Categorical Interpretations of Logics

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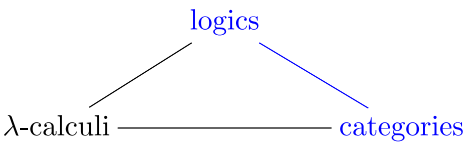
We give the ingredients of the correspondence between proof systems based on sequent calculus on one side, and categories on the other side. These are basic blocks of categorical logic. It is one of the components of the Curry-Howard-Lambek correspondence:



1 Categories

1.1 Identity Logic

Given a set $\mathcal{X}$ of propositional variables (whose elements are denoted X, Y, etc.), we start with a very simple notion of formula:

$$A ::= X$$

Sequents are pairs of formulas denoted $A \vdash B$, and proofs are built using two rules:

$$\frac{}{A \vdash A}\,ax \qquad\qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}\,cut$$

Proposition 1 (Cut Elimination)
Using the following proof transformations:

$$\frac{\dfrac{}{A \vdash A}\,ax \qquad \overset{\pi}{A \vdash B}}{A \vdash B}\,cut \qquad \mapsto \qquad \overset{\pi}{A \vdash B}$$

$$\frac{\overset{\pi}{A \vdash B} \qquad \dfrac{}{B \vdash B}\,ax}{A \vdash B}\,cut \qquad \mapsto \qquad \overset{\pi}{A \vdash B}$$

any proof can be turned into a cut-free one.

Proof. By induction on the number of (cut) rules in the proof by selecting a top-most (cut) rule. □

1.2 Category

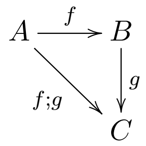
1.2.1 Definitions

Definition 1 (Category)
A *category* $\mathbb{C}$ is given by a class of *objects* $obj(\mathbb{C})$ and, for each pair of objects A and B in $obj(\mathbb{C})$, a class of *morphisms* (or arrows) $\mathbb{C}(A, B)$ from A to B together with:

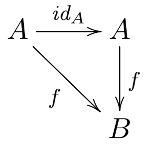
- *identities*: $id_A \in \mathbb{C}(A, A)$ for each object A:

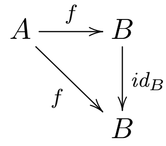
$$A \xrightarrow{id_A} A$$

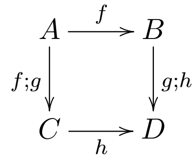
- *composition*: $\mathbb{C}(A, B) \times \mathbb{C}(B, C) \to \mathbb{C}(A, C)$, denoted by $(f, g) \mapsto f \,;\, g$:



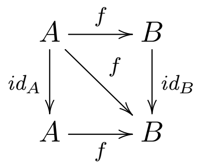
such that the following diagrams commute:

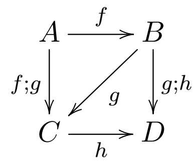






We can "summarize" these four diagrams into:





1.2.2 Category $\mathbb{R}\mathsf{el}$

The category $\mathbb{R}\mathsf{el}$ of relations have sets as objects and given two sets A and B, the morphisms are $\mathbb{R}\mathsf{el}(A, B) := \mathcal{P}(A \times B)$. The identity is the diagonal relation: $id_A = \{(a, a) \mid a \in A\}$. The composition is the composition of relations: $R \,;\, S = \{(a, c) \mid \exists b, (a, b) \in R \wedge (b, c) \in S\}$.

One can chech this is indeed a category since:

$$\begin{aligned}
id_A \,;\, R &= \{(a, b) \mid \exists a', (a, a') \in id_A \wedge (a', b) \in R\} \\
&= \{(a, b) \mid (a, a) \in id_A \wedge (a, b) \in R\} \\
&= R \\
R \,;\, id_B &= \{(a, b) \mid \exists b', (a, b') \in R \wedge (b', b) \in id_B\} \\
&= R \\
R \,;\, (S \,;\, T) &= \{(a, d) \mid \exists b, (a, b) \in R \wedge (b, d) \in S \,;\, T\} \\
&= \{(a, d) \mid \exists b, \exists c, (a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T\} \\
&= \{(a, d) \mid \exists c, (a, c) \in R \,;\, S \wedge (c, d) \in T\} \\
&= (R \,;\, S) \,;\, T
\end{aligned}$$

1.3 Interpretation

Given a category $\mathbb{C}$, and a function $\mathcal{V}$ from $\mathcal{X}$ to $obj(\mathbb{C})$, we interpret:

- each formula A as an object $[\![A]\!]$ of $\mathbb{C}$;
- each proof π of $A \vdash B$ as a morphism $[\![\pi]\!]$ from $[\![A]\!]$ to $[\![B]\!]$.

This is given by:

- $[\![X]\!] = \mathcal{V}(X)$;
- a proof π containing just an (ax) rule with conclusion $A \vdash A$ is interpreted as $[\![A]\!] \xrightarrow{id_{[\![A]\!]}} [\![A]\!]$;
- a proof π with conclusion $A \vdash C$ obtained by applying a (cut) rule to a proof π_1 with conclusion $A \vdash B$ and a proof π_2 with conclusion $B \vdash C$ is interpreted as $[\![A]\!] \xrightarrow{[\![\pi_1]\!]} [\![B]\!] \xrightarrow{[\![\pi_2]\!]} [\![C]\!]$.

Theorem 1 (Soundness)
If π maps to π' by cut elimination (Proposition 1) then $[\![\pi]\!] = [\![\pi']\!]$.

Proof. The two rewriting steps are interpreted as equalities in $\mathbb{C}$ since $id_{[\![A]\!]} ; f = f$ and $f ; id_{[\![B]\!]} = f$. □

2 Monoidal Categories

2.1 Non-Commutative Tensor Logic

We extend the grammar of formulas with a binary connective $\otimes$ and its unit 1:

$$A ::= X \mid A \otimes A \mid 1$$

The shape of sequents is $\Gamma \vdash A$ where Γ is a list of formulas and A is a formula. Proofs are built using the following rules:

$$\frac{}{A \vdash A}\,ax \qquad \frac{\Gamma \vdash A \qquad \Delta, A, \Sigma \vdash B}{\Delta, \Gamma, \Sigma \vdash B}\,cut$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes R \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}\,\otimes L \qquad \frac{}{\vdash 1}\,1R \qquad \frac{\Gamma, \Delta \vdash C}{\Gamma, 1, \Delta \vdash C}\,1L$$

Proposition 2 (Cut Elimination)
Using the following proof transformations:

$$\frac{\dfrac{}{A \vdash A}\,ax \qquad \overset{\pi}{\Gamma, A, \Delta \vdash B}}{\Gamma, A, \Delta \vdash B}\,cut \quad \mapsto \quad \overset{\pi}{\Gamma, A, \Delta \vdash B}$$

$$\frac{\overset{\pi}{\Gamma \vdash B} \qquad \dfrac{}{B \vdash B}\,ax}{\Gamma \vdash B}\,cut \quad \mapsto \quad \overset{\pi}{\Gamma \vdash B}$$

$$\frac{\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\,\otimes R \qquad \dfrac{\overset{\pi_3}{\Sigma, A, B, \Xi \vdash C}}{\Sigma, A \otimes B, \Xi \vdash C}\,\otimes L}{\Sigma, \Gamma, \Delta, \Xi \vdash C}\,cut \quad \mapsto \quad \frac{\overset{\pi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\Delta \vdash B} \qquad \overset{\pi_3}{\Sigma, A, B, \Xi \vdash C}}{\Sigma, A, \Delta, \Xi \vdash C}\,cut}{\Sigma, \Gamma, \Delta, \Xi \vdash C}\,cut$$

$$\frac{\dfrac{}{\vdash 1}\,1R \qquad \dfrac{\overset{\pi}{\Gamma, \Delta \vdash C}}{\Gamma, 1, \Delta \vdash C}\,1L}{\Gamma, \Delta \vdash C}\,cut \quad \mapsto \quad \overset{\pi}{\Gamma, \Delta \vdash C}$$

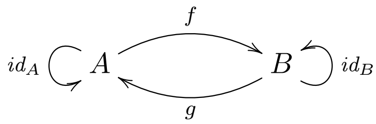
as well as various commutations of (cut) rules with other rules, any proof can be turned into a cut-free one.

2.2 Monoidal Category

2.2.1 Definitions

Definition 2 (Isomorphism)
An *isomorphism* f from the object A to the object B is a morphism from A to B such that there exists a morphism g from B to A (called the *inverse* of f, and often denoted f^{-1}) such that the following diagrams commute:

We can "summarize" these two diagrams into:



Lemma 1 (Inverse of an Isomorphism)
If f is an isomorphism from A to B, then f^{-1} is the unique morphim such that $f\,;f^{-1} = id_A$, f^{-1} is an isomorphism from B to A, and $(f^{-1})^{-1} = f$.

Proof. If we have a morphism g from B to A such that $f\,;g = id_A$, then:

$$g = id_B\,;g = (f^{-1}\,;f)\,;g = f^{-1}\,;(f\,;g) = f^{-1}\,;id_A = f^{-1}$$

The diagrams showing that f is an isomorphism with f^{-1} as an inverse are the same as those required to show that f^{-1} is an isomorphim with f as an inverse. □

Definition 3 (Functor)
A *functor* F between two categories $\mathbb{C}$ and $\mathbb{D}$ is:

- a function from the objects of $\mathbb{C}$ to the objects of $\mathbb{D}$;
- and for each A and B, a function from $\mathbb{C}(A,B)$ to $\mathbb{D}(FA,FB)$

such that the following diagrams in $\mathbb{D}$ commute:

The composition of two functors is a functor.

Definition 4 (Identity Functor)
If $\mathbb{C}$ is a category, the *identity functor* $Id_{\mathbb{C}}$ from $\mathbb{C}$ to $\mathbb{C}$ is defined by:

- for each $A \in obj(\mathbb{C})$, $Id_{\mathbb{C}}A = A$
- if A and B are in $obj(\mathbb{C})$ and $f \in \mathbb{C}(A,B)$, $Id_{\mathbb{C}}f = f$

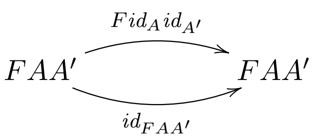
Definition 5 (Product Category)
The *product* $\mathbb{C} \times \mathbb{D}$ of two categories $\mathbb{C}$ and $\mathbb{D}$ is the category with:

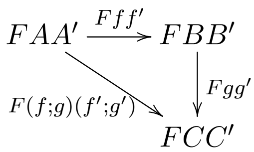
- objects are pairs of objects of $\mathbb{C}$ and objects of $\mathbb{D}$: $obj(\mathbb{C} \times \mathbb{D}) := obj(\mathbb{C}) \times obj(\mathbb{D})$;
- morphisms from (A, A') to (B, B') are pairs of morphisms of $\mathbb{C}$ from A to B and morphisms of $\mathbb{D}$ from A' to B': $\mathbb{C} \times \mathbb{D}((A, A'), (B, B')) = \mathbb{C}(A, B) \times \mathbb{D}(A', B')$;
- identity on (A, A') is the pair $(id_A, id_{A'})$;
- composition of (f, f') and (g, g') is $(f\, ;\, g, f'\, ;\, g')$.

Definition 6 (Bifunctor)
A *bifunctor* from two categories $\mathbb{C}$ and $\mathbb{D}$ to a category $\mathbb{E}$ is a functor from $\mathbb{C} \times \mathbb{D}$ to $\mathbb{E}$.
More concretely, it is given by:

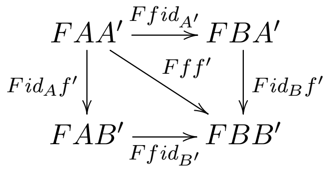
- a function from $obj(\mathbb{C}) \times obj(\mathbb{D})$ to $obj(\mathbb{E})$
- for each A and B in $obj(\mathbb{C})$ and A' and B' in $obj(\mathbb{D})$, a function from $\mathbb{C}(A, B) \times \mathbb{D}(A', B')$ to $\mathbb{E}(FAA', FBB')$

such that the following diagrams in $\mathbb{E}$ commute:





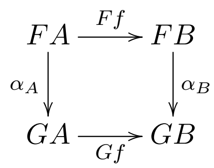
In particular, the following diagram commutes:



If A is an object of $\mathbb{C}$ and F is a bifunctor from $\mathbb{C}$ and $\mathbb{D}$ to $\mathbb{E}$, $FA\_$ is a functor from $\mathbb{D}$ to $\mathbb{E}$ which maps B to FAB and f' to Fid_Af'. As a consequence, one often uses the notations FAf' for Fid_Af' and FfB for $Ffid_B$, if A is an object of $\mathbb{C}$ and B is an object of $\mathbb{D}$.

All this can be generalized to notions of n-ary functors.

Definition 7 (Natural Transformation)
A *natural transformation* α between two functors F and G from a category $\mathbb{C}$ to a category $\mathbb{D}$ is a family $(\alpha_A)_{A \in obj(\mathbb{C})}$ of morphisms from FA to GA such that the following diagram in $\mathbb{D}$ commutes for all $f \in \mathbb{C}(A, B)$:



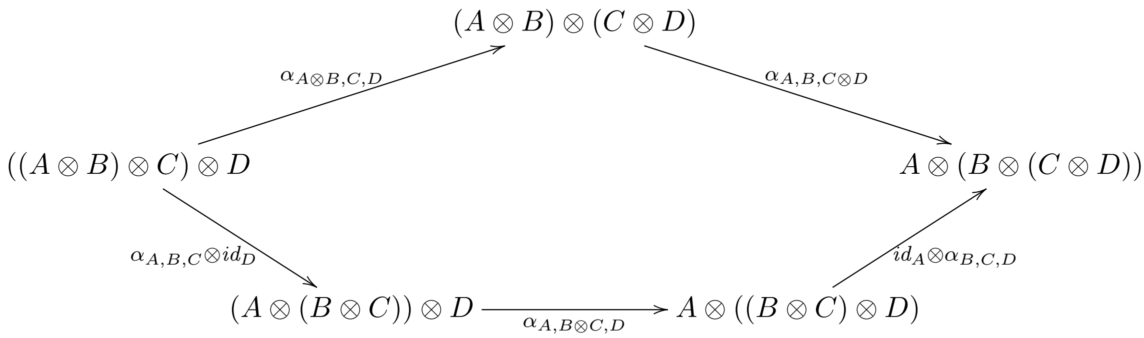
A *natural isomorphism* is a natural transformation α such that each element α_A is an isomorphism.

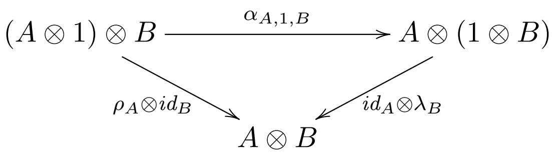
Definition 8 (Monoidal Category)
A *monoidal category* is a 6-tuple $(\mathbb{C}, \otimes, \mathit{1}, \alpha, \lambda, \rho)$ where:

- $\otimes$ is a bifunctor from $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{C}$

- 1 is an object of $\mathbb{C}$
- α is a natural isomorphism from $(\_ \otimes \_') \otimes \_''$ to $\_ \otimes (\_' \otimes \_'')$
- λ is a natural isomorphism from $1 \otimes \_$ to $Id_{\mathbb{C}}$
- ρ is a natural isomorphism from $\_ \otimes 1$ to $Id_{\mathbb{C}}$

such that the following diagrams commute:



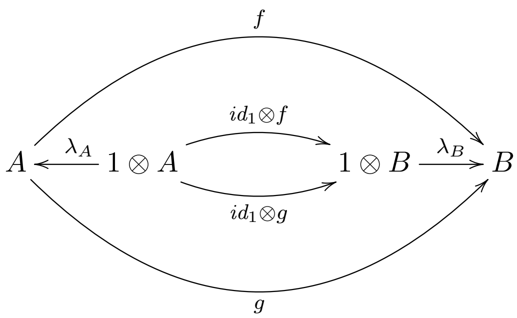


2.2.2 Properties

Let us consider a fixed monoidal category $(\mathbb{C}, \otimes, 1, \alpha, \lambda, \rho)$.

Lemma 2 (Equality up to $\_ \otimes id_1$ and $id_1 \otimes \_$)
Let A and B be two objects of $\mathbb{C}$ and f and g be two morphisms of $\mathbb{C}$ from A to B, $f \otimes id_1 = g \otimes id_1 \iff f = g \iff id_1 \otimes f = id_1 \otimes g$.

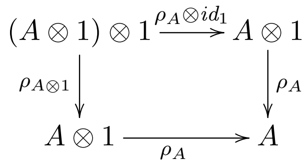
Proof. We have $f = g$ implies both $f \otimes id_1 = g \otimes id_1$ and $id_1 \otimes f = id_1 \otimes g$. Now assume $id_1 \otimes f = id_1 \otimes g$, the following diagram commutes:



since the two squares commute by naturality of λ. We conclude $f = \lambda_A^{-1} ; id_1 \otimes f ; \lambda_B = \lambda_A^{-1} ; id_1 \otimes g ; \lambda_B = g$ since λ_A is an isomorphism. Similarly, we obtain the implication $f \otimes id_1 = g \otimes id_1 \Longrightarrow f = g$ by naturality of ρ. ☐

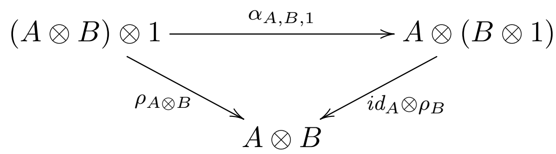
Lemma 3 (Unit of Unit)
Let A be an object of $\mathbb{C}$, $\rho_{A\otimes 1} = \rho_A \otimes id_1 : (A \otimes 1) \otimes 1 \to A \otimes 1$.

Proof. By naturality of ρ, we have:

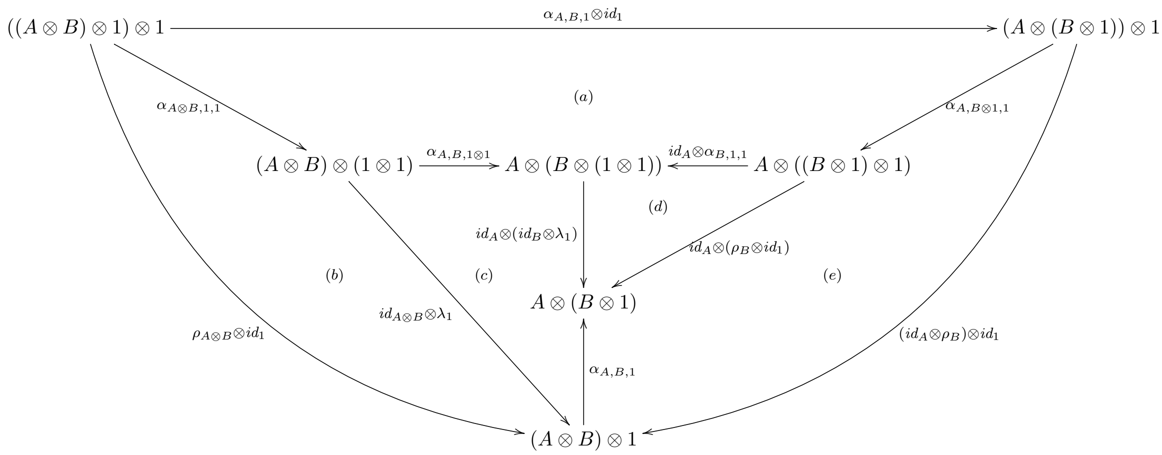


thus, since ρ_A is an isomorphism, $\rho_{A \otimes 1} = \rho_A \otimes id_1$. ☐

Lemma 4 (Associativity of Unit)
Let A and B be two objects of $\mathbb{C}$, the following diagram commutes:



Proof. The following diagram commutes:



by:

(a) pentagon of monoidal categories

(b) triangle of monoidal categories

(c) naturality of α

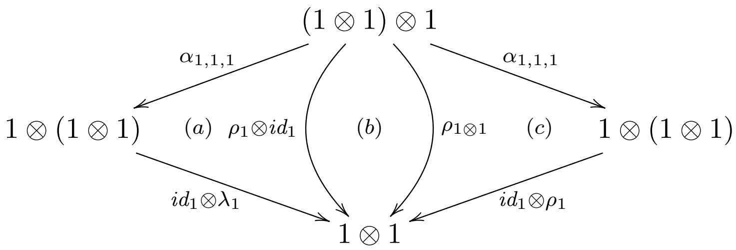
(d) triangle of monoidal categories

(e) naturality of α

And we conclude with Lemma 2 since $\alpha_{A,B,1}$ is an isomorphism. ☐

Lemma 5 (Unit at Unit)
In any monoidal category, $\rho_1 = \lambda_1$.

Proof. The following diagram commutes:



by:

(a) triangle of monoidal categories

(b) Lemma 3

(c) Lemma 4

We thus have $id_1 \otimes \lambda_1 = id_1 \otimes \rho_1$ since $\alpha_{1,1,1}$ is an isomorphism, and finally $\lambda_1 = \rho_1$ by Lemma 2. □

2.2.3 Category $\mathbb{R}$el

Lemma 6 (Bijective Relations)
Let A and B be two sets and R a relation between A and B, R is an isomorphisms if and only if it is the graph of a bijection from A to B.

Proof. If R is the graph of a bijection f, we define $S = \{(b, a) \mid b = f(a)\}$. We have:

$$\begin{aligned} R\,;S &= \{(a, a') \mid \exists b, (a, b) \in R \wedge (b, a') \in S\} \\ &= \{(a, a') \mid \exists b, f(a) = b \wedge b = f(a')\} \\ &= \{(a, a') \mid f(a) = f(a')\} \\ &= id_A \end{aligned}$$

and similarly $S\,;R = id_B$.

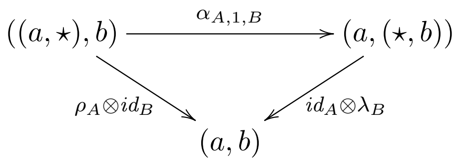
If R is an isomorphism, let S be its inverse. For each $a \in A$, since $(a, a) \in id_A = R\,;S$, there exists $b \in B$ such that $(a, b) \in R$ and $(b, a) \in S$. For any $a \in A$ and $b' \in B$ such that $(a, b') \in R$, we have $(b, b') \in S\,;R = id_B$ thus $b = b'$. This means that for any $a \in A$, there is a unique $b \in B$ such that $(a, b) \in R$. □

The monoidal structure of the category $\mathbb{R}$el is given by:

- The tensor product of two sets A and B is the product $A \otimes B := A \times B$. It is not a cartesian product (in the categorical sense) in the category $\mathbb{R}$el.
- Given two relations R between A and B, and S between A' and B', their tensor product is $R \otimes S := \{((a, a'), (b, b')) \mid (a, b) \in R \wedge (a', b') \in S\}$ between $A \times A'$ and $B \times B'$.
- The unit of $\otimes$ is $1 := \{\star\}$ (a fixed singleton).
- Following Lemma 6, $\alpha_{A,B,C}$ is obtained from the canonical bijection between $(A \times B) \times C$ and $A \times (B \times C)$: $((a, b), c) \mapsto (a, (b, c))$.
- Following Lemma 6, λ_A is obtained from the canonical bijection between $1 \times A$ and A: $(\star, a) \mapsto a$.

- Following Lemma 6, ρ_A is obtained from the canonical bijection between $A \times 1$ and A: $(a, \star) \mapsto a$.

$\otimes$ defines a bifunctor, α, λ and ρ are natural isomorphisms and the two diagrams of monoidal categories commute. For example:



2.3 Interpretation

We extend the approach of Section 1.3. Connectives are interpreted by using functors of the appropriate arity. In particular units are interpreted as objects, and binary connectives by using bifunctors. Rules operate on morphisms by means of the morphism parts of functors and associated natural transformations.

Given a monoidal category $(\mathbb{C}, \otimes, 1, \alpha, \lambda, \rho)$, and a function $\mathcal{V}$ from $\mathcal{X}$ to $obj(\mathbb{C})$, we interpret:

- each formula A as an object $[\![A]\!]$ of $\mathbb{C}$;
- each proof π of $A_1, \ldots, A_k \vdash B$ as a morphism $[\![\pi]\!]$ from 1 to $[\![B]\!]$ if $k = 0$, from $[\![A_1]\!]$ to $[\![B]\!]$ if $k = 1$, and from $[\![A_1]\!] \otimes ([\![A_2]\!] \otimes \cdots ([\![A_{k-1}]\!] \otimes [\![A_k]\!]))$ to $[\![B]\!]$ if $k \geq 2$.

It is important to notice that, thanks to the diagrams in the definition of monoidal categories and properties like Lemmas 2, 3, 4 and 5, different ways of associating $[\![A_1]\!] \otimes \cdots \otimes [\![A_k]\!]$, or of introducing some 1s in such a big $\otimes$, are all related through a unique isormorphism built from α, λ and ρ. As a consequence we will ignore such associativity/unit questions and write $[\![A_1, \ldots, A_k]\!] = [\![A_1]\!] \otimes \cdots \otimes [\![A_k]\!]$.

The interpretation is given by:

- $[\![X]\!] = \mathcal{V}(X)$, $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$, and $[\![1]\!] = 1$;
- a proof π containing just an (ax) rule with conclusion $A \vdash A$ is interpreted as $[\![A]\!] \xrightarrow{id_{[\![A]\!]}} [\![A]\!]$;
- a proof π with conclusion $\Delta, \Gamma, \Sigma \vdash B$ obtained by applying a (cut) rule to a proof π_1 with conclusion $\Gamma \vdash A$ and a proof π_2 with conclusion $\Delta, A, \Sigma \vdash B$ is interpreted as $[\![\Delta]\!] \otimes [\![\Gamma]\!] \otimes [\![\Sigma]\!] \xrightarrow{id_{[\![\Delta]\!]} \otimes [\![\pi_1]\!] \otimes id_{[\![\Sigma]\!]}} [\![\Delta]\!] \otimes [\![A]\!] \otimes [\![\Sigma]\!] \xrightarrow{[\![\pi_2]\!]} [\![B]\!]$.
- a proof π with conclusion $\Gamma, \Delta \vdash A \otimes B$ obtained by applying a $(\otimes R)$ rule to a proof π_1 with conclusion $\Gamma \vdash A$ and a proof π_2 with conclusion $\Delta \vdash B$ is interpreted as $[\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![\pi_1]\!] \otimes [\![\pi_2]\!]} [\![A]\!] \otimes [\![B]\!]$.
- a proof π with conclusion $\Gamma, A \otimes B, \Delta \vdash C$ obtained by applying a $(\otimes L)$ rule to a proof π_1 with conclusion $\Gamma, A, B, \Delta \vdash C$ is interpreted as $[\![\Gamma]\!] \otimes ([\![A]\!] \otimes [\![B]\!]) \otimes [\![\Delta]\!] \xrightarrow{[\![\pi_1]\!]} [\![C]\!]$. Note the use of α is hidden here, thanks to the remark above.
- a proof π with conclusion $\vdash 1$ containing just a $(1R)$ rule is interpreted as $1 \xrightarrow{id_1} [\![1]\!]$.
- a proof π with conclusion $\Gamma, 1, \Delta \vdash C$ obtained by applying a $(1L)$ rule to a proof π_1 with conclusion $\Gamma, \Delta \vdash C$ is interpreted as $[\![\Gamma]\!] \otimes [\![1]\!] \otimes [\![\Delta]\!] \xrightarrow{\rho_{[\![\Gamma]\!]} \otimes id_{[\![\Delta]\!]}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![\pi_1]\!]} [\![C]\!]$.

Theorem 2 (Soundness)
If π maps to π' by cut elimination (Proposition 2) then $[\![\pi]\!] = [\![\pi']\!]$.

3 Symmetry

3.1 Tensor Logic

We simply extend non-commutative tensor logic with an exchange rule:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ ex$$

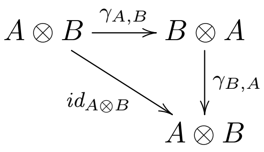
3.2 Symmetric Monoidal Category

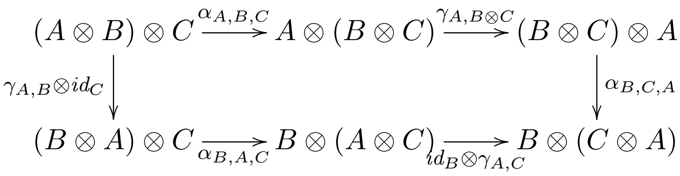
3.2.1 Definition

Definition 9 (Symmetric Monoidal Category)
A *symmetric monoidal category* is a 7-tuple $(\mathbb{C}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ where:

- $(\mathbb{C}, \otimes, 1, \alpha, \lambda, \rho)$ is a monoidal category
- γ is a natural isomorphism from $\_ \otimes \_'$ to $\_' \otimes \_$

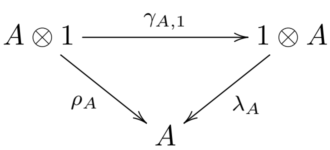
such that the following diagrams commute:



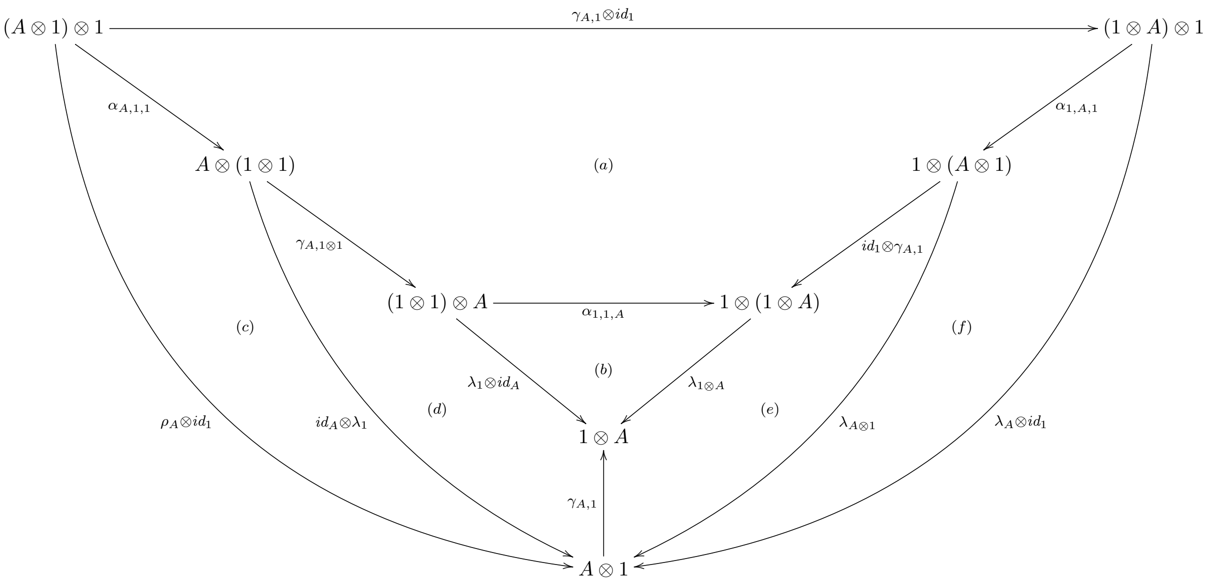


3.2.2 Properties

Lemma 7 (Symmetry of Unit)
In any symmetric monoidal category:



Proof. Thanks to Lemma 2, it is sufficient to prove the commutation of the following diagram (since $\gamma_{A,1}$ is an isomorphism):



which commutes by:

(a) hexagon of symmetric monoidal categories

(b) Lemma 4

(c) triangle of monoidal categories

(d) naturality of γ

(e) naturality of λ

(f) Lemma 4 □

3.2.3 Category $\mathbb{R}\mathsf{el}$

Following Lemma 6, the symmetry $\gamma_{A,B}$ is obtained from the canonical bijection between $A \times B$ and $B \times A$: $(a, b) \mapsto (b, a)$. It satisfies all the required conditions to make $\mathbb{R}\mathsf{el}$ a symmetric monoidal category.

3.3 Intepretation

We extend Section 2.3:

- a proof π with conclusion $\Gamma, B, A, \Delta \vdash C$ obtained by applying an (ex) rule to a proof π_1 with conclusion $\Gamma, A, B, \Delta \vdash C$ is interpreted as:

$$[\![\Gamma]\!] \otimes [\![B]\!] \otimes [\![A]\!] \otimes [\![\Delta]\!] \xrightarrow{id_{[\![\Gamma]\!]} \otimes \gamma_{[\![B]\!],[\![A]\!]} \otimes id_{[\![\Delta]\!]}} [\![\Gamma]\!] \otimes [\![A]\!] \otimes [\![B]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![\pi_1]\!]} [\![C]\!]$$

4 Closure

4.1 Intuitionistic Multiplicative Linear Logic

We extend the grammar of formulas with a binary connective $\multimap$:

$$A ::= X \mid A \otimes A \mid 1 \mid A \multimap A$$

We add the following two rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \multimap R \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta, B, \Sigma \vdash C}{\Delta, A \multimap B, \Gamma, \Sigma \vdash C} \multimap L$$

Proposition 3 (Cut Elimination)
By adding the following proof transformations:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \multimap B} \multimap R \qquad \dfrac{\begin{matrix}\pi_2\\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\pi_3\\ \Sigma, B, \Xi \vdash C\end{matrix}}{\Sigma, A \multimap B, \Delta, \Xi \vdash C} \multimap L}{\Sigma, \Gamma, \Delta, \Xi \vdash C} cut \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma, \Delta \vdash B} cut \qquad \begin{matrix}\pi_3\\ \Sigma, B, \Xi \vdash C\end{matrix}}{\Sigma, \Gamma, \Delta, \Xi \vdash C} cut$$

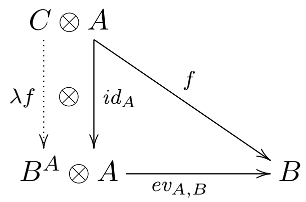
and some commutations of (cut) rules with other rules, any proof can be turned into a cut-free one.

4.2 Symmetric Monoidal Closed Category

4.2.1 Definitions

Definition 10 (Exponential Object)
If A and B are two objects of a symmetric monoidal category $\mathbb{C}$, an *exponential object* of A and B is a pair $(B^A, ev_{A,B})$ where B^A is an object of $\mathbb{C}$ and $ev_{A,B} \in \mathbb{C}(B^A \otimes A, B)$ such that, for any morphism $f \in \mathbb{C}(C \otimes A, B)$, there exists a unique morphism $\lambda f \in \mathbb{C}(C, B^A)$ (called the *curryfication* of f) such that $f = (\lambda f \otimes id_A) \,;\, ev_{A,B}$.

This can be written:



Definition 11 (Symmetric Monoidal Closed Category)
A *symmetric monoidal closed category* is a symmetric monoidal category such that each pair of objects A and B has an associated exponential object $(B^A, ev_{A,B})$.

4.2.2 Category $\mathbb{R}\mathsf{el}$

If A and B are two sets, we define $B^A := A \times B$ and $ev_{A,B} := \{(((a, b), a), b) \mid a \in A \wedge b \in B\}$. Given a relation R between $C \times A$ and B, we define $\lambda R := \{(c, (a, b)) \mid ((c, a), b) \in R\}$. We have:

$$\begin{aligned}(\lambda R \otimes id_A) \,;\, ev_{A,B} &= \{((c, a), b) \mid \exists (a', b'), (c, (a', b')) \in \lambda R \wedge (((a', b'), a), b) \in ev_{A,B}\} \\ &= \{((c, a), b) \mid \exists (a', b'), ((c, a'), b') \in R \wedge a' = a \wedge b' = b\} \\ &= R\end{aligned}$$

We can check it defines an exponential object of A and B.

4.3 Intepretation

We extend Section 3.3:

- formulas are interpreted by using $[\![A \multimap B]\!] := [\![B]\!]^{[\![A]\!]}$
- a proof π with conclusion $\Gamma \vdash A \multimap B$ obtained by applying a $(\multimap R)$ rule to a proof π_1 with conclusion $\Gamma, A \vdash B$ is interpreted as: $[\![\Gamma]\!] \xrightarrow{\lambda[\![\pi_1]\!]} [\![B]\!]^{[\![A]\!]}$
- a proof π with conclusion $\Delta, A \multimap B, \Gamma, \Sigma \vdash C$ obtained by applying a $(\multimap L)$ rule to a proof π_1 with conclusion $\Gamma \vdash A$ and a proof π_2 with conclusion $\Delta, B, \Sigma \vdash C$ is interpreted as:

$$[\![\Delta]\!] \otimes [\![B]\!]^{[\![A]\!]} \otimes [\![\Gamma]\!] \otimes [\![\Sigma]\!] \xrightarrow{id_{[\![\Delta]\!] \otimes [\![B]\!]^{[\![A]\!]}} \otimes [\![\pi_1]\!] \otimes id_{[\![\Sigma]\!]}} [\![\Delta]\!] \otimes [\![B]\!]^{[\![A]\!]} \otimes [\![A]\!] \otimes [\![\Sigma]\!] \xrightarrow{id_{[\![\Delta]\!]} \otimes ev_{[\![A]\!],[\![B]\!]} \otimes id_{[\![\Sigma]\!]}} [\![\Delta]\!] \otimes [\![B]\!] \otimes [\![\Sigma]\!] \xrightarrow{[\![\pi_2]\!]} [\![C]\!]$$

Theorem 3 (Soundness)
If π maps to π' by cut elimination (Proposition 3) then $[\![\pi]\!] = [\![\pi']\!]$.

5 Exponential Co-Monad

5.1 Intuitionistic Multiplicative Exponential Linear Logic

We extend the grammar of formulas with a unary connective !:

$$A ::= X \mid A \otimes A \mid 1 \mid A \multimap A \mid\, !A$$

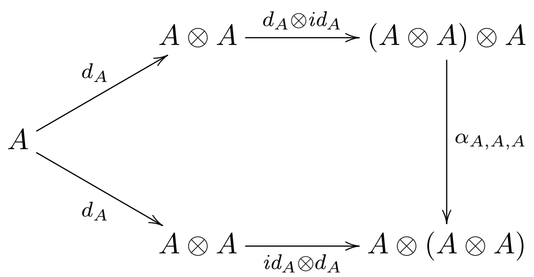
We add the following rules:

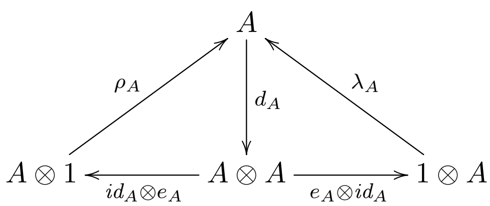
$$\frac{!\Gamma \vdash A}{!\Gamma \vdash\, !A}\; !R \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\; !L \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\; !w \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\; !c$$

5.2 Co-Monads and Co-Monoids

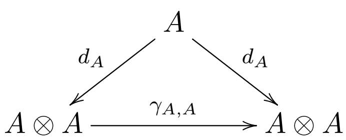
5.2.1 Definitions

Definition 12 (Co-Monoid)
A *co-monoid* in a monoidal category $\mathbb{C}$ is a triple (A, d_A, e_A) with A an object, d_A a morphism from A to $A \otimes A$ and e_A a morphism from A to 1 such that:

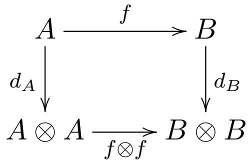


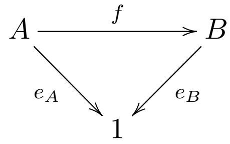


If $\mathbb{C}$ is symmetric monoidal, a co-monoid is *symmetric* if the following diagram commutes:



Definition 13 (Co-Monoidal Morphism)
A *co-monoidal morphism* f between two co-monoids (A, d_A, e_A) and (B, d_B, e_B) in a monoidal category is a morphism from A to B such that the following diagrams commute:

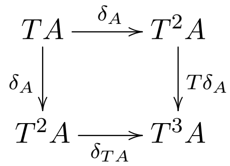


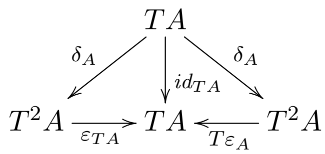


Definition 14 (Co-Monad)
A *co-monad* on a category $\mathbb{C}$ is a triple (T, ε, δ) where:

- T is a functor from $\mathbb{C}$ to $\mathbb{C}$
- ε is a natural transformation from T to $Id_{\mathbb{C}}$
- δ is a natural transformation from T to T^2 (the composition of T with itself)

such that the following diagrams commute:

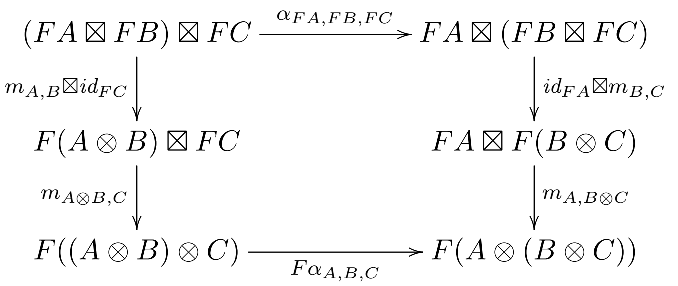


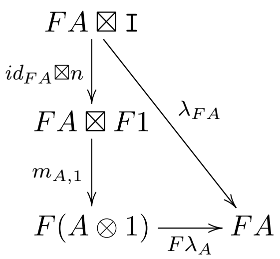


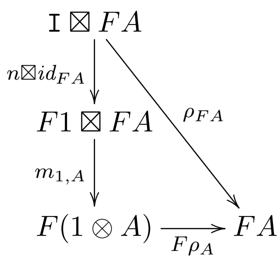
Definition 15 (Monoidal Functor)
A *monoidal functor* between two monoidal categories $(\mathbb{C}, \otimes, 1)$ and $(\mathbb{D}, \boxtimes, \mathtt{I})$ is a triple (F, m, n) where:

- F is a functor from $\mathbb{C}$ to $\mathbb{D}$
- m is a natural transformation from $F\_ \boxtimes F\_'$ to $F(\_ \otimes \_')$
- n is a morphism from $\mathtt{I}$ to $F1$

such that the following diagrams in $\mathbb{D}$ commute:







If $\mathbb{C}$ and $\mathbb{D}$ are symmetric monoidal, a *symmetric monoidal functor* is a monoidal functor such that the following diagram in $\mathbb{D}$ commutes:

$$\begin{array}{ccc} FA \boxtimes FB & \xrightarrow{\gamma_{FA,FB}} & FB \boxtimes FA \\ {\scriptstyle m_{A,B}}\downarrow & & \downarrow{\scriptstyle m_{B,A}} \\ F(A \otimes B) & \xrightarrow[F\gamma_{A,B}]{} & F(B \otimes A) \end{array}$$

Definition 16 (Monoidal Natural Transformation)
A *monoidal natural transformation* α between two monoidal functors F and G between the same two monoidal categories $(\mathbb{C}, \otimes, 1)$ and $(\mathbb{D}, \boxtimes, \mathtt{I})$ is a natural transformation such that the following diagrams in $\mathbb{D}$ commute:

$$\begin{array}{ccc} FA \boxtimes FB & \xrightarrow{m^F_{A,B}} & F(A \otimes B) \\ {\scriptstyle \alpha_A \boxtimes \alpha_B}\downarrow & & \downarrow{\scriptstyle \alpha_{A\otimes B}} \\ GA \boxtimes GB & \xrightarrow[m^G_{A,B}]{} & G(A \otimes B) \end{array} \qquad\qquad \begin{array}{ccc} \mathtt{I} & \xrightarrow{n^F} & F1 \\ & {\scriptstyle n^G}\searrow & \downarrow{\scriptstyle \alpha_1} \\ & & G1 \end{array}$$

Definition 17 (Monoidal Co-Monad)
A co-monad (T, ε, δ) on a monoidal category $\mathbb{C}$ is *monoidal* if T is a monoidal functor, and ε and δ are monoidal natural transformations.

If $\mathbb{C}$ is symmetric monoidal, the co-monad is *symmetric monoidal* if, moreover, T is a symmetric monoidal functor.

5.2.2 Category $\mathbb{R}\mathsf{el}$

If A is a set, $\mathcal{M}_{\text{fin}}(A)$ is the set of finite multisets over A (or sets with repetition, or unordered lists, or functions from A to $\mathbb{N}$ which only have a finite number of elements of A mapped to a non-zero value). $[a_1, \dots, a_k]$ denotes a finite multiset whose elements are $a_1, \dots, a_k$ (thus the order does not matter). The empty multiset is denoted $[\ ]$. The concatenation of two finite multisets μ and ν is denoted $\mu + \nu$.

If R is a relation between A and B, we define:

$$\mathcal{M}_{\text{fin}}(R) := \{([a_1, \dots, a_k], [b_1, \dots, b_k]) \mid \forall 1 \le i \le k, (a_i, b_i) \in R\} \in \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B)$$

This defines a functor from $\mathbb{R}\mathsf{el}$ to $\mathbb{R}\mathsf{el}$, which comes with various interesting morphims and natural transformations:

$$\begin{aligned} m_{A,B} &= \{(([a_1, \dots, a_k], [b_1, \dots, b_k]), [(a_1, b_1), \dots, (a_k, b_k)]) \mid \forall 1 \le i \le k, a_i \in A \wedge b_i \in B\} \\ &\quad \in \mathbb{R}\mathsf{el}(\mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B), \mathcal{M}_{\text{fin}}(A \times B)) \\ n &= \{(\star, [\star])\} \in \mathbb{R}\mathsf{el}(\{\star\}, \mathcal{M}_{\text{fin}}(\{\star\})) \\ \varepsilon_A &= \{([a], a) \mid a \in A\} \in \mathbb{R}\mathsf{el}(\mathcal{M}_{\text{fin}}(A), A) \\ \delta_A &= \left\{ \left(\sum_{i=1}^{k} \mu_i, [\mu_1, \dots, \mu_k] \right) \mid \forall 1 \le i \le k, \mu_i \in \mathcal{M}_{\text{fin}}(A) \right\} \in \mathbb{R}\mathsf{el}(\mathcal{M}_{\text{fin}}(A), \mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}(A))) \\ d_A &= \{(\mu + \nu, (\mu, \nu)) \mid \mu \in \mathcal{M}_{\text{fin}}(A) \wedge \nu \in \mathcal{M}_{\text{fin}}(A)\} \in \mathbb{R}\mathsf{el}(\mathcal{M}_{\text{fin}}(A), \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(A)) \\ e_A &= \{([\ ], \star)\} \in \mathbb{R}\mathsf{el}(\mathcal{M}_{\text{fin}}(A), \{\star\}) \end{aligned}$$

These data satisfy all the properties required for interpreting intuitionistic multiplicative exponential linear logic as in the next section.

5.3 Interpretation

We assume given a symmetric monoidal closed category $(\mathbb{C}, \otimes, 1, \alpha, \lambda, \rho, \gamma, (\_)^{\text{-}}, ev)$ equipped with:

- a symmetric monoidal co-monad $(T, m, n, \varepsilon, \delta)$
- for each object TA, a symmetric co-monoid (TA, d_A, e_A)

such that, for each morphism f from A to B, Tf is a co-monoidal morphism from (TA, d_A, e_A) to (TB, d_B, e_B).

We extend Section 4.3:

- formulas are interpreted by using $[\![!A]\!] := T[\![A]\!]$.
- a proof π with conclusion $\Gamma, !A \vdash B$ obtained by applying a $(!L)$ rule to a proof π_1 with conclusion $\Gamma, A \vdash B$ is interpreted as: $[\![\Gamma]\!] \otimes T[\![A]\!] \xrightarrow{id_{[\![\Gamma]\!]} \otimes \varepsilon_A} [\![\Gamma]\!] \otimes [\![A]\!] \xrightarrow{[\![\pi_1]\!]} [\![B]\!]$.
- a proof π with conclusion $\Gamma, !A \vdash B$ obtained by applying a $(!w)$ rule to a proof π_1 with conclusion $\Gamma \vdash B$ is interpreted as: $[\![\Gamma]\!] \otimes T[\![A]\!] \xrightarrow{id_{[\![\Gamma]\!]} \otimes e_A} [\![\Gamma]\!] \otimes 1 \xrightarrow{\rho_{[\![\Gamma]\!]}} [\![\Gamma]\!] \xrightarrow{[\![\pi_1]\!]} [\![B]\!]$.
- a proof π with conclusion $\Gamma, !A \vdash B$ obtained by applying a $(!c)$ rule to a proof π_1 with conclusion $\Gamma, !A, !A \vdash B$ is interpreted as: $[\![\Gamma]\!] \otimes T[\![A]\!] \xrightarrow{id_{[\![\Gamma]\!]} \otimes d_A} [\![\Gamma]\!] \otimes T[\![A]\!] \otimes T[\![A]\!] \xrightarrow{[\![\pi_1]\!]} [\![B]\!]$.
- a proof π with conclusion $!A_1, \dots, !A_k \vdash !B$ obtained by applying a $(!R)$ rule to a proof π_1 with conclusion $!A_1, \dots, !A_k \vdash B$ is interpreted as:

$$T[\![A_1]\!] \otimes \cdots \otimes T[\![A_k]\!] \xrightarrow{\delta_{[\![A_1]\!]} \otimes \cdots \otimes \delta_{[\![A_k]\!]}} TT[\![A_1]\!] \otimes \cdots \otimes TT[\![A_k]\!]$$

$$TT[\![A_1]\!] \otimes \cdots \otimes TT[\![A_k]\!] \xrightarrow{m_{T[\![A_1]\!],T[\![A_2]\!]} \otimes id_{TT[\![A_3]\!] \otimes \cdots \otimes TT[\![A_k]\!]}} T(T[\![A_1]\!] \otimes T[\![A_2]\!]) \otimes TT[\![A_3]\!] \otimes \cdots \otimes TT[\![A_k]\!]$$

$$\vdots$$

$$T(T[\![A_1]\!] \otimes \cdots \otimes T[\![A_{k-1}]\!]) \otimes TT[\![A_k]\!] \xrightarrow{m_{T[\![A_1]\!] \otimes \cdots \otimes T[\![A_{k-1}]\!], T[\![A_k]\!]}} T(T[\![A_1]\!] \otimes \cdots \otimes T[\![A_k]\!]) \xrightarrow{T[\![\pi_1]\!]} T[\![B]\!]$$

(n is used if $k = 0$).

6 Duality

6.1 Classical Logics

We focus here on Multiplicative Linear Logic with its one-sided presentation. The move from intuitionistic to classical logics as no particular impact on exponentials (simply define $?A$ as $(!A^{\perp})^{\perp}$).

Formulas are given by:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid 1 \mid A \parr A \mid \perp$$

Sequents are $\vdash \Gamma$ where Γ is a list of formulas. Proofs are built using the following rules:

$$\frac{}{\vdash A^{\perp}, A}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ cut \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp$$

6.2 $*$-Autonomous Category

6.2.1 Definition

Definition 18 ($*$-Autonomous Category)
A symmetric monoidal closed category $\mathbb{C}$ is $*$-*autonomous* if it contains a *dualizing object*, that is an object $\perp$ such that, for each object A of $\mathbb{C}$, the following morphism is an isomorphism between A and $(A \multimap \perp) \multimap \perp$:

$$\lambda\left(A \otimes (A \multimap \perp) \xrightarrow{\gamma_{A, A \multimap \perp}} (A \multimap \perp) \otimes A \xrightarrow{ev_{A,\perp}} \perp \right)$$

6.2.2 Category $\mathbb{R}\mathsf{el}$

We define $\perp := \{\star\}$ (a given singleton). We have:

$$(a, (a, \star)) \xrightarrow{\gamma_{A, A \multimap \perp}} ((a, \star), a) \xrightarrow{ev_{A,\perp}} \star$$

Thus its curryfication is the isomorphism $\{(a, ((a, \star), \star)) \mid a \in A\}$.

6.3 Interpretation

Given a $*$-autonomous category $(\mathbb{C}, \otimes, \mathit{1}, \alpha, \lambda, \rho, \gamma, (\_)^{-}, ev, \perp)$, and a function $\mathcal{V}$ from $\mathcal{X}$ to $obj(\mathbb{C})$, we interpret:

- each formula A as an object $[\![A]\!]$ of $\mathbb{C}$;
- each proof π of $\vdash A_1, \ldots, A_k$ as a morphism $[\![\pi]\!]$ from 1 to $(([\![A_1]\!] \multimap \perp) \otimes \cdots \otimes ([\![A_k]\!] \multimap \perp)) \multimap \perp$.

The interpretation of formulas is given by:

- $[\![X]\!] = \mathcal{V}(X)$ and $[\![X^{\perp}]\!] = \mathcal{V}(X) \multimap \perp$;
- $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$ and $[\![A \parr B]\!] = (([\![A]\!] \multimap \perp) \otimes ([\![B]\!] \multimap \perp)) \multimap \perp$;
- $[\![1]\!] = 1$ and $[\![\perp]\!] = \perp$.

As a consequence $[\![A^{\perp}]\!]$ is isomorphic to $[\![A]\!] \multimap \perp$.