Subject 3: Basic game semantics

The category of partial functions

We write **PFn** for the category of sets and partial functions.

Question 1. Give an example that shows why the Cartesian product

$$A \times B := \{(a, b) \mid a \in A, b \in B\}$$

equipped with the projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ is *not* a categorical product in **PFn**.

Question 2. Can you find an alternative definition that *does* provide a categorical product in **PFn**? What is the terminal object?

The category of simple games

We write **G** for the category of games and strategies defined in class.

Question 3*. Prove that $\otimes$ is a bi-functor, *i.e.* for strategies $\sigma : A \multimap B$, $\sigma' : A' \multimap B'$, $\tau : B \multimap C$ and $\tau' : B' \multimap C'$, we have $(\sigma \otimes \tau); (\sigma' \otimes \tau') = (\sigma; \tau) \otimes (\sigma'; \tau')$; and $1_{A \otimes B} = 1_A \otimes 1_B$.

Question 4. Consider the game $\mathbb{B} := (\{q\}, \{t, f\}, \{\varepsilon, q, qt, qf\})$. What are the strategies on this game?

Question 5. What is the game tree of $(\mathbb{B} \otimes \mathbb{B}) \multimap \mathbb{B}$?

Question 6. Define a left-strict strategy on $(\mathbb{B} \otimes \mathbb{B}) \multimap \mathbb{B}$ that calculates the logical OR function. Define all possible input strategies on $\mathbf{1} \multimap (\mathbb{B} \otimes \mathbb{B})$ and calculate the composite strategies.

Question 7. What is the game tree of $((\mathbb{B} \otimes \mathbb{B}) \multimap \mathbb{B}) \multimap \mathbb{B}$? Can you think of a strategy on this game that distinguishes between left- and right-strict OR?

The ! exponential

Question 8. Show that $!A \otimes !B$ is isomorphic to $!(A \& B)$ in $\mathbf{G}$.

Question 9*. Show (i) that ! is a symmetric monoidal functor; (ii) that the dereliction strategies $\varepsilon_A \colon\ !A \multimap A$ are the components of a monoidal natural transformation; and (iii) that the co-multiplication strategies $\delta_A \colon\ !A \multimap !!A$ are also the components of a monoidal natural transformation.