Notes on categories
incomplete and never to be finished

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Chapter 1

Category theory I

Categories, functors and natural transformations.

1.1 Categories

A category \( \mathcal{C} \) consists of a class \( \mathcal{O}_\mathcal{C} \) of unstructured objects and a class \( \mathcal{A}_\mathcal{C} \) of arrows of the form \( f : c \rightarrow c' \) where \( f \) is our name for the arrow and \( c \) and \( c' \) are objects; we write \( \text{dom} f \) for \( c \) and \( \text{codom} f \) for \( c' \). We require there to be, for every object \( c \), an identity arrow \( 1_c : c \rightarrow c \) and, for all composable arrows \( f : c \rightarrow c' \) and \( f' : c' \rightarrow c'' \), i.e. where \( \text{codom} f = \text{dom} f' \), a composite arrow \( f' \circ f : c \rightarrow c'' \) satisfying \( f \circ 1_c = f = 1_{c'} \circ f \) and, for all arrows \( f'' : c'' \rightarrow c' \), \( (f'' \circ f') \circ f = f'' \circ (f' \circ f) \). The operation \( \circ \) is the composition law of \( \mathcal{C} \); the two requirements are the identity and the associativity properties.

An arrow \( f : c \rightarrow c' \) is an isomorphism iff, for some \( f' : c' \rightarrow c \), \( f' \circ f = 1_c \) and \( f \circ f' = 1_{c'} \). If \( f' \) exists, it is unique since, for any other candidate \( f'' : c' \rightarrow c \), \( f'' = f'' \circ f \circ f' = f' \). We then say that \( f \) is invertible, or is an isomorphism, and define the operation \( f^{-1} := f' \); we also say that the objects \( c \) and \( c' \) are isomorphic [in the category \( \mathcal{C} \)], written \( c \cong c' \) [or \( f : c \cong c' \) if we wish to stress the specific witness \( f \)].

A subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) consists of a subclass \( \mathcal{O}_\mathcal{C}' \) of \( \mathcal{O}_\mathcal{C} \) and a subclass \( \mathcal{A}_\mathcal{C}' \) of \( \mathcal{A}_\mathcal{C} \) such that, for all arrows \( f \) in \( \mathcal{A}_\mathcal{C}' \), \( \text{dom} f \) and \( \text{codom} f \) are both in \( \mathcal{O}_\mathcal{C}' \), for all objects \( c \) in \( \mathcal{O}_\mathcal{C}' \), \( \mathcal{A}_\mathcal{C}' \) contains \( 1_c \) and, for all arrows \( f : c \rightarrow c' \) and \( f' : c' \rightarrow c'' \) in \( \mathcal{A}_\mathcal{C}' \), their composite \( f' \circ f \) is also in \( \mathcal{A}_\mathcal{C}' \). The identities and composition law of \( \mathcal{C}' \) are inherited from \( \mathcal{C} \), i.e. \( 1_c' := 1_c \), for all \( c \) in \( \mathcal{O}_\mathcal{C}' \), and \( f' \circ f := f' \circ f \), for all \( f : c \rightarrow c' \) and \( f' : c' \rightarrow c'' \) in \( \mathcal{A}_\mathcal{C}' \).
The isomorphisms of \( C \) form a subcategory: if \( f : c \cong c' \) and \( f' : c' \cong c'' \) then \( f' \circ f : c \cong c'' \) since, setting \((f' \circ f)^{-1} := f^{-1} \circ f'^{-1}, \) we have \((f' \circ f)^{-1} \circ (f' \circ f) = f^{-1} \circ (f'^{-1} \circ f' \circ f) \circ f = 1_c \) and, similarly, \((f' \circ f) \circ (f^{-1} \circ f'^{-1}) = 1_{c''}. \)

‘The’ category generally known as \( \textbf{Set} \) has all sets as objects and all total functions between them as arrows [where ‘all’ depends on your choice of set theory].

An arrow \( f : c \to c' \) of \( \textbf{Set} \) is an isomorphism if, and only if, \( f \) [viewed as a set-theoretic function] is a bijection; so, in particular, \( c \) and \( c' \) [viewed as sets] are isomorphic.

The category \( C \) is \textit{small} iff its class of arrows is a set; its collection of objects is then necessarily also a set. A category where all the arrows are identity arrows is called \textit{discrete}. A small discrete category is a set.

The category \( C \) is \textit{locally small} iff, for all pairs of objects \( c \) and \( c' \), the class \( \text{hom}(c, c') \) of all arrows \( f : c \to c' \) is an object of \( \textbf{Set} \), \textit{i.e.} actually a set, not a class.

A small category with one object, \textit{i.e.} where all arrows are composable, is a \textit{monoid}. A monoid where all arrows are invertible is a \textit{group}. A small category where all arrows are invertible is a \textit{groupoid}.

More generally, if \( C \) is a locally small category containing objects \( c \) and \( c' \), we define \( \text{Iso}(c, c') \) to be the set of all isomorphisms \( f : c \cong c' \) and write \( \text{Aut}(c) \) for \( \text{Iso}(c, c) \). Given \( f \in \text{Iso}(c, c') \), define a total function from \( g \in \text{Aut}(c) \) to \( \text{Iso}(c, c') \) by \( g \mapsto f \circ g \). This is an isomorphism in \( \textbf{Set} \) witnessed by the total function from \( f' \in \text{Iso}(c, c') \) to \( \text{Aut}(c) \) defined as \( f' \mapsto f'^{-1} \circ f' \); clearly \( g \mapsto f'^{-1} \circ (f \circ g) = g \) and \( f' \mapsto f \circ (f'^{-1} \circ f') = f' \). So, provided that \( c \cong c' \), there are always exactly as many automorphisms of \( c \) (or indeed \( c' \)) as there are witnesses of the isomorphism of \( c \) and \( c' \).

The category of groups and group homomorphisms is called \( \textbf{Grp} \); that of graphs and graph homomorphisms is called \( \textbf{Grph} \).

The \textit{opposite} category \( C^{\text{op}} \) of the category \( C \) is defined to have the same objects as \( C \) and the ‘same’ arrows as \( C \) but going in the other direction: \( f^{\text{op}} : c \to c' \) is an arrow of \( C^{\text{op}} \) iff \( f : c' \to c \) is an arrow of \( C \). Concomitantly, \( f^{\text{op}} \circ f'^{\text{op}} := (f \circ f')^{\text{op}} \) and \( 1_c^{\text{op}} := 1_{c^{\text{op}}} \).

The \textit{product category} \( C_1 \times C_2 \) of the categories \( C_1 \) and \( C_2 \) has as objects all pairs \( \langle c_1, c_2 \rangle \), where \( c_1 \) is an object of \( C_1 \) and \( c_2 \) is an object of \( C_2 \), and as arrows all pairs \( \langle f_1, f_2 \rangle : \langle c_1, c_2 \rangle \to \langle c'_1, c'_2 \rangle \), where \( f_1 : c_1 \to c'_1 \) is an arrow of \( C_1 \) and \( f_2 : c_2 \to c'_2 \) is an arrow of \( C_2 \). Concomitantly, composable arrows \( \langle f_1, f_2 \rangle \) and \( \langle f'_1, f'_2 \rangle \) are composed component-wise, \textit{i.e.} \( \langle f'_1, f'_2 \rangle \circ_{1 \times 2} \langle f_1, f_2 \rangle := \langle f'_1 \circ f_1, f'_2 \circ f_2 \rangle \), and \( 1_{\langle c_1, c_2 \rangle}^{1 \times 2} := \langle 1_{c_1}, 1_{c_2} \rangle \).
1.2 Functors

A functor $F$ from the category $C$ to the category $C'$ consists of two mappings [one sending $c$ in $O_C$ to $Fc$ in $O_{C'}$ and the other sending $f : c \to c'$ in $A_C$ to $Ff : Fc \to Fc'$ in $A_{C'}$] satisfying, for all objects $c$ of $C$, $F1_c = 1'_{Fc}$ and, for all arrows $f : c \to c'$ and $f' : c' \to c''$ of $C$, $F(f' \circ f) = Ff' \circ Ff$.

A functor from a monoid [viewed as the category $C$] to a second monoid [viewed as the category $C'$] is a standard monoid homomorphism. Likewise for groups and groupoids.

If the arrow $f : c \to c'$ of $C$ is invertible then $(Ff)^{-1} := Ff^{-1}$ inverts $Ff : Fc \to Fc'$ in $C'$ since $Ff^{-1} \circ Ff = F(f^{-1} \circ f) = F1c = 1'_{Fc}$ and $Ff \circ Ff^{-1} = F(f \circ f^{-1}) = F1_{c'}$.

An alternative possible definition of functor would specify only the arrow mapping: any mapping $F$ from the arrows of $C$ to the arrows of $C'$ such that, for all arrows $f_1 : c_1 \to c'_1$ and $f_2 : c_2 \to c'_2$ of $C$, $dom f_1 = dom Ff_1$, $codom f_1 = codom Ff_1$, $dom f_2 = dom Ff_2$, and $codom f_2 = codom Ff_2$ immediately induces a functor; the induced object mapping can be defined as $Fc := c'$ iff $F1_c = 1'_{Fc}$.

A third (and final) possible definition of functor is as a family of functions, $F_{c,c'} : \text{hom}(c,c') \to \text{hom}(Fc,Fc')$, indexed by all pairs of objects of $C$.

The functor $F : C \to C'$ is (i) faithful iff, for every pair $c,c'$ of objects of $C$, $F_{c,c'}$ is injective; (ii) full iff, for every pair $c,c'$ of objects of $C$, $F_{c,c'}$ is surjective; and (iii) essentially surjective iff, for every object $c'$ of $C'$, there is some object $c$ of $C$ such that $Fc \cong c'$. A full and faithful functor is sometimes called fully faithful.

If $C'$ is a subcategory of $C$, the inclusion functor $I : C' \to C$ sends each object and arrow of $C'$ to ‘itself’ in $C$. This functor is always faithful. If it is full, we say that $C'$ is a full subcategory of $C$; a full subcategory is therefore uniquely determined by its subclass $O_{C'}$ of objects.

A category $C$ is concrete iff there is a faithful functor $U : C \to \text{Set}$.

A locally small category $C$ induces a hom functor $\text{hom}_C : C^{op} \times C \to \text{Set}$ where $\langle c,c' \rangle \mapsto \text{hom}(c,c')$ and $\langle f_1^{op}, f_2 \rangle : \langle c_1, c_2 \rangle \to \langle c'_1, c'_2 \rangle \mapsto (f : c_1 \to c_2 \mapsto f_2 \circ f \circ f_1) : \text{hom}(c_1, c_2) \to \text{hom}(c'_1, c'_2)$:

- Clearly, $\text{hom}_C 1_{\langle c_1,c_2 \rangle} = (f : c_1 \to c_2 \mapsto 1_{c_2} \circ f \circ 1_{c_1}) = 1_{\text{hom}_{\langle c_1,c_2 \rangle}}$.
- Moreover, given additionally $\langle f_1^{op}, f_2 \rangle : \langle c_1', c_2' \rangle \to \langle c'_1, c'_2 \rangle$, we have that $\text{hom}_C(\langle f_1^{op}, f_2 \rangle \circ \langle f_1^{op}, f_2 \rangle) = (f : c_1 \to c_2 \mapsto (f_2 \circ f_2) \circ f \circ (f_1 \circ f_1)) = (f : c_1 \to c_2 \mapsto f_2 \circ (f_2 \circ f \circ f_1) \circ f_1) = \text{hom}_C(\langle f_1^{op}, f_2 \rangle \circ \text{hom}_C(\langle f_1^{op}, f_2 \rangle)$.
The functors $F : C \to C'$ and $F' : C' \to C''$ can be composed by defining $c \mapsto F'(Fc)$, for all objects $c$ of $C$, and $f \mapsto F'(Ff)$, for all arrows $f : c \to c'$ of $C$. This yields a functor $F' \circ F : C \to C''$ since (i) $(F' \circ F)c = F'1_{Fc} = 1_{(F'F)c}$; and, given $f' : c' \to c''$ of $C$, (ii) $(F' \circ F)(f' \circ f) = F'(F(f' \circ f)) = F'(Ff') \circ F'(Ff) = (F' \circ F)f' \circ (F' \circ F)f$.

The identity functor $1_C : C \to C$ sends every object and arrow to itself; clearly $F1_C = F = 1_{C'}F$. Moreover, given a third functor $F'' : C'' \to C'''$, clearly $((F'' \circ F') \circ F) = (F'' \circ (F' \circ F))$. So we have a category, known as $\mathbf{Cat}$, with objects all small categories and arrows all functors between them; two small categories are isomorphic iff they are isomorphic in $\mathbf{Cat}$.

### 1.3 Natural transformations

A natural transformation $\alpha : F \to F'$ from the functor $F : C \to C'$ to the functor $F' : C \to C'$ is a family of arrows $\alpha_c :Fc \to F'c$ of $C'$, indexed by the objects of $C$, such that

$$
\begin{array}{ccc}
Fc & \xrightarrow{\alpha_c} & F'c \\
Ff & \downarrow & F'f \\
Fc' & \xrightarrow{\alpha_{c'}} & F'c'
\end{array}
$$

commutes for all arrows $f : c \to c'$ of $C$.

The natural transformation $\alpha : F \to F'$ is a natural isomorphism iff, for every object $c$ of $C$, the arrow $\alpha_c :Fc \to F'c$ is an isomorphism. This is equivalent to saying that there is a natural transformation $\alpha^{-1} : F' \to F$ satisfying, for all objects $c$ of $C$, $\alpha^{-1}_c \circ \alpha_c = 1_{Fc}$ and $\alpha_c \circ' \alpha'_c = 1_{F'c}$.

Given a further natural transformation $\alpha' : F' \to F''$, where $F'' : C \to C'$ is a third functor, the vertical composite of $\alpha$ and $\alpha'$, defined component-wise as $(\alpha' \bullet \alpha)_c := \alpha'_c \circ' \alpha_c$, is clearly a natural transformation $\alpha' \bullet \alpha : F \to F''$. The identity natural transformation $1_F : F \to F$, defined as $(1_F)_c := 1_{Fc}$, satisfies $\alpha \bullet 1_F = \alpha = 1_{F'} \bullet \alpha$ and, given a fourth functor $F'''$ and a third natural transformation $\alpha'' : F'' \to F'''$, we have $(\alpha'' \bullet \alpha') \bullet \alpha = \alpha'' \bullet (\alpha' \bullet \alpha)$. Provided no ‘problems of size’ arise, we thus obtain a functor category $C^{C'}$ with objects all functors from $C$ to $C'$ and arrows all natural transformations between these functors; it is sufficient that $C$ be a small category.

Let $1$ and $2$ be the (small) discrete categories with one and two objects respectively. Clearly $C^1 \cong C$ and $C^2 \cong C \times C$ in $\mathbf{Cat}$. 

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The categories $C$ and $C'$ are equivalent iff there exist functors $F : C \to C'$
and $G : C' \to C$ and natural isomorphisms $\varepsilon : FG \cong 1_{C'}$ and $\eta : 1_C \cong GF$.

Since $\varepsilon_{c'} : F(Gc') \cong c'$, for all $c'$ in $O_{C'}$, the functor $F$ is essentially
surjective. Moreover, for any arrow $f : c_1 \to c_2$ of $C$, $f = \eta_{c_2} \circ GFf \circ \eta_{c_1}^{-1}$ and
$GFf = \eta_{c_2}^{-1} \circ f \circ \eta_{c_1}$, so that $\hom(c_1, c_2)$ is in bijection with $\hom(GFc_1, GFc_2)$.

[If $C$ is locally small, there is therefore an isomorphism in $\textbf{Set}$ witnessing
$\hom(c_1, c_2) \cong \hom(GFc_1, GFc_2)$.] So $F_{c_1, c_2}$ must be injective and $G_{Fc_1,Fc_2}$
must be surjective, for all pairs of objects $c_1, c_2$, i.e. $F$ is faithful and $G$ is
full. The symmetric argument establishes that $G$ is essentially surjective and
faithful and that $F$ is full.

The functor $F : C \to C'$ is a weak equivalence iff, for some $G : C' \to C$,
there exist natural isomorphisms $\epsilon : FG \cong 1_{C'}$ and $\eta : 1_C \cong GF$. If $F$ is
a weak equivalence then, from the above, we know that it is fully faithful
and essentially surjective; the converse is also true—with the caveat that it
depends on the axiom of choice:

Suppose that the functor $F : C \to C'$ is fully faithful and essentially
surjective. By essential surjectivity, for any $c'$ in $O_{C'}$, there is at least one
$c$ in $O_C$ such that $Fc \cong c'$; an application of the axiom of choice then picks
out a choice of $c$, for each $c'$ in $O_{C'}$, allowing us to define $Gc' := c$.

1.4 Cat as a 2-category

Given functors $F'_1, F'_2 : C' \to C''$, a natural transformation $\alpha' : F'_1 \to F'_2$ and
functors $F : C \to C'$ and $F'' : C'' \to C'''$, we define, for each object $c$ of $C$, an
arrow $(F'' \circ \alpha' \circ F)_c := F''(\alpha'_F_c)$ of $C'''$. This defines a natural transformation
$F'' \circ \alpha' \circ F : F'' \circ F'_1 \circ F \to F'' \circ F'_2 \circ F$ since, for any $f : c \to c'$ in $C$,

$$F''(F'_1(Fc)) \xrightarrow{F''(\alpha'_F_c)} F''(F'_2(Fc))$$

commutes (because $\alpha'$ is a natural transformation and $F''$ is a functor).

This hybrid composition of two functors and a natural transformation is
usually called whiskering. It is a special case of the horizontal composition of
natural transformations if we replace the functors $F$ and $F''$ by the identity
natural transformations $1_F : F \to F$ and $1_{F''} : F'' \to F''$ respectively:
Given functors $F_1, F_2 : C \to C'$ and $F'_1, F'_2 : C' \to C''$ together with natural transformations $\alpha : F_1 \to F_2$ and $\alpha' : F'_1 \to F'_2$, the horizontal composite $\alpha' \circ \alpha : F'_1 \circ F_1 \to F'_2 \circ F_2$ of $\alpha$ and $\alpha'$ is defined, for each object $c$ of $C$, to be the diagonal of

$$
\begin{align*}
F'_1(F_1c) & \xrightarrow{\alpha'_{F_1c}} F'_2(F_1c) \\
F'_1(F_2c) & \xrightarrow{\alpha'_{F_2c}} F'_2(F_2c)
\end{align*}
$$

(this square necessarily commutes because $\alpha'$ is a natural transformation). This defines a natural transformation since, for any arrow $f : c \to c'$ in $C$, the two internal squares of

$$
\begin{array}{ccc}
F'_1(F_1c) & \xrightarrow{\alpha'_{F_1c}} & F'_2(F_1c) \\
\downarrow F'_1(F_1f) & & \downarrow F'_2(F_1f) \\
F'_1(F_1c') & \xrightarrow{\alpha'_{F_1c'}} & F'_2(F_1c')
\end{array}
$$

commute (because $\alpha'$ and $\alpha$ are natural transformations and $F'_2$ is a functor) and so the outer square commutes as required.

Horizontal composition has identities, specifically the identity natural transformations $1_{C'}$ and $1_{C''}$ for the identity functors $1_{C'}$ and $1_{C''}$. (Note that this differs from the identities for vertical composition.) It is also (strictly) associative as all the faces of the cube below commute.
A 2-category $\mathbf{C}$ consists of a class $\mathcal{O}_\mathbf{C}$ of objects, also called 0-cells, where (i) for each ordered pair $c, c'$ of 0-cells, there is a category $\text{hom}_{\mathbf{C}}(c, c')$ whose objects and arrows are called 1-cells and 2-cells respectively; the composition of 2-cells $\alpha_1 : f_1 \to f_2$ and $\alpha_2 : f_2 \to f_3$ is called vertical composition and is denoted by $\alpha_2 \circ \alpha_1$; (ii) for each 0-cell $c$, there is a functor $I_c : \mathbf{1} \to \text{hom}_{\mathbf{C}}(c, c)$; and (iii) for each ordered triple $c, c', c''$ of 0-cells, there is a functor $C_{c,c',c''} : \text{hom}_{\mathbf{C}}(c', c'') \times \text{hom}_{\mathbf{C}}(c, c') \to \text{hom}_{\mathbf{C}}(c, c'')$.

These data will be required to satisfy further conditions but let us first unpack what they mean: (i) the 1-cells of $\text{hom}_{\mathbf{C}}(c, c')$ are the ‘arrows’ of $\mathbf{C}$ from $c$ to $c'$; the 2-cells of $\text{hom}_{\mathbf{C}}(c, c')$ are ‘arrows between arrows’; (ii) $I_c$ picks out a 1-cell $1$ and its identity arrow $1_c$ in $\text{hom}_{\mathbf{C}}(c, c)$; this 1-cell will be the ‘identity arrow’ for $c$ in $\mathbf{C}$; (iii) $C_{c,c',c''}$ defines the horizontal composition of 1- and 2-cells; its object part takes ‘composable arrows’ $f : c \to c'$ and $f' : c' \to c''$ to $f' \circ f := C_{c,c',c''}(f', f) : c \to c''$; and its arrow part takes 2-cells $\alpha : f_1 \to f_2$ and $\alpha' : f'_1 \to f'_2$ between ‘composable arrows’ $f_1, f_2 : c \to c'$ and $f'_1, f'_2 : c' \to c''$ to $\alpha' \circ \alpha := C_{c,c',c''}(\alpha', \alpha) : f'_1 \circ f_1 \to f'_2 \circ f_2$; and (iv) finally, functoriality of $C_{c,c',c''}$ imposes the interchange law relating the vertical and horizontal compositions of 2-cells $\alpha_1 : f_1 \to f_2$, $\alpha_2 : f_2 \to f_3$, $\alpha'_1 : f'_1 \to f'_2$ and $\alpha'_2 : f'_2 \to f'_3$ between the 1-cells $f_1, f_2, f_3 : c \to c'$ and $f'_1, f'_2, f'_3 : c' \to c''$:

$$(\alpha'_2 \circ \alpha'_1) \circ (\alpha_2 \circ \alpha_1) = (\alpha'_2 \circ \alpha_2) \circ (\alpha'_1 \circ \alpha_1).$$

We complete the definition of 2-category by asking that (i) for any ordered quadruple of 0-cells $c, c', c'', c'''$ together with 1-cells $f_1, f_2 : c \to c'$, $f'_1, f'_2 : c' \to c''$ and 2-cells $\alpha : f_1 \to f_2$, $\alpha' : f'_1 \to f'_2$ and $\alpha'' : c' \to c'''$, we have $f''_1 \circ (f'_1 \circ f_1) = (f''_1 \circ f'_1) \circ f_1$ (and likewise for $f_2$, $f''_2$ and $f''_3$) and $\alpha'' \circ (\alpha' \circ \alpha) = (\alpha'' \circ \alpha'') \circ \alpha$; and (ii) for any 0-cells $c$ and $c'$ together with 1-cells $f_1, f_2 : c \to c'$ and a 2-cell $\alpha : f_1 \to f_2$, we have $f_1 \circ 1_c = f_1 = 1_{c'} \circ f_1$ (and likewise for $f_2$) and $\alpha \circ 1_c = \alpha = 1_{c'} \circ \alpha$.

These conditions guarantee that, in accordance with the above intuition, the 0-cells and 1-cells of $\mathbf{C}$ are indeed the objects and the arrows of a category.

The category $\mathbf{Cat}$ can be given the structure of a 2-category by setting $\mathcal{O}_{\mathbf{Cat}}$ to be the class of small categories; then (i) $\text{hom}_{\mathbf{Cat}}(c, c') := c^c$, the functor category from $c$ to $c'$; (ii) $I_c$ selects the identity functor $1_c$ on $c$ and its identity natural transformation $1_{1_c}$; and (iii) $C_{c,c',c''}(F', F) := F' \circ F$ and $C_{c,c',c''}(\alpha', \alpha) := \alpha' \circ \alpha$, the horizontal composition of natural transformations.

We have already proved above that these data satisfy all the conditions required of a 2-category. Clearly, the induced category of 0-cells and 1-cells is just $\mathbf{Cat}$. 7
1.5 Equivalences in a 2-category

In a category, we have a notion of isomorphism of objects but not of arrows. In a 2-category, we say that a 2-cell $\alpha : f_1 \to f_2$ is a 2-isomorphism [or just an isomorphism when we can get away with it] of the 1-cells $f_1, f_2 : c \to c'$ iff there exists a 2-cell $\alpha' : f_2 \to f_1$ such that $\alpha' \circ \alpha = 1_{f_1}$ and $\alpha \circ \alpha' = 1_{f_2}$. As $\alpha$ and $\alpha'$ are just isomorphisms in a category, i.e. $\text{hom}(c, c')$, $\alpha'$ is unique and we define $\alpha^{-1} := \alpha'$.

If $\alpha_1 : f_1 \to f_2$ and $\alpha_2 : f_2 \to f_3$ are 2-isomorphisms for $f_1, f_2, f_3 : c \to c'$ then $(\alpha_2 \circ \alpha_1)^{-1} := \alpha_1^{-1} \circ \alpha_2^{-1}$.

If $\alpha : f_1 \to f_2$ and $\alpha' : f_1' \to f_2'$ are 2-isomorphisms for $f_1, f_2 : c \to c'$ and $f_1', f_2' : c' \to c''$ then $(\alpha'^{-1} \circ \alpha^{-1}) \circ (\alpha' \circ \alpha) = (\alpha'^{-1} \circ \alpha') \circ (\alpha^{-1} \circ \alpha) = 1_{f_1'} \circ 1_{f_1} = \text{hom}(f_1, f_2 \circ f_1')$ and $(\alpha' \circ \alpha) \circ (\alpha'^{-1} \circ \alpha^{-1}) = (\alpha' \circ \alpha'^{-1}) \circ (\alpha \circ \alpha^{-1}) = 1_{f_2'} \circ 1_{f_2} = 1_{f_2'} \circ f_2$; so we can define $(\alpha' \circ \alpha)^{-1} := \alpha'^{-1} \circ \alpha^{-1}$ [beware the subtle trap].

The 2-isomorphisms of $\text{Cat}$ are precisely natural isomorphisms: clearly, any 2-isomorphism defines a natural isomorphism; conversely, each $\alpha : F_1 c \to F_2 c$ of a natural isomorphism $\alpha : F_1 \to F_2$ [of functors $F_1, F_2 : \text{Cat} \to \text{Cat}$] is invertible, so $\alpha^{-1} \circ \alpha = 1_{F_1 c}$ and $\alpha \circ \alpha^{-1} = 1_{F_2 c}$, i.e. $\alpha' \circ \alpha = 1_{F_1}$ and $\alpha \circ \alpha' = 1_{F_2}$ as required.

An equivalence in a 2-category consists of 1-cells $f : c \to c'$ and $f' : c' \to c$ and 2-isomorphisms $\eta : 1_c \to f' \circ f$ and $\varepsilon : f \circ f' \to 1_c$. An equivalence in $\text{Cat}$ is precisely an equivalence of categories as defined previously.

An equivalence is adjoint iff the so-called ‘triangle identities’ hold:

$$
\begin{array}{ccc}
  f' & \xrightarrow{\eta \circ \varepsilon} & f' \\
  1_{f'} \circ f' \circ f & \xrightarrow{\varepsilon \circ 1_f} & f' \\
  f & \xrightarrow{1_{f}} & f \\
\end{array}
$$

If $(f, f', \eta, \varepsilon)$ is an adjoint equivalence then so is $(f', f, \varepsilon^{-1}, \eta^{-1})$: $1_{f'} = 1_{f'} = ((\eta \circ 1_f \circ (1_{f'} \circ \varepsilon))^{-1} = (\eta \circ 1_f \circ (1_{f'} \circ \varepsilon))^{-1} = (\eta \circ 1_f \circ (1_{f'} \circ \varepsilon))^{-1}$; and $1_f = (1_f \circ \eta)^{-1} \circ (\varepsilon \circ 1_f)^{-1} = (1_f \circ \eta^{-1}) \circ (\varepsilon^{-1} \circ 1_f)$.

If $(f, f', \eta, \varepsilon)$ is an equivalence then either triangle identity holds if, and only if, the other one does: ...

If $(f, f', \eta, \varepsilon)$ is an equivalence then there exist $f'' : c' \to c$ and $\varepsilon' : f \circ f'' \to 1_c$ such that $(f, f'', \eta, \varepsilon')$ is an adjoint equivalence: ...
Chapter 2

Category theory II

Diagrams, limits, comma categories, universal arrows.

2.1 Categories of diagrams

A diagram [more properly, a J-diagram] in $\mathsf{C}$ is a functor $F : J \to \mathsf{C}$ where $J$ is a small, often even finite, category.

A cone to $F$ is an object $c$ of $\mathsf{C}$ together with arrows $\alpha_j : c \to Fj$ of $\mathsf{C}$, where $j$ ranges over the objects of $J$, such that $Ff \circ \alpha_j = \alpha_{j'}$ for all arrows $f : j \to j'$ of $J$. A cone is thus a natural transformation from the constant functor $\Delta_c : J \to \mathsf{C}$ [defined by $\Delta_c(j) := c$ for all objects $j$ of $J$; and $\Delta_c(f : j \to j') := 1_c$ for all arrows $f$ of $J$] to $F$.

We call the functor category $\mathsf{C}^J$ the category of $J$-diagrams in $\mathsf{C}$; a cone to $F$ is thus an arrow of $\mathsf{C}^J$ of the form $\alpha : \Delta_c \to F$. The category $\mathsf{C}^J/F$ of $J$-diagrams over $F$ is defined to have arrows of $\mathsf{C}^J$ of the form $\alpha : G \to F$ [any $G$] as objects; and arrows of $\mathsf{C}^J$ of the form $\beta : G \to G'$, such that

\[
\begin{array}{ccc}
G & \xrightarrow{\beta} & G' \\
\alpha \downarrow & & \downarrow \alpha' \\
F & & F'
\end{array}
\]

commutes, as arrows.

A cone $\nu : \Delta_u \to F$ to $F$ is universal iff, for any cone $\alpha : \Delta_c \to F$, there is a unique arrow $f : c \to u$ of $\mathsf{C}$ such that $\nu_j \circ f = \alpha_j$ for all objects $j$ of $J$.

A universal cone to $F$, if it exists, is called a limit of [the diagram] $F$ and is unique up to unique isomorphism.
2.2 Comma categories

If $F_1 : C_1 \to C$ and $F_2 : C_2 \to C$ are functors, the \textit{comma category} $F_1 \downarrow F_2$ has, as objects, all triples $(c_1, c_2, f : F_1 c_1 \to F_2 c_2)$ where $c_1$ and $c_2$ are objects of $C_1$ and $C_2$ respectively and $f$ is an arrow of $C$; and, as arrows from $(c_1, c_2, f)$ to $(c'_1, c'_2, f')$, all pairs $(g_1 : c_1 \to c'_1, g_2 : c_2 \to c'_2)$, where $g_1$ and $g_2$ are arrows of $C_1$ and $C_2$ respectively, such that

$$
\begin{array}{ccc}
F_1 c_1 & \xrightarrow{F_1 g_1} & F_1 c'_1 \\
\downarrow f & & \downarrow f' \\
F_2 c_2 & \xrightarrow{F_2 g_2} & F_2 c'_2
\end{array}
$$

commutes. Given $(g_1 : c_1 \to c'_1, g_2 : c_2 \to c'_2)$ and $(g'_1 : c'_1 \to c''_1, g'_2 : c'_2 \to c''_2)$, their composite is $(g'_1 \circ g_1, g'_2 \circ g_2)$; this is well-defined since $F_1$ and $F_2$ are functors and associative because $C$ is a category. The identity arrow for $(c_1, c_2, f : F_1 c_1 \to F_2 c_2)$ is $(1_{c_1}, 1_{c_2})$; this indeed satisfies the identity property since $F_1$ and $F_2$ are functors and $C$ is a category.

Comma categories are a very general concept that enable a unification of many otherwise seemingly ad hoc concepts: in the above discussion of limits, we had to define a notion of category of ‘arrows to $F$’ and, moreover, restrict to ‘arrows from objects of the form $\Delta c$’. This can be elegantly presented using comma categories:

If $c : 1 \to C$ is the constant functor selecting the object $c$ in $C$ then $1_C \downarrow c$ is the \textit{slice category} over $c$, written $C/c$, of arrows into $c$. More generally, $F_1 \downarrow c$ is the \textit{category of arrows from $F_1$ to $c$}. The \textit{category of cones to [the diagram] $F$} can therefore be expressed as $\Delta \downarrow F$ where $\Delta : C \to C^J$ sends $c$ to $\Delta c$ and $f : c \to c'$ to the natural transformation $\Delta f : \Delta c \to \Delta c'$ whose components are all $f$; and $F : C \to C^J$ is the constant functor selecting $F$.

2.3 Universal arrows

An object $1$ of $C$ is \textit{terminal} iff, for all objects $c$ of $C$, there is exactly one arrow from $c$ to $1$. Dually, an object $0$ is \textit{initial} in $C$ iff, for all objects $c$ of $C$, there is exactly one arrow from $0$ to $c$.

Any singleton set is terminal in $\text{Set}$; the category $1$ is terminal in $\text{Cat}$. The empty set is initial in $\text{Set}$; the empty category $0$, with no objects, is initial in $\text{Cat}$.
Initial and terminal objects need not be unique but they are always unique up to isomorphism: if \( t \) and \( t' \) are both terminal objects in \( C \), there must be an arrow \( f' : t' \to t \) from \( t' \) to \( t \) and an arrow \( f : t \to t' \) from \( t \) to \( t' \); so \( f' \circ f = 1_t \), the unique arrow from \( t \) to itself, and \( f \circ f' = 1_{t'} \), the unique arrow from \( t' \) to itself. Furthermore, \( t \) and \( t' \) are isomorphic up to a unique isomorphism: \( f' \) and \( f \) are themselves unique since \( t \) and \( t' \) are terminal.

A terminal arrow from a functor \( F : C \to C' \) to an object \( c' \) of \( C' \) is a terminal object in \( F \circ c' \). In other words, a terminal arrow is an object \( c_t \) of \( C \) and an arrow \( f'_t : Fc_t \to c' \) of \( C' \) such that, for any arrow \( f' : Fc \to c' \) of \( C' \), there is a unique arrow \( f : c \to c_t \) of \( C \) such that

\[
\begin{array}{c}
\text{commutes.} \\
\end{array}
\]

An initial arrow from \( c' \) to \( F \) is defined dually. We speak of a universal arrow when we do not care to stress whether it is initial or terminal.

A universal cone [limit] is therefore the particular case of a terminal arrow from a diagonal functor \( \Delta_J : C \to C^J \). An initial arrow to a diagonal functor is called a co-limit.

**Products** A terminal arrow from \( \Delta_2 : C \to C^2 \) to the object* \( (c_1, c_2) \) consists of an object, that we write as \( c_1 \times c_2 \), of \( C \) and an arrow \( (\pi_1, \pi_2) : (c_1 \times c_2, c_1 \times c_2) \to (c_1, c_2) \) of \( C^2 \) such that, for any arrow \( (f_1, f_2) : (c, c) \to (c_1, c_2) \) of \( C^2 \), there is a unique arrow \( f : c \to c_1 \times c_2 \) such that

\[
\begin{array}{c}
\text{commutes.} \\
\end{array}
\]

If the object \( c_1 \times c_2 \) and the arrows \( \pi_1 : c_1 \times c_2 \to c_1 \) and \( \pi_2 : c_1 \times c_2 \to c_2 \) exist in \( C \) then we say that \( c_1 \times c_2 \) is the product of \( c_1 \) and \( c_2 \); \( \pi_1 \) and \( \pi_2 \) are known as the projections (from \( c_1 \times c_2 \)) and \( f \) as the pairing of \( f_1 \) and \( f_2 \).

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*We have exploited the isomorphism \( C \times C \cong C^2 \), between the product of \( C \) with itself and the functor category from the discrete category \( 2 \), in order to have a more elementary description of the objects and arrows of \( C^2 \) as pairs of objects and pairs of arrows of \( C \).
Pull-backs and push-outs  More generally, consider the category $\Lambda$ with three objects and two non-identity arrows:

The category $\mathcal{C}^\Lambda$ has spans, i.e. diagrams of the form $c \xrightarrow{f} c' \xrightarrow{f'} c''$ in $\mathcal{C}$, as objects and triples $(f, f', f''$) of arrows of $\mathcal{C}$ satisfying $c \xrightarrow{f} c' \xrightarrow{f'} c''$ as arrows.

The category of co-spans of $\mathcal{C}$, i.e. diagrams of the form $c \xleftarrow{f} c' \xleftarrow{f'} c''$ in $\mathcal{C}$, is defined as $\mathcal{C}^\vee$ where $\vee := \Lambda^{op}$.

A terminal arrow from $\Delta : \mathcal{C} \rightarrow \mathcal{C}^\vee$ to the co-span $c \xrightarrow{f_{21}} c_2 \xrightarrow{f_{34}} c_3$ is a pull-back of the co-span. Concretely, this consists of an object $c_1$ and arrows $f_{1i} : c_1 \rightarrow c_i$ [for $i = 2, 3, 4$] such that $f_{21} \circ f_{12} = f_{14} = f_{34} \circ f_{13}$. i.e. a span making the resulting square commute which additionally satisfies the universal property that any other span making the square commute factors uniquely through it. If $c_4$ is a terminal object, this degenerates to the product of $c_2$ and $c_3$.

Dually, an initial arrow from the span $c \xleftarrow{f_{12}} c_2 \xleftarrow{f_{13}} c_3$ to $\Delta$ is a push-out from the span. If $c_1$ is an initial object, this defines a co-product of $c_2$ and $c_3$: an object $c_2 + c_3$ of $\mathcal{C}$ and injections $i_2 : c_2 \rightarrow c_2 + c_3$ and $i_3 : c_3 \rightarrow c_2 + c_3$ in $\mathcal{C}$ such that any pair of arrows $f_2 : c_2 \rightarrow c$ and $f_3 : c_3 \rightarrow c$ factorizes uniquely through the injections via their co-pairing $[f_2, f_3] : c_2 + c_3 \rightarrow c$.

Suppose we have commuting squares

where, as indicated, the right-hand inner square is a pull-back. It follows that the left-hand inner square is a pull-back if, and only if, the outer rectangle is a pull-back. This is called the pasting lemma for pull-backs.
2.4 Monos and pull-backs

An arrow \( f : c \to c' \) is a mono iff, for any pair of parallel arrows \( g_1, g_2 : c'' \to c \), if \( f \circ g_1 = f \circ g_2 \), then \( g_1 = g_2 \), i.e. \( f \) is post-cancellable. We write \( f : c \to c' \) to specify that \( f \) is a mono. The arrow \( f : c \to c' \) is a mono if, and only if,

\[
\begin{array}{c}
  c \\
  \downarrow 1_c \\
  f \\
  \downarrow h \\
  c' \\
\end{array}
\]

is a pull-back: given \( f_1, f_2 : c'' \to c \) where \( f \circ f_1 = f \circ f_2 \), we have a unique \( f' : c'' \to c \) such that \( f_1 = 1_c \circ f' = f' = 1_c \circ f' = f_2 \); and, for any \( f_1, f_2 : c'' \to c \) such that \( f \circ f_1 = f \circ f_2 \), we have that \( f_1 = f_2 \) which defines the unique arrow that makes the commuting square \( f \circ 1_c = f \circ 1_c \) a pull-back.

If \( f : c' \to c' \) and \( f' : c' \to c' \) then \( f' \circ f \) is a mono: if \( g_1, g_2 : c'' \to c \) satisfy \( (f' \circ f) \circ g_1 = (f' \circ f) \circ g_2 \), then \( f \circ g_1, f \circ g_2 : c'' \to c' \) and \( f \circ g_1 = f \circ g_2 \), since \( f' \) is a mono, whereupon \( g_1 = g_2 \) since \( f \) is a mono.

If \( f : c \to c' \), \( f_1 : c \to c'' \) and \( f_2 : c'' \to c' \) satisfy \( f = f_2 \circ f_1 \) then \( f_1 \) is a mono: if \( g_1, g_2 : c'' \to c \) satisfy \( f_1 \circ g_1 = f_1 \circ g_2 \) then \( f_2 \circ (f_1 \circ g_1) = f_2 \circ (f_1 \circ g_2) \), whereupon \( g_1 = g_2 \) since \( f \) is a mono.

Monos are preserved by pull-backs in the following sense: given a co-span \( f : c' \to c \) and \( g : c'' \to c \) such that the span \( f' : c''' \to c'' \) and \( g' : c''' \to c' \) is a pull-back thereof, it follows that \( g' \) is a mono. To see this, suppose that \( h_1, h_2 : c''' \to c'' \) such that \( g' \circ h_1 = g' \circ h_2 \); then \( f \circ (g' \circ h_1) = f \circ (g' \circ h_2) \) and so \( f' \circ h_1 = f' \circ h_2 \) since the square commutes and \( g \) is a mono. Moreover, the span \( g' \circ h_1 : c''' \to c' \) and \( f' \circ h_2 : c''' \to c' \) makes the square commute; so there is a unique \( h : c''' \to c'' \) such that \( g' \circ h = g \circ h_1 \) and \( f' \circ h = f' \circ h_2 \). But both \( h_1 \) and \( h_2 \) satisfy these conditions on \( h \), so \( h_1 = h_2 \).

An arrow \( f : c' \to c \) of \( C \) is an epi iff \( f^{op} : c \to c' \) is a mono in \( C^{op} \). An epi is thus pre-cancellable. We write \( f : c' \to c \) to specify that \( f \) is an epi. By definition, epis are preserved by push-outs in the dual of the preceding sense.
Chapter 3

Category theory III

Adjoint functors, ...

3.1 Adjoint functors

Let \( F : C \to C' \) be a functor and suppose that, for every object \( c' \) of \( C' \), we have an object \( c := Gc' \) and a given terminal arrow \( \varepsilon_{c'} : Fc \to c' \).

We extend the object mapping \( G \) to a functor \( G : C_0 \to C \) by sending each arrow \( f_0 : c_0 \to c_1 \) of \( C_0 \) to the unique arrow \( f : c_1 \to c_2 \) of \( C \) [where \( c_1 := Gc'_1 \) and \( c_2 := Gc'_2 \)] such that \( Ff_0 \circ \varepsilon_{c_0} = \varepsilon_{c_1} \).

This is indeed a functor since (i) \( 1' : c' \to c' \) is sent to the unique arrow \( f : c \to c \) such that \( \varepsilon_{c'} = \varepsilon_{c'} \circ Ff \), so \( f = 1_c \) as \( F \) is a functor; and (ii) for \( f'_1 : c'_1 \to c'_2 \) and \( f'_2 : c'_2 \to c'_3 \), there is a unique arrow \( G(f'_2 \circ f'_1) := g : c_1 \to c_3 \) such that \( (f'_2 \circ f'_1) \circ \varepsilon_{c'_1} = \varepsilon_{c'_3} \circ Fg \); and unique arrows \( Gf'_1 := g_1 : c_1 \to c_2 \) and \( Gf'_2 := g_2 : c_2 \to c_3 \) such that \( f'_1 \circ \varepsilon_{c'_1} = \varepsilon_{c'_2} \circ Fg_1 \) and \( f'_2 \circ \varepsilon_{c'_2} = \varepsilon_{c'_3} \circ Fg_2 \); so \( \varepsilon_{c'_3} \circ F(g_2 \circ g_1) = \varepsilon_{c'_3} \circ Fg_2 \circ Fg_1 = f'_2 \circ \varepsilon_{c'_2} \circ Fg_1 = f'_2 \circ f'_1 \circ \varepsilon_{c'_1} \), i.e. \( g = g_2 \circ g_1 \) as required. [Draw the diagrams!]

We say that \( F \) is left adjoint to \( G \); note that \( F \) does not determine \( G \) without the additional data of the terminal arrows \( \varepsilon_{c'} \).
The fact that \( G \) is a functor means that the terminal arrows \( \varepsilon_{c'} \) are in fact the co-ordinates of a natural transformation \( \varepsilon : F \circ G \to 1_{C'} \): given an arrow \( f' : c'_1 \to c'_2 \) of \( C' \), the required naturality square

\[
\begin{array}{ccc}
(F \circ G)c'_1 & \xrightarrow{\varepsilon_{c'_1}} & c'_1 \\
(F \circ G)f' \downarrow & & \downarrow f' \\
(F \circ G)c'_2 & \xrightarrow{\varepsilon_{c'_2}} & c'_2
\end{array}
\]

is simply the above triangle. The natural transformation \( \varepsilon \) is called the co-unit of the adjunction, a remarkably confusing terminology [from universal algebra] since it is induced by terminal, not initial, properties.

Given an object \( c \) of \( C \), define \( \eta_c : c \to (G \circ F)c \) to be the unique arrow of \( C \) such that

\[
\begin{array}{ccc}
Fc & \xrightarrow{1_{Fc}} & Fc \\
F\eta_c \downarrow & & \downarrow \varepsilon_{Fc} \\
(F \circ G \circ F)c & \xrightarrow{\varepsilon_{Fc}} & (F \circ G)c
\end{array}
\]

commutes, i.e. \( \eta_c := 1_{Fc}^\circ \varepsilon_c \). Given \( g' : Fc \to c' \), we have that \( F(Gg' \circ \eta_c) = FGg' \circ F\eta_c \), since \( F \) is a functor, and \( \varepsilon_{c'} \circ FGg' = g' \circ \varepsilon_{Fc} \), since \( \varepsilon \) is a natural transformation; therefore \( \varepsilon_{c'} \circ F(Gg' \circ \eta_c) = g' \circ (\varepsilon_{Fc} \circ F\eta_c) = g' \) which we can rephrase as \( g'^\circ = Gg' \circ \eta_c \).

If \( f : c \to Gc' \) is an arrow of \( C \), its left adjoint \( f^\circ := \varepsilon_{c'} \circ Ff \) factors, by definition, through \( Ff \) so that \( f = f^\circ \circ Ff \), i.e. \( f \) factors through \( Gf^\circ \).
If another \( g' : Fc \to c' \) satisfies \( f = Gg' \circ \eta_c \) then \( g'^\circ = Gg' \circ \eta_c = f = f^\circ \) and so \( f^\circ = g' \). This Establishes that \( \eta_c \) is an initial arrow from \( c \) to \( G \).

Moreover, given an arrow \( f : c_1 \to c_2 \) of \( C \), the left adjoint \( (\eta_{c_2} \circ f)^\circ := \varepsilon_{Fc_2} \circ F(\eta_{c_2} \circ f) = \varepsilon_{Fc_2} \circ F\eta_{c_2} \circ Ff = Ff \) so that \( Ff \circ \eta_{c_1} = \eta_{c_2} \circ Ff \), i.e. \( \eta \) is a natural transformation.

Finally, the left adjoint \( 1^\circ_{Gc'} = \varepsilon_{c'} \) so that \( G\varepsilon_{c'} \circ \eta_{Gc'} = 1_{Gc'} \); this is the so-called triangle identity for \( \eta \):
Let us recap: starting from a functor $F : C \to C'$ and a family of terminal arrows $\varepsilon_{c'}$ (in $C'$, indexed by the objects of $C'$), we can define (i) a functor $G : C' \to C$ for which $\varepsilon : F \circ G \to 1_C$ becomes a natural transformation; and (ii) a family of initial arrows $\eta_c$ (in $C$, indexed by the objects of $C$) that form a natural transformation $\eta : 1_C \to G \circ F$ that satisfies the triangle identities: one by definition; and the other as shown just above.

Recall that the left adjunct $f^1 : Fc \to c'$ of an arrow $f : c \to Gc'$ of $C$ is defined to be $f^1 := \varepsilon_{c'} \circ Ff$. The induced mapping from $\hom_C(Fc, c')$ to $\hom_C(c, Gc')$ is (i) surjective, since every $f : c \to Gc'$ gives rise to some left adjunct; and (ii) injective, since $\varepsilon_{c'}$ being terminal means that $f^2$ is $f$'s unique left adjunct.

This bijection $\phi_{c,c'}^{-1} : \hom_C(c, Gc') \cong \hom_C(Fc, c')$ is ‘natural’ in the sense that, given $g : c_0 \to c$, $\phi_{c_0,c'}^{-1}(f \circ g) := \varepsilon_{c'} \circ F(f \circ g) = \varepsilon_{c'} \circ Ff \circ Fg =: \phi_{c_0,c'}^{-1}(f) \circ Fg$; and, given $g' : c' \to c'_0$, $\phi_{c,c_0}^{-1}(Gg' \circ f) := \varepsilon_{c'_0} \circ F(Gg' \circ f) = \varepsilon_{c'_0} \circ FGg' \circ Ff = g \circ \varepsilon_{c'} \circ Ff =: g \circ \phi_{c_0}^{-1}(f)$.

If $C$ and $C'$ are locally small categories, this gives us a bona fide natural isomorphism $\phi_{c,c'} : \hom_C(Fc, c') \cong \hom_C(c, Gc')$ in $\textbf{Set}$ with naturality in $c$ [with respect to $g : c_0 \to c$]

$$
\begin{array}{ccc}
\hom_C(Fc, c') & \xrightarrow{\phi_{c,c'}} & \hom_C(c, Gc') \\
\lambda f'. f' \circ Fg & & \lambda f. f \circ g \\
\hom_C(Fc_0, c') & \xrightarrow{\phi_{c_0,c'}^{-1}} & \hom_C(c_0, Gc')
\end{array}
$$

and naturality in $c'$ [with respect to $g' : c' \to c'_0$]

$$
\begin{array}{ccc}
\hom_C(Fc, c') & \xrightarrow{\phi_{c,c'}} & \hom_C(c, Gc') \\
\lambda f'. g \circ f' & & \lambda f. Gg' \circ f \\
\hom_C(Fc, c'_0) & \xrightarrow{\phi_{c,c'_0}^{-1}} & \hom_C(c, Gc'_0)
\end{array}
$$

dually to the above.

Note how the terminal arrows $\varepsilon_{c'}$ allow a generalization of the case, found in equivalences of categories, where $F$ being a fully faithful functor induces a bijection $\hom_C(c_1, c_2) \cong \hom_C(Fc_1, Fc_2)$. In the case of an adjunction, despite the lack of the assumption that $F$ be full and faithful, we obtain our bijection by virtue of the universal property of $\varepsilon_{c'}$. 

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