# Rappels

## LK monolatère

- $A ::= X \mid \neg X \mid A \wedge A \mid A \vee A \mid \top \mid \bot$

$\neg$ : formules $\longrightarrow$ formules

$$X \longmapsto \neg X$$

$$\neg X \longmapsto X$$

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg \top = \bot$$

...

- $\vdash \Gamma$ $\leftarrow$ liste de formules

## • Règles

### gpe identité

$$\frac{}{\vdash A, \neg A}\,\text{ax} \qquad \left(\frac{\vdash A, \Gamma \qquad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta}\,\text{cut}\right)$$

### gpe structurel

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,\text{ex} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\,\text{wk} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,\text{ctr}$$

### gpe logique

- additif

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge^{\text{add}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,\vee_1^{\text{add}} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}\,\vee_2^{\text{add}}$$

$$\frac{}{\vdash \Gamma, \top}\,\top^{\text{add}}$$

- multiplicatif

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge^{\text{mul}} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee^{\text{mul}}$$

$$\frac{}{\vdash \top}\,\top^{\text{mul}} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot^{\text{mul}}$$

<u>ex:</u>

$$A \vee (B \wedge C) \;\underset{\dashv}{\vdash}\; (A \vee B) \wedge (A \vee C)$$

$$\neg\big((A \vee B) \wedge (A \vee C)\big) \;?$$

$$\vdash (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C),\; A \vee (B \wedge C)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg A, A}\,\text{ax}}{\vdash \neg A, A, B \wedge C}\,\text{wk}}{\vdash \neg A, \neg A \wedge \neg C, A, B \wedge C}\,\text{wk} \qquad \dfrac{\vdash \neg B, \neg A, A, B \wedge C \qquad \dfrac{\dfrac{\dfrac{}{\vdash \neg B, B}\,\text{ax} \quad \dfrac{}{\vdash \neg C, C}\,\text{ax}}{\vdash \neg B, \neg C, B \wedge C}\,\wedge\text{mul}}{\vdash \neg B, \neg C, A, B \wedge C}\,\text{wk}}{\vdash \neg B, \neg A \wedge \neg C, A, B \wedge C}\,\wedge\text{add}}{\vdash \neg A \wedge \neg B, \neg A \wedge \neg C, A, B \wedge C}\,\wedge\text{add}}{\vdash \neg A \wedge \neg B, \neg A \wedge \neg C, A \vee (B \wedge C)}\,\vee\text{mul}}{\vdash (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C), A \vee (B \wedge C)}\,\vee\text{mul}$$

# Elimination des Coupures

- règle dérivable

- règle <u>admissible</u>

$$\frac{J_1 \quad \dots \quad J_k}{J}\,\Omega$$

$\Omega$ est <u>admissible</u> si :
lorsque $J_1, \dots J_k$ sont prouvables alors $J$ aussi

$$\frac{\vdash X}{\vdash Y}$$

## Théorème :

La règle (cut) est admissible dans LK sans (cut).

$$\frac{\vdash A, \Gamma \qquad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta}$$

## cas clef

$$\frac{\dfrac{\cdots}{\vdash A}\,\text{règle logique} \qquad \dfrac{\cdots}{\vdash \neg A}\,\text{règle logique}}{\vdash}$$

$$\frac{\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A} \quad \dfrac{\pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B}\,\wedge\text{add} \qquad \dfrac{\dfrac{\pi_3}{\vdash \Delta, \neg A}}{\vdash \Delta, \neg A \vee \neg B}\,\vee_1\text{add}}{\vdash \Gamma, \Delta}\,\text{cut}$$

$\leadsto$

$$\frac{\dfrac{\pi_1}{\vdash \Gamma, A} \qquad \dfrac{\pi_3}{\vdash \Delta, \neg A}}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdash \Gamma, B\end{matrix}}{\vdash \Gamma, A \wedge B}\,\wedge^{add} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdash \Delta, \neg A, \neg B\end{matrix}}{\vdash \Delta, \neg A \vee \neg B}\,\vee^{mul}}{\vdash \Gamma, \Delta}\,cut
$$

$$\rightsquigarrow$$

$$
\cfrac{\cfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_3\\ \vdash \Delta, \neg A, \neg B\end{matrix}}{\vdash \Gamma, \Delta, \neg B}\,cut \qquad \begin{matrix}\pi_2\\ \vdash \Gamma, B\end{matrix}}{\vdash \Gamma, \Delta, \Gamma}\,cut}{\vdash \Gamma, \Delta}\,ctr
$$

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdash \Delta, B\end{matrix}}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge^{mul} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdash \Sigma, \neg A, \neg B\end{matrix}}{\vdash \Sigma, \neg A \vee \neg B}\,\vee^{mul}}{\vdash \Gamma, \Delta, \Sigma}\,cut
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_3\\ \vdash \Sigma, \neg A, \neg B\end{matrix}}{\vdash \Gamma, \Sigma, \neg B}\,cut \qquad \begin{matrix}\pi_2\\ \vdash \Delta, B\end{matrix}}{\vdash \Gamma, \Delta, \Sigma}\,cut
$$

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdash \Delta, B\end{matrix}}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge^{mul} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdash \Sigma, \neg A\end{matrix}}{\vdash \Sigma, \neg A \vee \neg B}\,\vee^{add}_1}{\vdash \Gamma, \Delta, \Sigma}\,cut
$$

$$\rightsquigarrow$$

$$
\cfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_3\\ \vdash \Sigma, \neg A\end{matrix}}{\vdash \Gamma, \Sigma}\,cut}{\vdash \Gamma, \Delta, \Sigma}\,wk
$$

$$
\dfrac{\dfrac{}{\vdash \Gamma, \top}\,\top^{add} \qquad \dfrac{\begin{matrix}\pi\\ \vdash \Delta\end{matrix}}{\vdash \Delta, \bot}\,\bot^{mul}}{\vdash \Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \cfrac{\begin{matrix}\pi\\ \vdash \Delta\end{matrix}}{\vdash \Gamma, \Delta}\,wk
$$

$$
\dfrac{\dfrac{}{\vdash \top}\,\top^{mul} \qquad \dfrac{\begin{matrix}\pi\\ \vdash \Delta\end{matrix}}{\vdash \Delta, \bot}\,\bot^{mul}}{\vdash \Delta}\,cut \quad \rightsquigarrow \quad \begin{matrix}\pi\\ \vdash \Delta\end{matrix}
$$

## Cas commutatif

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{\vdash \Delta, \neg A, B, C}{\vdash \Delta, \neg A, B \vee C}\,\vee\text{ mul}}{\vdash \Gamma, \Delta, B \vee C}\,\text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A, B, C}{\vdash \Gamma, \Delta, B, C}\,\text{cut}}{\vdash \Gamma, \Delta, B \vee C}\,\vee\text{ mul}$$

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{\vdash \Delta, B, \neg A \qquad \vdash \Delta, C, \neg A}{\vdash \Delta, B \wedge C, \neg A}\,\wedge\text{ add}}{\vdash \Gamma, \Delta, B \wedge C}\,\text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B, \neg A}{\vdash \Gamma, \Delta, B}\,\text{cut} \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, C, \neg A}{\vdash \Gamma, \Delta, C}\,\text{cut}}{\vdash \Gamma, \Delta, B \wedge C}\,\wedge\text{ add}$$

## Cas structurel

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, A}\,\text{weak} \qquad \begin{matrix}\vdots\,\pi_2\\ \vdash \Delta, \neg A\end{matrix}}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, \Delta}\,\text{weak}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma, A, A\end{matrix}}{\vdash \Gamma, A}\,\text{ctr} \qquad \begin{matrix}\vdots\,\pi_2\\ \vdash \Delta, \neg A\end{matrix}}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma, A, A\end{matrix} \qquad \begin{matrix}\vdots\,\pi_2\\ \vdash \Delta, \neg A\end{matrix}}{\vdash \Gamma, \Delta, A} \qquad \begin{matrix}\vdots\,\pi_2\\ \vdash \Delta, \neg A\end{matrix}}{\vdash \Gamma, \Delta, \Delta}}{\vdash \Gamma, \Delta}\,\text{ctr}$$

## Cas identité

$$\dfrac{\dfrac{}{\vdash \neg A, A}\,\text{ax} \qquad \begin{matrix}\vdots\,\pi\\ \vdash \Gamma, \neg A\end{matrix}}{\vdash \neg A, \Gamma}\,\text{cut} \quad \rightarrow \quad \begin{matrix}\vdots\,\pi\\ \vdash \Gamma, \neg A\end{matrix}$$

poids d'une coupure

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\ \text{cut}$$

$|A|$ : nb de connecteurs
$X, \neg X \mapsto 1$
$\bot, \top, \vee, \wedge$

0, 1, 2 : nb règles "pas clef"
↳ cas clef

$\left( |A|, \begin{smallmatrix} 0 \\ 1 \\ 2 \end{smallmatrix} \right)$ lexico

$[a, a, b, c] > [b, b, b, b, b, b, c, c, c, a]$

$a > b$
$a > c$

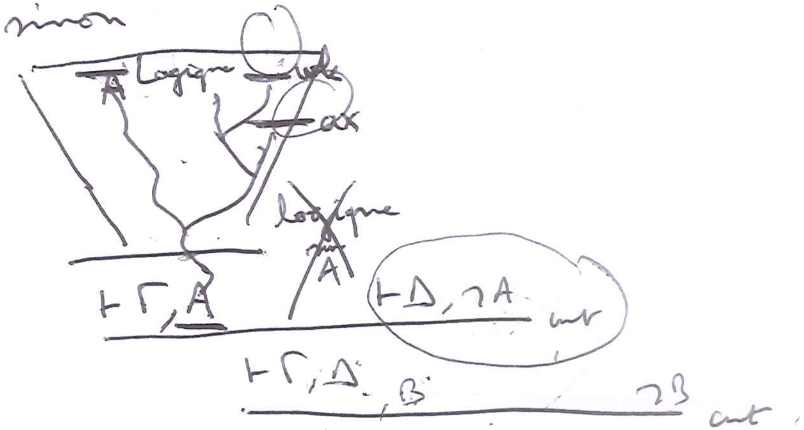
multi-ensemble des poids de coupures de $\pi$
↳ poids de $\pi$

si $\pi$ preuve avec au moins 1 coupure
$\longmapsto \pi'$ de poids ↘

→ choisir 1 coupure "la plus haute"
- cas clef

  OK

- sinon



# Conséquences

- **Propriété de la sous-formule**

  si $\vdash \Gamma$ est prouvable, il existe une preuve de $\vdash \Gamma$ qui n'utilise que des sous-formules des formules de $\Gamma$.

- **Cohérence**

  ⊛ $\vdash$ n'est pas prouvable

  ⊛ $\vdash \bot$ n'est pas prouvable

$$\dfrac{\dfrac{\pi}{\vdash \bot} \qquad \dfrac{}{\vdash \top}\,\top\ \mathrm{mul}}{\vdash}\,\mathrm{cut}$$

  ⊛ qqst $A$, on n'a pas $\vdash A$ et $\vdash \neg A$

$$\dfrac{\vdash A \qquad \vdash \neg A}{\vdash}\,\mathrm{cut}$$

  ⊛ qqst $A$, $\nvdash A \wedge \neg A$