Rappels

MALL

$$
\begin{aligned}
A := {} & X \mid \underbrace{A \otimes A \mid 1}_{} \mid A \,\&\, A \mid \top \\
& \mid X^\perp \mid \underbrace{A \parr A \mid \bot}_{M} \mid \underbrace{A \oplus A \mid 0}_{A}
\end{aligned}
$$

$$
\dfrac{}{\vdash A, A^\perp}\,ax
\qquad
\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut
\qquad
\dfrac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,ex
$$

$$
\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes
\qquad
\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr
\qquad
\dfrac{}{\vdash 1}\,1
\qquad
\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot
$$

$$
\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\&
\qquad
\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1
\qquad
\dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2
\qquad
\dfrac{}{\vdash \Gamma, \top}\,\top
$$

Propriétés

$A \dashv\vdash B \;:=\; A \vdash B$ et $B \vdash A$

$\qquad\qquad\qquad\quad \vdash A^\perp, B \qquad \vdash B^\perp, A$

$$A \dashv\vdash B \iff A^\perp \dashv\vdash B^\perp$$

- **Associativités**

$\otimes, \parr, \&, \oplus$

$$A \parr (B \parr C) \dashv\vdash (A \parr B) \parr C$$

- **Éléments neutres**

$A \otimes 1 \dashv\vdash A \qquad A \parr \bot \dashv\vdash A$

$A \,\&\, \top \dashv\vdash A \qquad A \oplus 0 \dashv\vdash A$

- **Distributivités**

$$A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$$

$$A \parr (B \,\&\, C) \dashv\vdash (A \parr B) \,\&\, (A \parr C)$$

- **Absorption**

$$A \otimes 0 \dashv\vdash 0$$
$$A \parr \top \dashv\vdash \top$$

- **Distributivité linéaire**

$$A \otimes (B \parr C) \vdash (A \otimes B) \parr C$$

Elimination des coupures

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\otimes \qquad \dfrac{\vdash \Sigma, A^\perp, B^\perp}{\vdash \Sigma, A^\perp \parr B^\perp}\parr}{\vdash \Gamma, \Delta, \Sigma}\,\mathrm{cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Sigma, A^\perp, B^\perp}{\vdash \Gamma, \Sigma, B^\perp}\,\mathrm{cut} \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, \Sigma}\,\mathrm{cut}$$

Expansion des axiomes

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp} \qquad \dfrac{}{\vdash B, B^\perp}}{\vdash A \otimes B, A^\perp, B^\perp}\otimes}{\vdash A \otimes B, A^\perp \parr B^\perp}\parr$$

Réversibilité

règles rev. : ax, 1, $\top$

ex

$\&$, $\parr$, $\perp$

$$\dfrac{\vdash \Gamma, A \parr B \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,\mathrm{ax} \qquad \dfrac{}{\vdash B^\perp, B}\,\mathrm{ax}}{\vdash A^\perp \otimes B^\perp, A, B}\otimes}{\vdash \Gamma, A, B}\,\mathrm{cut}$$

$$\dfrac{\vdash \Gamma, A \,\&\, B \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,\mathrm{ax}}{\vdash A^\perp \oplus B^\perp, A}\oplus_1}{\vdash \Gamma, A}\,\mathrm{cut}$$

Connecteur réversible

avec dernière règle sur $*$

$$\Uparrow$$

$\vdash \Gamma, A * B$ prouvable

$\parr$, & : rev. $\bot, \top$ rev

1 : irréversible

$\vdash \Gamma, 1$ ou $\vdash \bot, 1$

$\otimes, \oplus$: irréversibles

0 : irréversible

$\vdash \top, 0$

Implication linéaire

$$A \to B := \neg A \vee B$$

$$A \multimap B := A^\perp \ \oplus \mid \parr \ B$$

$$\vdash A \multimap A = \vdash A^\perp \parr A$$

$\nvdash A^\perp$ $\nvdash A$

$$\vdash A^\perp \oplus A$$

Connecteurs exponentiels

(LL)

$$A ::= \dots \mid ?A \mid !A \qquad (?A)^\perp = !A^\perp \qquad (!A)^\perp = ?A^\perp$$

$?$: pourquoi pas $!$: bien sûr (bang)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?d$$

$?d$: déréliction

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \text{promotion}$$

$$\frac{\vdash ?B_1, \dots, ?B_n, A}{\vdash ?B_1, \dots, ?B_n, !A}\,!$$

$$\frac{\vdash}{!A \vdash} \qquad \frac{!A, !A \vdash}{!A \vdash} \qquad \frac{A \vdash}{!A \vdash} \qquad \frac{!B_1, \dots, !B_n \vdash A}{!B_1, \dots, !B_n \vdash !A}$$

$$!A \vdash A \otimes A \otimes A$$

$$!(A \& B) \dashv\vdash !A \otimes !B$$

$$e^{a+b} = e^a \times e^b$$

$$?(A \oplus B) \dashv\vdash ?A \parr ?B.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash A^\perp \oplus B^\perp, A}\oplus_1}{\vdash ?(A^\perp \oplus B^\perp), A}?d}{\vdash ?(A^\perp \oplus B^\perp), !A}! \qquad \dfrac{\vdots}{\vdash ?(A^\perp \oplus B^\perp), !B}}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B}\otimes}{\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B}?c$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash ?A^\perp, A}?d}{\vdash ?A^\perp, ?B^\perp, A.}?w \qquad \dfrac{\vdots}{\vdash ?A, ?B^\perp, B}}{\vdash ?A^\perp, ?B^\perp, A \& B}\&}{\vdash ?A^\perp, ?B^\perp, !(A \& B)}!}{\vdash ?A^\perp \parr ?B^\perp, !(A \& B)}\parr$$

$$!\top \dashv\vdash 1 \qquad ?0 \dashv\vdash \bot$$

ex: $!A \otimes !B \vdash !(A \otimes B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash B^\perp, B}}{\vdash A^\perp, B^\perp, A \otimes B}\otimes}{\vdash ?A^\perp, ?B^\perp, A \otimes B}?d}{\vdash ?A^\perp, ?B^\perp, !(A \otimes B)}!}{\vdash ?A^\perp \parr ?B^\perp, !(A \otimes B)}\parr$$

Elimination des coupures

$$\dfrac{\vdash \Gamma, !A \qquad \vdash ?A^\perp, \Delta}{\vdash \Gamma, \Delta}\text{cut} \longmapsto \;????$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}! \qquad \vdash ?A^\perp, \Delta}{\vdash ?\Gamma, \Delta}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}! \qquad \dfrac{\vdash A^\perp, \Delta}{\vdash ?A^\perp, \Delta}?d}{\vdash ?\Gamma, \Delta}\text{cut} \longmapsto \dfrac{\vdash ?\Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash ?\Gamma, \Delta}\text{cut}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta}{\vdash ?A^\perp, \Delta}\,?w}{\vdash ?\Gamma, \Delta}\,\text{cut} \quad \longmapsto \quad \dfrac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash ?A^\perp, ?A^\perp, \Delta}{\vdash ?A^\perp, \Delta}\,?c}{\vdash ?\Gamma, \Delta}\,\text{cut} \quad \longmapsto \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash ?A^\perp, ?A^\perp, \Delta}{\vdash ?\Gamma, ?A^\perp, \Delta}\,\text{cut}}{\vdash ?\Gamma, ?\Gamma, \Delta}\,\text{cut}}{\vdash ?\Gamma, \Delta}\,?c$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash ?A^\perp, ?\Delta, B}{\vdash ?A^\perp, ?\Delta, !B}\,!}{\vdash ?\Gamma, ?\Delta, !B}\,\text{cut} \quad \longmapsto \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad \vdash ?A^\perp, ?\Delta, B}{\vdash ?\Gamma, ?\Delta, B}\,\text{cut}}{\vdash ?\Gamma, ?\Delta, !B}\,!$$

Expansion des axiomes

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{ax}}{\vdash ?A^\perp, A}\,?d}{\vdash ?A^\perp, !A}\,!$$

Règles réversibles

?c $\qquad \cancel{?w} \qquad \vdash A^\perp, ?A$

$$\dfrac{\vdash ?\Gamma, !A \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{ax}}{\vdash ?A^\perp, A}\,?d}{\vdash ?\Gamma, A}\,\text{cut}$$

!

$$\dfrac{\vdash \cancel{A^\perp, !A}}{\vdash ?A^\perp, !A}\,?d \qquad \cancel{?d}$$

Connecteurs réversibles !

$\cancel{?} \qquad \cancel{!}$

$\vdash \bot \parr ?A^\perp, !A$

$$\begin{array}{ccc} \pi & & \\ \vdash \Gamma & \longmapsto & \vdash \widetilde{\Gamma} \\ LL & & LK \end{array}$$

$$\left.\begin{array}{c}\otimes \\ \&\end{array}\right\} \longrightarrow \wedge$$

$$\left.\begin{array}{c}\oplus \\ \parr\end{array}\right\} \longrightarrow \vee$$

$$\left.\begin{array}{c}1 \\ \top\end{array}\right\} \longrightarrow \top$$

$$\left.\begin{array}{c}0 \\ \bot\end{array}\right\} \longrightarrow \bot$$

$$\left.\begin{array}{c}! \\ ?\end{array}\right\} \longrightarrow \emptyset$$

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \text{cut}$$

$$A \multimap B := A^{\perp} \parr B$$

$$\left\{\begin{array}{l} \dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \multimap D \\[3em] \dfrac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ \multimap G \end{array}\right.$$

Logique Linéaire Intuitionniste (ILL)

$$\Gamma \vdash A$$

$$\begin{aligned} A ::= X \mid A \otimes A \mid 1 \mid A \& A \mid \top \\ \mid A \multimap A \mid A \oplus A \mid 0 \\ \mid\, !A \end{aligned}$$

$$\left(\frac{\Gamma \vdash A}{\Gamma \vdash ?A} \qquad \frac{!\Gamma, A \vdash ?B}{!\Gamma, ?A \vdash ?B} \right)$$

$$\frac{\vdash \overbrace{?A, \ldots, ?A}^{n}, \Gamma}{\vdash ?A, \Gamma}$$