Rappels

- Elimination des coupures
  - ↳ propriété de la sous-formule
    - ↳ $\vdash$ n'est pas prouvable

$\vdash A \wedge \neg A$ n'est pas prouvable ?

$$\dfrac{\dfrac{}{\vdash \neg A, A}\,ax}{\vdash \neg A \vee A}\,\vee^{mul} \qquad \vdash A \wedge \neg A$$
$$\vdash$$

Réversibilité

$$\dfrac{\mathcal{J}_1 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}}$$ réversible

si $\dfrac{\mathcal{J}}{\mathcal{J}_1} \quad \cdots \quad \dfrac{\mathcal{J}}{\mathcal{J}_k}$ sont admissible

$\vee^{mul}$ / $\wedge^{add}$ / $\bot^{mul}$ / règles 0-aires

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee^{mul}$$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,\vee_1^{add}$$

$\vdash \neg Y, X$

$\vdash \neg Y, X \vee Y$

$$\dfrac{\vdash \Gamma, A \vee B \qquad \dfrac{\dfrac{}{\vdash \neg A, A}\,ax \quad \dfrac{}{\vdash \neg B, B}\,ax}{\vdash \neg A \wedge \neg B, A, B}\,\wedge^{mul}}{\vdash \Gamma, A, B}\,cut$$

$$\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge^{add}$$

$$\dfrac{\vdash \Gamma, A \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge^{mul}$$

$\vdash \neg Y$

$\vdash \neg X, \neg Y, X \wedge Y$

$$\dfrac{\vdash \Gamma, A \wedge B \qquad \dfrac{\dfrac{}{\vdash \neg A, A}\,ax}{\vdash \neg A \vee \neg B, A}\,\vee_1^{add}}{\vdash \Gamma, A}\,cut$$

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot^{mul}$$

$$\dfrac{\vdash \Gamma, \bot \qquad \dfrac{}{\vdash \top}\,\top^{mul}}{\vdash \Gamma}\,cut$$

$$\frac{\vdash \ldots\ A, B\ \ldots}{\vdash \ldots\ A \vee B\ \ldots}$$

Expension des axiomes

$$\frac{}{\vdash A, \neg A}\ ax \quad \Leftarrow \quad \frac{}{\vdash X, \neg X}\ ax_{at}$$

admissible

preuve : par induction sur A : dérivable

- $A = X$ ou $\neg X$
- $A = B \vee C$

$$\dfrac{\dfrac{\dfrac{}{\vdash B, \neg B}\ IH \quad \dfrac{}{\vdash C, \neg C}\ IH}{\vdash B, C, \neg B \wedge \neg C}\ \wedge^{mul}}{\vdash B \vee C, \neg B \wedge \neg C}\ \vee^{mul}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash B, \neg B}\ IH}{\vdash B \vee C, \neg B}\ \vee_1^{add} \quad \dfrac{\dfrac{}{\vdash C, \neg C}\ IH}{\vdash B \vee C, \neg C}\ \vee_2^{add}}{\vdash B \vee C, \neg B \wedge \neg C}\ \wedge^{add}$$

- $A = \top$

$$\dfrac{\dfrac{}{\vdash \top}\ \top^{mul}}{\vdash \top, \bot}\ \bot^{mul} \qquad \dfrac{}{\vdash \top, \bot}\ \top^{add}$$

□

LK réversible

type logique

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee^{mul} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge^{add}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot^{mul} \qquad \frac{}{\vdash \Gamma, \top}\ \top^{add}$$

type identité

$\vdash X_1, \ldots, X_p, \neg Y_1, \ldots, \neg Y_q$ ssi $X_i = Y_j$

$$\frac{}{\vdash X, \neg X, \Gamma}\ ax^{add} \qquad \frac{}{\vdash X, \neg X, X_1, \ldots, X_p, \neg Y_1, \ldots, \neg Y_q}\ ax^{add}$$

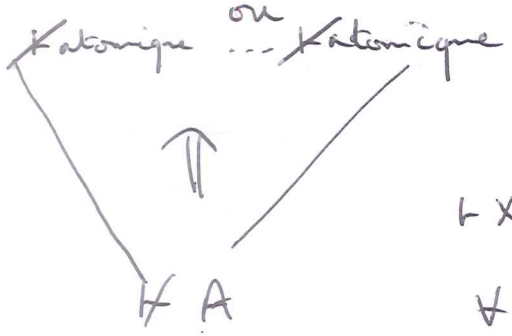
type structurel

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex$$

Prop : LK réversible est complet pour la validité classique.

ie A vrai $\Rightarrow \vdash A$ prouvable dans LK rev

preuve : contraposée



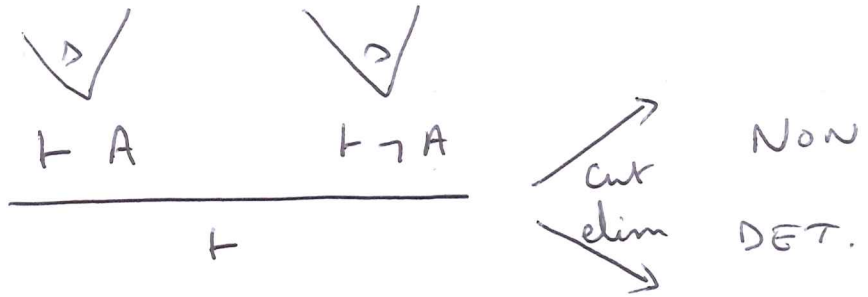
$\vdash X_1, \dots, X_p, \neg Y_1, \dots, \neg Y_q$

$\forall i,j \quad X_i \neq Y_j$

valuation ν : FAUX / VRAI

valuation "FAUX" : présenté par les règles de haut en bas

$\Rightarrow \nu$ donne FAUX sur $\vdash A$ □



$\rightsquigarrow$ LJ : intuitionniste

β - réduction — Élim. cut.

η - expansion — expansion des ax.

$\lambda x.(t)x =_\eta t$
$x \notin t$

LJ formules

$A ::= X \mid A \wedge A \mid A \vee A \mid \top \mid \bot \mid \neg A \mid A \to A$

séquents $\Gamma \vdash A$

Élimination des coupures ✓

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to D \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \to G$$

$\neg A := A \to$ ⬛ $\rightsquigarrow$ $\dfrac{\Gamma \vdash A \cdot}{\Gamma, \neg A \vdash}$ ⬛ $\qquad \dfrac{\Gamma, A \vdash}{\Gamma \vdash \neg A}$ ⬛

$\vdash_{LJ} A \vee B \Rightarrow \vdash_{LJ} A$ ou $\vdash_{LJ} B$

$\not\vdash_{LJ} X \vee \neg X$

$\vdash_{LB} A \Rightarrow \vdash_{LK} A$

$\not\Leftarrow$

$A \longmapsto A^\bullet$ tq $\boxed{\Gamma \vdash_{LK} \Delta \Rightarrow \Gamma^\bullet, \underline{\Delta^\bullet} \vdash_{LJ} F}$

$\vdash_{LK} A \Rightarrow \underline{A^\bullet} \vdash_{LJ} F$

$\vdash_{LJ} A^\bullet$

et $\vdash_{LJ} A^\bullet \Rightarrow \vdash_{LK} A^\bullet \Leftrightarrow \vdash_{LK} A$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\Gamma^\bullet, \underline{A^\bullet}, \underline{\Delta^\bullet} \vdash F}{\Gamma^\bullet, \underline{\Delta^\bullet} \vdash A^\bullet}\neg D \qquad \dfrac{\Gamma^\bullet, \underline{B^\bullet}, \underline{\Delta^\bullet} \vdash F}{\Gamma^\bullet, \underline{\Delta^\bullet} \vdash B^\bullet}\neg D}{\Gamma^\bullet, \underline{\Delta^\bullet} \vdash A^\bullet \wedge B^\bullet}\wedge}{\Gamma^\bullet, \underline{\Delta^\bullet}, \neg(A^\bullet \wedge B^\bullet) \vdash F}\neg G$$

enlever des $\neg\neg$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\text{ax}}{\neg\neg A, \neg A \vdash F}\neg G}{\neg\neg A \wedge \neg\neg B, \neg A \vdash F}\wedge G}{\neg(\neg\neg A \wedge \neg\neg B), \neg A \vdash F}\neg D$$

$$\dfrac{\dfrac{\neg(\neg\neg A \wedge \neg\neg B), \neg A \vdash F}{\neg A \vdash \neg(\neg\neg A \wedge \neg\neg B)}\neg G \qquad \neg(\neg\neg A \wedge \neg\neg B) \vdash \neg\neg A}{\neg(\neg\neg A \wedge \neg\neg B) \vdash \neg\neg A \wedge \neg\neg B}\wedge D$$

$X^\bullet = \neg\neg X$

$(A \wedge B)^\bullet = \neg\neg(A^\bullet \wedge B^\bullet)$

$(A \vee B)^\bullet = \neg\neg(A^\bullet \vee B^\bullet)$

$T^\bullet = \neg\neg T$

$\bot^\bullet = \neg\neg\bot$

$(A \to B)^\bullet = \neg\neg(A^\bullet \to B^\bullet)$

$A^\bullet = \neg\neg B$

$\underline{A^\bullet} := \neg B$

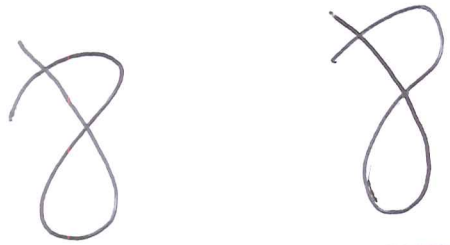
Logique Linéaire

LaTeX
cmll

formules orthogonal tensem avec par plus un top bottom zéro

$$A ::= X \mid X^\perp \mid A \otimes A \mid A \,\&\, A \mid A \parr A \mid A \oplus B \mid 1 \mid \top \mid \bot \mid 0$$

$\neg$ $\wedge^{mul}$ $\wedge^{add}$ $\vee^{mul}$ $\vee^{add}$ $\top^{unit}$ $\top^{add}$ $\bot^{mul}$ $\bot^{add}$



$A^\perp$:

$(X^\perp)^\perp := X$

$(A \otimes B)^\perp := A^\perp \parr B^\perp$

$(A \parr B)^\perp := A^\perp \otimes B^\perp$

$(A \,\&\, B)^\perp := A^\perp \oplus B^\perp$

$(A \oplus B)^\perp := A^\perp \,\&\, B^\perp$

$1^\perp := \bot$

$\bot^\perp := 1$

$\top^\perp := 0$

$0^\perp := \top$

lemme : $\forall A \quad A^{\perp\perp} = A$

séquents

$\vdash \Gamma$

règles

$$\frac{}{\vdash A, A^\perp}\,ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,ex \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot$$

additif

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2 \qquad \frac{}{\vdash \Gamma, \top}\,\top$$

multiplicatif

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$