the strategies interpreting the η-long normal forms:

(a) $\lambda f^{\alpha\to\alpha} g^{\alpha\to\alpha} x^{\alpha}.\, f\,(g\,(f\,x))$

(b) $\lambda f^{\alpha\to\alpha\to\alpha} x^{\alpha}.\, f\,x\,(f\,\Omega\,x)$

(hint: you don't necessarily have to unfold the definition of interpretation – the correspondence between η-long normal forms and total finite strategies goes both ways: you should be able to write down P-views that mimic the structure of the normal form and map back to it by definability)

A strategy $\sigma : A$ is **total** if it reacts to any Opponent move: for all $s \in \sigma$, for all $sa \in \mathcal{L}_A$, there exists $sab \in \sigma$. For $\sigma : A$ innocent, we say that it **has total P-views** iff for all $s \in \ulcorner\sigma\urcorner$, if $sa \in \mathcal{L}_A$ (with a pointing to the last move of s), then there is $sab \in \ulcorner\sigma\urcorner$.

Q. 5. Show that σ innocent is total iff it has total P-views.

So to check that an innocent strategy is total, it suffices to check that it has a response to any Opponent extension of one of its P-views. Recall that an innocent strategy $\sigma : A$ is **finite** iff $\ulcorner\sigma\urcorner$ is a finite set.

Q. 6. Give a bijection between natural numbers and total finite innocent strategies on $[\![N]\!]$.

Q. 7. Describe (by drawing its maximal P-views) an innocent strategy on $[\![N \to N]\!]$ representing the square function $(-)^2 : \mathbb{N} \to \mathbb{N}$.

The types of higher Church numerals are obtained by $N_0 = N$, and $N_{n+1} = N_n[\alpha \to \alpha/\alpha]$. Any natural number $n \in \mathbb{N}$ can be represented in N_k, by $\underline{n}_0 = \underline{n}$, and $\underline{n}_{k+1} = \underline{n}_k[\alpha \to \alpha/\alpha]$.

Q. 8. Is there a bijection between natural numbers and total finite innocent strategies on N_1?

Q. 9. Give the η-long normal form of $\underline{2}_1$. Deduce the maximal P-view(s) of $[\![\underline{2}_1]\!]$.

Q. 10. Describing the corresponding interaction(s), give the maximal P-view(s) of $[\![\underline{2}_0]\!];[\![\underline{2}_1]\!]$.

2 Ω-free terms and total finite strategies

In the definability result, Ω is used exactly when Player does not answer. An immediate consequence is that finite innocent strategies *with total P-views* correspond with η-long normal forms of the usual, Ω-free simply-typed λ-calculus. From now on, we drop Ω. It is a basic result on the λ-calculus that if $\Gamma \vdash M : A$ then there exists an η-long normal form $\Gamma \vdash N : A$ such that $M \simeq_{\beta\eta} N$. In this subject we refer to it as the *syntactic normalization theorem.*

Q. 11. Assuming the syntactic normalization theorem, show that for all $\Gamma \vdash M : A$, $[\![M]\!]$ is total finite.

We will refer to this fact as the *semantic normalization theorem.* Here we proved it using the syntactic one; in the rest of this subject, we aim to prove it semantically. This amounts to checking that our basic strategies (projections and evaluation) are total finite, and that being total finite is preserved by our constructions on strategies (composition, pairing, curryfication).

Q. 12. Prove that if A is a simple type, $\mathrm{id}_{[\![A]\!]}$ is total finite.

The fact that evaluation is total finite is similar, and it is easily checked that pairing and curryfication preserve total finite strategies (we admit it here). The difficult part is stability under composition.

Q. 13. Find two total innocent strategies whose composition is not total.

($\star$) **The rest of this subject is optional, and potentially more difficult.**
We will prove that total finite strategies do not suffer from the issue above, and are stable under composition. An **infinite dialogue** on alphabet Σ is just an infinite word $s \in \Sigma^\omega$, where each move s_i with $i > 0$ is equipped with *exactly one* pointer to an earlier move.
We define two *view* functions $\ulcorner - \lrcorner$ and $\llcorner - \urcorner$ on finite pointing strings on Σ, by:

$$\ulcorner s \quad s_i \lrcorner \;=\; \llcorner s \urcorner \;\cup\; \{i\}$$

$$\llcorner s \quad s_i \overset{\frown}{\cdots} s_j \urcorner \;=\; \ulcorner s \quad s_i \lrcorner \;\cup\; \{j\}$$

$$\ulcorner s_0 \lrcorner \;=\; \llcorner s_0 \urcorner \;=\; \{0\} \qquad\qquad \text{(if } s_0 \text{ is the first move)}$$

We say that an infinite dialogue s is **bounded** by $N \in \mathbb{N}$ for all $n \in \mathbb{N}$, $|\ulcorner s_{\leq n} \lrcorner| \leq N$ (recall that $s_{\leq n}$ is the prefix of s of length $n+1$). It is **visible** if when s_j points to s_i, $i \in \ulcorner s_j \lrcorner$.

Q. 14. Assuming there are total finite strategies $\sigma : A$ and $\tau : A \Rightarrow \bot$ (so necessarily $\sigma;\tau : \bot$ is not total), show that there exists a bounded visible infinite dialogue.

A similar – but a bit longer – argument shows that if there are total finite $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ such that $\sigma;\tau$ is not *total finite*, then there exists a bounded visible infinite dialogue. We admit that, and will now prove that no such visible bounded infinite dialogue can exist.

On an infinite dialogue s, we define the sets of **winning** and **losing** moves as the smallest pair of sets satisfying:

- A move s_i is **winning** iff there is $s_j \to s_i$ such that s_j is losing,
- A move s_j is **losing** iff for all $s_j \to s_i$, s_j is winning.

These sets are well-defined by Knaster-Tarski's theorem. Note in particular that, as a base case, moves that are not pointed to are losing. This assignment need not be total in general (there could be moves that are neither winning nor losing).

Q. 15. Show that if s is a visible bounded (by N) infinite dialogue, then for all $n \in \mathbb{N}$, there exists $p \leq n$ with the same parity as n and such that $p \in \llcorner s_n \urcorner$, and s_p is winning.

(hint: reason by induction on $N - n$)

Q. 16. Show that there is no bounded visible infinite dialogue. Deduce that the composition of two total finite strategies is total finite.

(hint: is the initial move winning?)

Q. 17. Deduce (without using syntactic normalization) that for all simply-typed λ-term $\Gamma \vdash M : A$, there exists an η-long normal form $\Gamma \vdash N : A$ such that $[\![M]\!] = [\![N]\!]$.