ILL

$\Gamma \vdash A$

$A ::= X \mid A \otimes A \mid 1 \mid A \& A \mid \top$
$\quad \mid A \multimap A \quad \mid A \oplus A \mid 0$
$\quad \mid\, !A$

$$\frac{}{A \vdash A}\,\text{ax} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\,\text{cut}$$

$$\frac{\Gamma \vdash A}{\sigma(\Gamma) \vdash A}\,\text{ex} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,!w \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,!c$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,!d \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,!$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap D \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap G$$

$X \longmapsto X$

$! \longmapsto \emptyset$

$A \multimap B \longmapsto \tilde{A} \to \tilde{B}$

$$\overset{\pi}{\underset{ILL}{\Gamma \vdash A}} \rightsquigarrow \overset{\tilde{\pi}}{\underset{LJ}{\tilde{\Gamma} \vdash \tilde{A}}}$$

Traduction de Girard

$$\underset{LJ(\to)}{\Gamma \vdash A} \longrightarrow \underset{ILL(\multimap, !)}{!\Gamma^\circ \vdash A^\circ}$$

$\boxed{\widetilde{A^\circ} = A}$

$$\begin{cases} (A \to B)^\circ = (!A^\circ) \multimap B^\circ \\ X^\circ = X \end{cases}$$

$\boxed{(A \vee B)^\circ = !A^\circ \oplus !B^\circ}$

$(\Gamma \vdash A)^\circ = !\Gamma^\circ \vdash A^\circ$

règles

$$\frac{\overline{A^\circ \vdash A^\circ}}{!A^\circ \vdash A^\circ}\,!d$$

$$\frac{\dfrac{!\Gamma^\circ \vdash A^\circ}{!\Gamma^\circ \vdash !A^\circ}\,! \qquad !A^\circ, !\Delta^\circ \vdash B^\circ}{!\Gamma^\circ, !\Delta^\circ \vdash B^\circ}\,\text{cut}$$

$$\frac{!\Gamma^\circ \vdash B^\circ}{!\Gamma^\circ, !A^\circ, B^\circ}\,!w$$

$$\frac{!\Gamma^\circ, !A^\circ \vdash B^\circ}{!\Gamma^\circ \vdash !A^\circ \multimap B^\circ}\,\multimap D$$

ctr : idem

$$\cancel{\dfrac{\dfrac{!\Gamma^\circ \vdash A^\circ \qquad !\Delta^\circ, !B^\circ \vdash C^\circ \qquad !A^\circ \multimap B^\circ \vdash !B^\circ}{!\Gamma^\circ, !\Delta^\circ, !A^\circ \multimap B^\circ \vdash C^\circ}\,!d}{!\Gamma^\circ, !\Delta^\circ, !(!A^\circ \multimap B^\circ) \vdash C^\circ}}$$

$$\dfrac{\dfrac{!\Gamma^\circ \vdash A^\circ}{!\Gamma^\circ \vdash !A^\circ}\,! \qquad \dfrac{!\Delta^\circ, !B^\circ \vdash C^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{!A^\circ \vdash !A^\circ}\,ax \qquad \dfrac{}{B^\circ \vdash B^\circ}\,ax}{!A^\circ \multimap B^\circ, !A^\circ \vdash B^\circ}\,\multimap G}{!(!A^\circ \multimap B^\circ), !A^\circ \vdash B}\,!d}{!(!A^\circ \multimap B^\circ), !A^\circ \vdash !B^\circ}\,!}{!\Delta^\circ, !(!A^\circ \multimap B^\circ), !A^\circ \vdash C^\circ}\,cut}{!\Gamma^\circ, !\Delta^\circ, !(!A^\circ \multimap B^\circ) \vdash C^\circ}\,cut$$

$$\Rightarrow \quad \Gamma \vdash_{LJ} A \underset{\text{squelette}}{\overset{\text{Girard}}{\Longleftrightarrow}} !\Gamma^\circ \vdash_{ILL} A^\circ \Longleftrightarrow \vdash_{LL} ?(\Gamma^\circ)^\perp, A^\circ$$

$$ILL \longrightarrow LL$$

$$\Gamma \vdash A \longmapsto \vdash \Gamma^\perp, A$$

$$A \multimap B \longmapsto A^\perp \parr B$$

$A ::= X \mid !A \multimap A$ ILL

$A ::= X \mid ?A^\perp \parr A$ LL $\subseteq O$

$$\left.\begin{array}{l} O ::= X \mid I \parr O \mid !O \\ I ::= X^\perp \mid O \otimes I \mid ?I \end{array}\right\}$$

<u>lemme :</u>

$\vdash_{LL} \Gamma$ avec Γ ne contient que des I, O

$\Rightarrow \Gamma$ contient exactement 1 O

par induction sur la preuve de Γ

$$\dfrac{}{\vdash \underset{I}{A^\perp}, \underset{O}{A}}\ ax \qquad \dfrac{\overset{(IH)}{\cancel{\mathrm{II}}} \in I \quad O \atop \vdash \Gamma, A, B}{\vdash \Gamma, \underbrace{A \parr B}_{O}}\ \parr$$

$$\dfrac{\overset{\vec{I} \in O}{\vdash \Gamma, A} \qquad \overset{\vec{I}, O \in I}{\vdash \Delta, B}}{\vdash \underbrace{\Gamma}_{\vec{I}}, \underbrace{\Delta}_{\vec{I}, O}, \underbrace{A \otimes B}_{I}}\ \otimes$$

...

Théorème (Conservativité)

$$\Gamma \vdash_{ILL} A \iff \vdash_{LL} \Gamma^\perp, A$$

$\vdash_{ILL} A \iff \vdash_{LL} A$

si Γ, A n'utilisent que $!, \multimap$

Preuve

$\Rightarrow$ ✓

$\Leftarrow$ $\quad \Gamma^\perp : I$

$\quad A : O$

LL $\quad$ lemme

$I, O \in \quad \vdash I_1, \dots, I_n, O \longmapsto I_1^\perp, \dots, I_n^\perp \vdash O$ $\quad$ ILL

$\vdash \Gamma^\perp, A$ $\qquad \underbrace{I_1^\perp, \dots, I_n^\perp}_{\in ILL} \vdash \underset{ILL}{\overset{\in}{O}}$

Devoir

1) Définir $LJ(\to, \wedge, \vee)$

2) Définir $ILL(\multimap, \otimes, \&, \oplus, !)$

3) Définir traduction Girard

$$LJ \xrightarrow{()^\circ} ILL$$

a) formules

b) séquents

c) preuves

4) $!\Gamma^\circ \vdash_{ILL} A^\circ \implies \Gamma \vdash_{LJ} A$

5) $\Gamma \vdash_{ILL} A \iff \vdash_{LL} \Gamma^\perp, A$

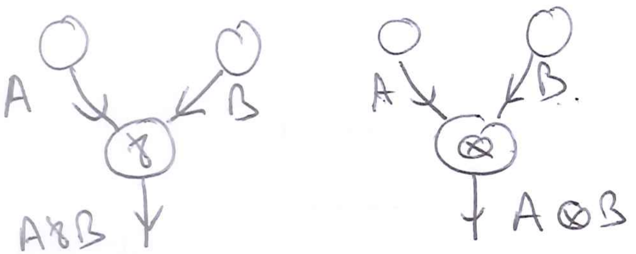
6) conclure

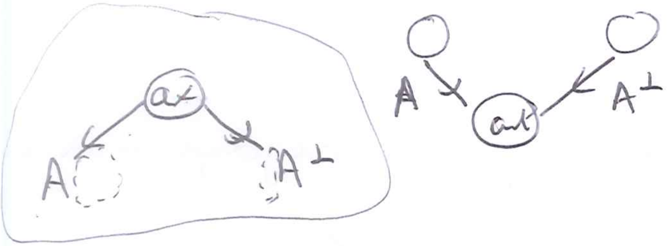
$$\Gamma \vdash_{LJ} A \iff \vdash_{LL} ?(\Gamma^\circ)^\perp, A^\circ$$

Réseaux de preuve

$\vdash A, B, C, D$

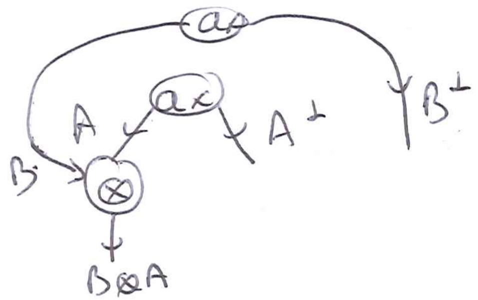
$$\dfrac{\vdash A \parr B,\ C, D}{\vdash A \parr B,\ C \parr D}$$





Structures de preuve

- graphe orienté avec demi-arêtes sortantes ⟶ conclusion
- arête ⟼ formule
- nœud ⟼ nom de règle
  - $\otimes, \parr$: 2 entrants (A, B) / 1 sortant ($A \otimes B$, $A \parr B$)
  - ax : 2 sortantes : A, $A^\perp$ ⟶ prémisse
  - cut : 2 entrantes : A, $A^\perp$



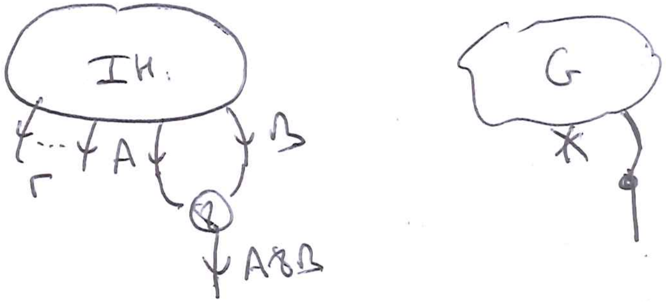
MLL

$$A ::= X \mid X^\perp \mid A \otimes A \mid A \parr A$$

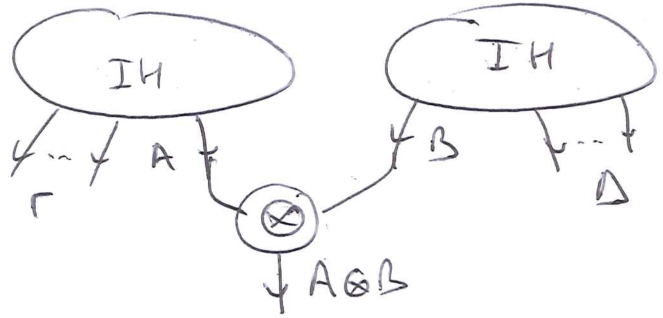
$$\dfrac{}{\vdash A, A^\perp}\ \text{ax} \qquad \dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

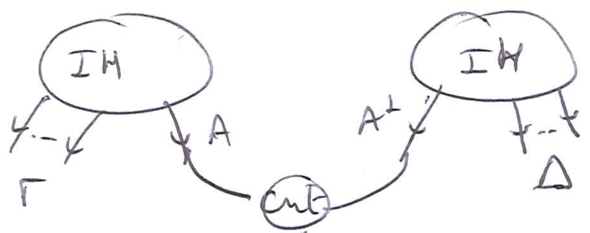
- $\dfrac{}{\vdash A, A^\perp}\ \text{ax}$ ⤳ 

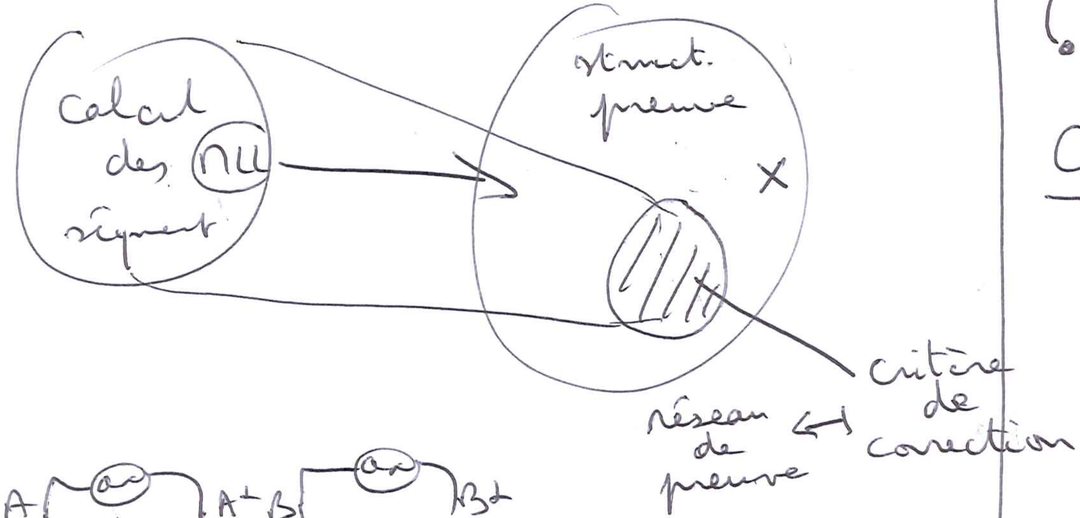
- $\parr$ 

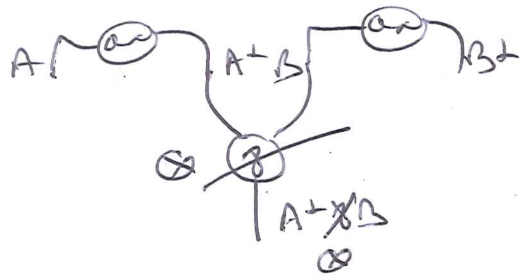
• ⊗



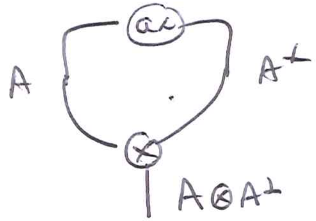
• cut



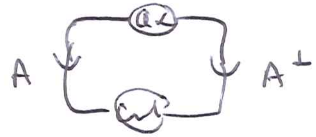




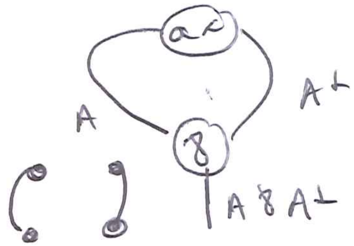
$$\frac{\vdash A, A^\perp, B, B^\perp}{\vdash A, A^\perp ⅋ B, B^\perp}$$



$\vdash A \otimes A^\perp$



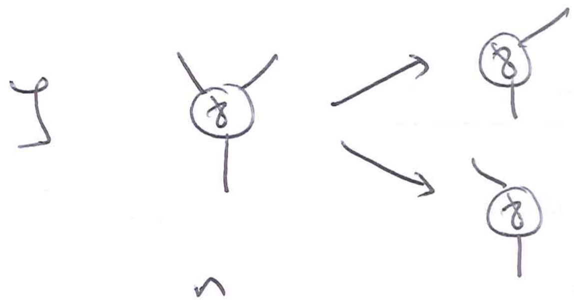
$\vdash$



$$\dfrac{\overline{\vdash A, A^\perp}\ ax}{\vdash A ⅋ A^\perp}\ ⅋$$

$\vdash A \multimap A$

Critère Danos-Regnier acyclique-connexe



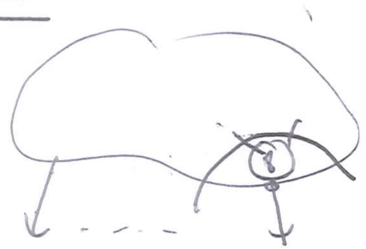
(oubli orientation)

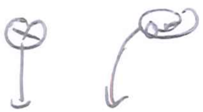
2^n graphes de correction acycliques & connexes

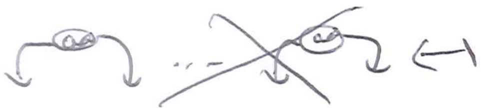
Théorème (Séquentialisation)

si $\mathcal{R}$ est un réseau de preuve de conclusion $A_1, \dots, A_n$ alors il existe une preuve π de $\vdash A_1, \dots, A_n$ dans MLL telle que $\pi \longmapsto \mathcal{R}$

Preuve







$\longleftarrow \overline{\vdash A, A^\perp}\,ax$