Notes on categories

incomplete and never to be finished

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Chapter 1

Category theory I

Categories, functors and natural transformations.

1.1 Categories

A *category* $\mathbf{C}$ consists of a class $\mathcal{O}_\mathbf{C}$ of unstructured *objects* and a class $\mathcal{A}_\mathbf{C}$ of *arrows* of the form $f : c \to c'$ where f is our name for the arrow and c and c' are objects; we write $\operatorname{dom} f$ for c and $\operatorname{codom} f$ for c'. We require there to be, for every object c, an *identity* arrow $1_c : c \to c$ and, for all *composable* arrows $f : c \to c'$ and $f' : c' \to c''$, *i.e.* where $\operatorname{codom} f = \operatorname{dom} f'$, a *composite* arrow $f' \circ f : c \to c''$ satisfying $f \circ 1_c = f = 1_{c'} \circ f$ and, for all arrows $f'' : c'' \to c'''$, $(f'' \circ f') \circ f = f'' \circ (f' \circ f)$. The operation $\circ$ is the *composition law* of $\mathbf{C}$; the two requirements are the *identity* and the *associativity* properties.

An arrow $f : c \to c'$ is an *isomorphism* iff, for some $f' : c' \to c$, $f' \circ f = 1_c$ and $f \circ f' = 1_{c'}$. If f' exists, it is unique since, for any other candidate $f'' : c' \to c$, $f'' = f'' \circ f \circ f' = f'$. We then say that f is *invertible*, or *is an isomorphism*, and define the operation $f^{-1} := f'$; we also say that the objects c and c' are *isomorphic* [in the category $\mathbf{C}$], written $c \cong c'$ [or $f : c \cong c'$ if we wish to stress the specific *witness* f].

A *subcategory* $\mathbf{C}'$ of $\mathbf{C}$ consists of a subclass $\mathcal{O}_{\mathbf{C}'}$ of $\mathcal{O}_\mathbf{C}$ and a subclass $\mathcal{A}_{\mathbf{C}'}$ of $\mathcal{A}_\mathbf{C}$ such that, for all arrows f in $\mathcal{A}_{\mathbf{C}'}$, $\operatorname{dom} f$ and $\operatorname{codom} f$ are both in $\mathcal{O}_{\mathbf{C}'}$, for all objects c in $\mathcal{O}_{\mathbf{C}'}$, $\mathcal{A}_{\mathbf{C}'}$ contains 1_c and, for all arrows $f : c \to c'$ and $f' : c' \to c''$ in $\mathcal{A}_{\mathbf{C}'}$, their composite $f' \circ f$ is also in $\mathcal{A}_{\mathbf{C}'}$. The identities and composition law of $\mathbf{C}'$ are inherited from $\mathbf{C}$, *i.e.* $1'_c := 1_c$, for all c in $\mathcal{O}_{\mathbf{C}'}$, and $f' \circ' f := f' \circ f$, for all $f : c \to c'$ and $f' : c' \to c''$ in $\mathcal{A}_{\mathbf{C}'}$.

The isomorphisms of $\mathbf{C}$ form a subcategory: if $f : c \cong c'$ and $f' : c' \cong c''$ then $f' \circ f : c \cong c''$ since, setting $(f' \circ f)^{-1} := f^{-1} \circ f'^{-1}$, we have $(f^{-1} \circ f'^{-1}) \circ (f' \circ f) = f^{-1} \circ (f'^{-1} \circ f') \circ f = 1_c$ and, similarly, $(f' \circ f) \circ (f^{-1} \circ f'^{-1}) = 1_{c''}$.

'The' category generally known as $\mathbf{Set}$ has all sets as objects and all total functions between them as arrows [where 'all' depends on your choice of set theory].

An arrow $f : c \to c'$ of $\mathbf{Set}$ is an isomorphism if, and only if, f [viewed as a set-theoretic function] is a bijection; so, in particular, c and c' [viewed as sets] are isomorphic.

The category $\mathbf{C}$ is *small* iff its class of arrows is a set; its collection of objects is then necessarily also a set. A category where all the arrows are identity arrows is called *discrete*. A small discrete category is a set.

The category $\mathbf{C}$ is *locally small* iff, for all pairs of objects c and c', the class $\hom(c, c')$ of all arrows $f : c \to c'$ is an object of $\mathbf{Set}$, *i.e.* actually a set, not a class.

A small category with one object, *i.e.* where *all* arrows are composable, is a *monoid*. A monoid where *all* arrows are invertible is a *group*. A small category where *all* arrows are invertible is a *groupoid*.

More generally, if $\mathbf{C}$ is a locally small category containing objects c and c', we define $\mathrm{Iso}(c, c')$ to be the set of all isomorphisms $f : c \cong c'$ and write $\mathrm{Aut}(c)$ for $\mathrm{Iso}(c, c)$. Given $f \in \mathrm{Iso}(c, c')$, define a total function from $g \in \mathrm{Aut}(c)$ to $\mathrm{Iso}(c, c')$ by $g \mapsto f \circ g$. This is an isomorphism in $\mathbf{Set}$ witnessed by the total function from $f' \in \mathrm{Iso}(c, c')$ to $\mathrm{Aut}(c)$ defined as $f' \mapsto f^{-1} \circ f'$: clearly $g \mapsto f^{-1} \circ (f \circ g) = g$ and $f' \mapsto f \circ (f^{-1} \circ f') = f'$. So, provided that $c \cong c'$, there are always exactly as many automorphisms of c (or indeed c') as there are witnesses of the isomorphism of c and c'.

The category of groups and group homomorphisms is called $\mathbf{Grp}$; that of graphs and graph homomorphisms is called $\mathbf{Grph}$.

The *opposite* category $\mathbf{C}^{op}$ of the category $\mathbf{C}$ is defined to have the same objects as $\mathbf{C}$ and the 'same' arrows as $\mathbf{C}$ but going in the other direction: $f^{op} : c \to c'$ is an arrow of $\mathbf{C}^{op}$ iff $f : c' \to c$ is an arrow of $\mathbf{C}$. Concomitantly, $f'^{op} \circ^{op} f^{op} := (f \circ f')^{op}$ and $1_c^{op} := 1_c$.

The *product category* $\mathbf{C_1} \times \mathbf{C_2}$ of the categories $\mathbf{C_1}$ and $\mathbf{C_2}$ has as objects all pairs $\langle c_1, c_2 \rangle$, where c_1 is an object of $\mathbf{C_1}$ and c_2 is an object of $\mathbf{C_2}$, and as arrows all pairs $\langle f_1, f_2 \rangle : \langle c_1, c_2 \rangle \to \langle c_1', c_2' \rangle$, where $f_1 : c_1 \to c_1'$ is an arrow of $\mathbf{C_1}$ and $f_2 : c_2 \to c_2'$ is an arrow of $\mathbf{C_2}$. Concomitantly, composable arrows $\langle f_1, f_2 \rangle$ and $\langle f_1', f_2' \rangle$ are composed component-wise, *i.e.* $\langle f_1', f_2' \rangle \circ_{1\times 2} \langle f_1, f_2 \rangle := \langle f_1' \circ_1 f_1, f_2' \circ_2 f_2 \rangle$, and $1^{1\times 2}_{\langle c_1, c_2 \rangle} := \langle 1^1_{c_1}, 1^2_{c_2} \rangle$.

1.2 Functors

A *functor* F from the category $\mathbf{C}$ to the category $\mathbf{C}'$ consists of two mappings [one sending c in $\mathcal{O}_\mathbf{C}$ to Fc in $\mathcal{O}_{\mathbf{C}'}$ and the other sending $f : c \to c'$ in $\mathcal{A}_\mathbf{C}$ to $Ff : Fc \to Fc'$ in $\mathcal{A}_{\mathbf{C}'}$] satisfying, for all objects c of $\mathbf{C}$, $F1_c = 1'_{Fc}$ and, for all arrows $f : c \to c'$ and $f' : c' \to c''$ of $\mathbf{C}$, $F(f' \circ f) = Ff' \circ' Ff$.

A functor from a monoid [viewed as the category $\mathbf{C}$] to a second monoid [viewed as the category $\mathbf{C}'$] is a standard *monoid homomorphism*. Likewise for groups and groupoids.

If the arrow $f : c \to c'$ of $\mathbf{C}$ is invertible then $(Ff)^{-1} := Ff^{-1}$ inverts $Ff : Fc \to Fc'$ in $\mathbf{C}'$ since $Ff^{-1} \circ' Ff = F(f^{-1} \circ f) = F1_c = 1'_{Fc}$ and $Ff \circ' Ff^{-1} = F(f \circ f^{-1}) = F1_{c'} = 1'_{Fc'}$.

An alternative possible definition of functor would specify only the arrow mapping: any mapping F from the arrows of $\mathbf{C}$ to the arrows of $\mathbf{C}'$ such that, for all arrows $f_1 : c_1 \to c'_1$ and $f_2 : c_2 \to c'_2$ of $\mathbf{C}$, $\operatorname{dom} Ff_1 = \operatorname{dom} Ff_2$ if $\operatorname{dom} f_1 = \operatorname{dom} f_2$ and $\operatorname{codom} Ff_1 = \operatorname{codom} Ff_2$ if $\operatorname{codom} f_1 = \operatorname{codom} f_2$ immediately induces a functor; the induced object mapping can be defined as $Fc := c'$ iff $F1_c = 1'_{c'}$.

A third (and final) possible definition of functor is as a family of functions, $F_{c,c'} : \hom(c, c') \to \hom(Fc, Fc')$, indexed by all pairs of objects of $\mathbf{C}$.

The functor $F : \mathbf{C} \to \mathbf{C}'$ is (i) *faithful* iff, for every pair c, c' of objects of $\mathbf{C}$, $F_{c,c'}$ is injective; (ii) *full* iff, for every pair c, c' of objects of $\mathbf{C}$, $F_{c,c'}$ is surjective; and (iii) *essentially surjective* iff, for every object c' of $\mathbf{C}'$, there is some object c of $\mathbf{C}$ such that $Fc \cong c'$. A full *and* faithful functor is sometimes called *fully faithful*.

If $\mathbf{C}'$ is a subcategory of $\mathbf{C}$, the *inclusion* functor $I : \mathbf{C}' \to \mathbf{C}$ sends each object and arrow of $\mathbf{C}'$ to 'itself' in $\mathbf{C}$. This functor is always faithful. If it is full, we say that $\mathbf{C}'$ is a *full subcategory* of $\mathbf{C}$; a full subcategory is therefore uniquely determined by its subclass $\mathcal{O}_{\mathbf{C}'}$ of objects.

A category $\mathbf{C}$ is *concrete* iff there is a faithful functor $U : \mathbf{C} \to \mathbf{Set}$.

A locally small category $\mathbf{C}$ induces a *hom* functor $\hom_\mathbf{C} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ where $\langle c, c' \rangle \mapsto \hom(c, c')$ and $\langle f_1^{op}, f_2 \rangle : \langle c_1, c_2 \rangle \to \langle c'_1, c'_2 \rangle \mapsto (f : c_1 \to c_2 \mapsto f_2 \circ f \circ f_1) : \hom(c_1, c_2) \to \hom(c'_1, c'_2)$:

- Clearly, $\hom_\mathbf{C} 1_{\langle c_1, c_2 \rangle} = (f : c_1 \to c_2 \mapsto 1_{c_2} \circ f \circ 1_{c_1}) = 1_{\hom(c_1, c_2)}$.
- Moreover, given additionally $\langle f_1'^{op}, f'_2 \rangle : \langle c'_1, c'_2 \rangle \to \langle c''_1, c''_2 \rangle$, we have that $\hom_\mathbf{C}(\langle f_1'^{op}, f'_2 \rangle \circ \langle f_1^{op}, f_2 \rangle) = (f : c_1 \to c_2 \mapsto (f'_2 \circ f_2) \circ f \circ (f_1 \circ f'_1)) = (f : c_1 \to c_2 \mapsto f'_2 \circ (f_2 \circ f \circ f_1) \circ f'_1) = \hom_\mathbf{C}\langle f_1'^{op}, f'_2 \rangle \circ \hom_\mathbf{C}\langle f_1^{op}, f_2 \rangle$.

The functors $F : \mathbf{C} \to \mathbf{C}'$ and $F' : \mathbf{C}' \to \mathbf{C}''$ can be composed by defining $c \mapsto F'(Fc)$, for all objects c of $\mathbf{C}$, and $f \mapsto F'(Ff)$, for all arrows $f : c \to c'$ of $\mathbf{C}$. This yields a functor $F' \circ F : \mathbf{C} \to \mathbf{C}''$ since (i) $(F' \circ F)1_c = F'1'_{Fc} = 1''_{(F'F)c}$; and, given $f' : c' \to c''$ of $\mathbf{C}$, (ii) $(F' \circ F)(f' \circ f) = F'(F(f' \circ f)) = F'(Ff' \circ' Ff) = F'(Ff') \circ'' F'(Ff) = (F' \circ F)f' \circ'' (F' \circ F)f$.

The *identity* functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ sends every object and arrow to itself; clearly $F1_{\mathbf{C}} = F = 1'_{\mathbf{C}'}F$. Moreover, given a third functor $F'' : \mathbf{C}'' \to \mathbf{C}'''$, clearly $((F'' \circ F') \circ F) = (F'' \circ (F' \circ F))$. So we have a category, known as **Cat**, with objects all *small* categories and arrows all functors between them; two small categories are *isomorphic* iff they are isomorphic in **Cat**.

1.3 Natural transformations

A *natural transformation* $\alpha : F \to F'$ from the functor $F : \mathbf{C} \to \mathbf{C}'$ to the functor $F' : \mathbf{C} \to \mathbf{C}'$ is a family of arrows $\alpha_c : Fc \to F'c$ of $\mathbf{C}'$, indexed by the objects of $\mathbf{C}$, such that

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & F'c \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle F'f} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & F'c' \end{array}$$

commutes for all arrows $f : c \to c'$ of $\mathbf{C}$.

The natural transformation $\alpha : F \to F'$ is a *natural isomorphism* iff, for every object c of $\mathbf{C}$, the arrow $\alpha_c : Fc \to F'c$ is an isomorphism. This is equivalent to saying that there is a natural transformation $\alpha^{-1} : F' \to F$ satisfying, for all objects c of $\mathbf{C}$, $\alpha_c^{-1} \circ' \alpha_c = 1'_{Fc}$ and $\alpha_c \circ' \alpha'_c = 1'_{F'c}$.

Given a further natural transformation $\alpha' : F' \to F''$, where $F'' : \mathbf{C} \to \mathbf{C}'$ is a third functor, the *vertical* composite of α and α', defined component-wise as $(\alpha' \bullet \alpha)_c := \alpha'_c \circ' \alpha_c$, is clearly a natural transformation $\alpha' \bullet \alpha : F \to F''$. The *identity* natural transformation $1_F : F \to F$, defined as $(1_F)_c := 1'_{Fc}$, satisfies $\alpha \bullet 1_F = \alpha = 1_{F'} \bullet \alpha$ and, given a fourth functor F''' and a third natural transformation $\alpha'' : F'' \to F'''$, we have $(\alpha'' \bullet \alpha') \bullet \alpha = \alpha'' \bullet (\alpha' \bullet \alpha)$. Provided no 'problems of size' arise, we thus obtain a *functor category* $\mathbf{C}'^{\mathbf{C}}$ with objects all functors from $\mathbf{C}$ to $\mathbf{C}'$ and arrows all natural transformations between these functors; it is sufficient that $\mathbf{C}$ be a small category.

Let $\mathbf{1}$ and $\mathbf{2}$ be the (small) discrete categories with one and two objects respectively. Clearly $\mathbf{C}^{\mathbf{1}} \cong \mathbf{C}$ and $\mathbf{C}^{\mathbf{2}} \cong \mathbf{C} \times \mathbf{C}$ in **Cat**.

The categories $\mathbf{C}$ and $\mathbf{C}'$ are *equivalent* iff there exist functors $F : \mathbf{C} \to \mathbf{C}'$ and $G : \mathbf{C}' \to \mathbf{C}$ and natural isomorphisms $\varepsilon : FG \cong 1_{\mathbf{C}'}$ and $\eta : 1_{\mathbf{C}} \cong GF$.

Since $\varepsilon_{c'} : F(Gc') \cong c'$, for all c' in $\mathcal{O}_{\mathbf{C}'}$, the functor F is essentially surjective. Moreover, for any arrow $f : c_1 \to c_2$ of $\mathbf{C}$, $f = \eta_{c_2} \circ GFf \circ \eta_{c_1}^{-1}$ and $GFf = \eta_{c_2}^{-1} \circ f \circ \eta_{c_1}$, so that $\hom(c_1, c_2)$ is in bijection with $\hom(GFc_1, GFc_2)$. [If $\mathbf{C}$ is locally small, there is therefore an isomorphism in **Set** witnessing $\hom(c_1, c_2) \cong \hom(GFc_1, GFc_2)$.] So F_{c_1,c_2} must be injective and G_{Fc_1,Fc_2} must be surjective, for all pairs of objects c_1, c_2, *i.e.* F is faithful and G is full. The symmetric argument establishes that G is essentially surjective and faithful and that F is full.

The functor $F : \mathbf{C} \to \mathbf{C}'$ is a *weak equivalence* iff, for some $G : \mathbf{C}' \to \mathbf{C}$, there exist natural isomorphisms $\epsilon : FG \cong 1_{\mathbf{C}'}$ and $\eta : 1_{\mathbf{C}} \cong GF$. If F is a weak equivalence then, from the above, we know that it is fully faithful and essentially surjective; the converse is also true—with the caveat that it depends on the axiom of choice:

Suppose that the functor $F : \mathbf{C} \to \mathbf{C}'$ is fully faithful and essentially surjective. By essential surjectivity, for any c' in $\mathcal{O}_{\mathbf{C}'}$, there is at least one c in $\mathcal{O}_{\mathbf{C}}$ such that $Fc \cong c'$; an application of the axiom of choice then picks out a choice of c, for each c' in $\mathcal{O}_{\mathbf{C}'}$, allowing us to define $Gc' := c$.

1.4 Cat as a 2-category

Given functors $F_1', F_2' : \mathbf{C}' \to \mathbf{C}''$, a natural transformation $\alpha' : F_1' \to F_2'$ and functors $F : \mathbf{C} \to \mathbf{C}'$ and $F'' : \mathbf{C}'' \to \mathbf{C}'''$, we define, for each object c of $\mathbf{C}$, an arrow $(F'' \circ \alpha' \circ F)_c := F''(\alpha'_{Fc})$ of $\mathbf{C}'''$. This defines a natural transformation $F'' \circ \alpha' \circ F : F'' \circ F_1' \circ F \to F'' \circ F_2' \circ F$ since, for any $f : c \to c'$ in $\mathbf{C}$,

$$\begin{array}{ccc}
F''(F_1'(Fc)) & \xrightarrow{F''\alpha'_{Fc}} & F''(F_2'(Fc)) \\
{\scriptstyle F''(F_1'(Ff))}\downarrow & & \downarrow{\scriptstyle F''(F_2'(Ff))} \\
F''(F_1'(Fc')) & \xrightarrow[F''\alpha'_{Fc'}]{} & F''(F_2'(Fc'))
\end{array}$$

commutes (because α' is a natural transformation and F'' is a functor).

This hybrid composition of two functors and a natural transformation is usually called *whiskering*. It is a special case of the *horizontal* composition of natural transformations if we replace the functors F and F'' by the identity natural transformations $1_F : F \to F$ and $1_{F''} : F'' \to F''$ respectively:

Given functors $F_1, F_2 : \mathbf{C} \to \mathbf{C}'$ and $F'_1, F'_2 : \mathbf{C}' \to \mathbf{C}''$ together with natural transformations $\alpha : F_1 \to F_2$ and $\alpha' : F'_1 \to F'_2$, the *horizontal* composite $\alpha' \circ \alpha : F'_1 \circ F_1 \to F'_2 \circ F_2$ of α and α' is defined, for each object c of $\mathbf{C}$, to be the diagonal of

$$\begin{tikzcd}
F'_1(F_1c) \arrow[r, "\alpha'_{F_1c}"] \arrow[d, "F'_1\alpha_c"'] \arrow[dr, dotted, "(\alpha'\circ\alpha)_c" description] & F'_2(F_1c) \arrow[d, "F'_2\alpha_c"] \\
F'_1(F_2c) \arrow[r, "\alpha'_{F_2c}"'] & F'_2(F_2c)
\end{tikzcd}$$

(this square necessarily commutes because α' is a natural transformation). This defines a natural transformation since, for any arrow $f : c \to c'$ in $\mathbf{C}$, the two internal squares of

$$\begin{tikzcd}
F'_1(F_1c) \arrow[r, "\alpha'_{F_1c}"] \arrow[d, "F'_1(F_1f)"'] \arrow[rr, bend left, "(\alpha'\circ\alpha)_c" description] & F'_2(F_1c) \arrow[r, "F'_2\alpha_c"] \arrow[d, "F'_2(F_1f)" description] & F'_2(F_2c) \arrow[d, "F'_2(F_2f)"] \\
F'_1(F_1c') \arrow[r, "\alpha'_{F_1c'}"'] \arrow[rr, bend right, "(\alpha'\circ\alpha)_{c'}" description] & F'_2(F_1c') \arrow[r, "F'_2\alpha_{c'}"'] & F'_2(F_2c')
\end{tikzcd}$$

commute (because α' and α are natural transformations and F'_2 is a functor) and so the outer square commutes as required.

Horizontal composition has identities, specifically the identity natural transformations $1_{1_{\mathbf{C}'}}$ and $1_{1_{\mathbf{C}''}}$ for the identity functors $1_{\mathbf{C}'}$ and $1_{\mathbf{C}''}$. (Note that this differs from the identities for vertical composition.) It is also (strictly) associative as all the faces of the cube below commute.

$$\begin{tikzcd}
F''_1(F'_1(F_1c)) \arrow[rr, "\alpha''_{F'_1(F_1c)}"] \arrow[dd, "F''_1(F'_1\alpha_c)"'] \arrow[dr, "F''_1(\alpha'_{F_1c})"] & & F''_2(F'_1(F_1c)) \arrow[dd, "F''_2(F'_1\alpha_c)" near end] \arrow[dr, "F''_2(\alpha'_{F_1c})"] & \\
& F''_1(F'_2(F_1c)) \arrow[rr, crossing over, "\alpha''_{F'_2(F_1c)}" near start] & & F''_2(F'_2(F_1c)) \arrow[dd, "F''_2(F'_2\alpha_c)"] \\
F''_1(F'_1(F_2c)) \arrow[rr, "\alpha''_{F'_1(F_2c)}" near end] \arrow[dr, "F''_1(\alpha'_{F_2c})"'] & & F''_2(F'_1(F_2c)) \arrow[dr, "F''_2(\alpha'_{F_2c})"] & \\
& F''_1(F'_2(F_2c)) \arrow[rr, "\alpha''_{F'_2(F_2c)}"'] \arrow[from=uu, crossing over, "F''_1(F'_2\alpha_c)" near start] & & F''_2(F'_2(F_2c))
\end{tikzcd}$$

A *2-category* $\mathbf{C}$ consists of a class $\mathcal{O}_\mathbf{C}$ of objects, also called *0-cells*, where (i) for each ordered pair c, c' of 0-cells, there is a category $\hom_\mathbf{C}(c, c')$ whose objects and arrows are called *1-cells* and *2-cells* respectively; the composition of 2-cells $\alpha_1 : f_1 \to f_2$ and $\alpha_2 : f_2 \to f_3$ is called *vertical* composition and is denoted by $\alpha_2 \bullet \alpha_1$; (ii) for each 0-cell c, there is a functor $I_c : \mathbf{1} \to \hom_\mathbf{C}(c, c)$; and (iii) for each ordered triple c, c', c'' of 0-cells, there is a functor $C_{c,c',c''} : \hom_\mathbf{C}(c', c'') \times \hom_\mathbf{C}(c, c') \to \hom_\mathbf{C}(c, c'')$.

These data will be required to satisfy further conditions but let us first unpack what they mean: (i) the 1-cells of $\hom_\mathbf{C}(c, c')$ are the 'arrows' of $\mathbf{C}$ from c to c'; the 2-cells of $\hom_\mathbf{C}(c, c')$ are 'arrows between arrows'; (ii) I_c picks out a 1-cell 1_c and its identity arrow 1_{1_c} in $\hom_\mathbf{C}(c, c)$; this 1-cell will be the 'identity arrow' for c in $\mathbf{C}$; (iii) $C_{c,c',c''}$ defines the *horizontal* composition of 1- and 2-cells; its object part takes 'composable arrows' $f : c \to c'$ and $f' : c' \to c''$ to $f' \circ f := C_{c,c',c''}(f', f) : c \to c''$; and its arrow part takes 2-cells $\alpha : f_1 \to f_2$ and $\alpha' : f'_1 \to f'_2$ between 'composable arrows' $f_1, f_2 : c \to c'$ and $f'_1, f'_2 : c' \to c''$ to $\alpha' \circ \alpha := C_{c,c',c''}(\alpha', \alpha) : f'_1 \circ f_1 \to f'_2 \circ f_2$; and (iv) finally, functoriality of $C_{c,c',c''}$ imposes the *interchange law* relating the vertical and horizontal compositions of 2-cells $\alpha_1 : f_1 \to f_2$, $\alpha_2 : f_2 \to f_3$, $\alpha'_1 : f'_1 \to f'_2$ and $\alpha'_2 : f'_2 \to f'_3$ between the 1-cells $f_1, f_2, f_3 : c \to c'$ and $f'_1, f'_2, f'_3 : c' \to c''$:

$$(\alpha'_2 \bullet \alpha'_1) \circ (\alpha_2 \bullet \alpha_1) = (\alpha'_2 \circ \alpha_2) \bullet (\alpha'_1 \circ \alpha_1).$$

We complete the definition of 2-category by asking that (i) for any ordered quadruple of 0-cells c, c', c'', c''' together with 1-cells $f_1, f_2 : c \to c'$, $f'_1, f'_2 : c' \to c''$ and $f''_1, f''_2 : c'' \to c'''$ and 2-cells $\alpha : f_1 \to f_2$, $\alpha' : f'_1 \to f'_2$ and $\alpha'' : c'' \to c'''$, we have $f''_1 \circ (f'_1 \circ f_1) = (f''_1 \circ f'_1) \circ f_1$ (and likewise for f_2, f'_2 and f''_2) and $\alpha'' \circ (\alpha' \circ \alpha) = (\alpha'' \circ \alpha') \circ \alpha$; and (ii) for any 0-cells c and c' together with 1-cells $f_1, f_2 : c \to c'$ and a 2-cell $\alpha : f_1 \to f_2$, we have $f_1 \circ 1_c = f_1 = 1_{c'} \circ f_1$ (and likewise for f_2) and $\alpha \circ 1_{1_c} = \alpha = 1_{1_{c'}} \circ \alpha$.

These conditions guarantee that, in accordance with the above intuition, the 0-cells and 1-cells of $\mathbf{C}$ are indeed the objects and the arrows of a category.

The category $\mathbf{Cat}$ can be given the structure of a 2-category by setting $\mathcal{O}_\mathbf{Cat}$ to be the class of small categories; then (i) $\hom_\mathbf{Cat}(c, c') := c'^c$, the functor category from c to c'; (ii) I_c selects the identity functor 1_c on c and its identity natural transformation 1_{1_c}; and (iii) $C_{c,c',c''}(F', F) := F' \circ F$ and $C_{c,c',c''}(\alpha', \alpha) := \alpha' \circ \alpha$, the horizontal composition of natural transformations.

We have already proved above that these data satisfy all the conditions required of a 2-category. Clearly, the induced category of 0-cells and 1-cells is just $\mathbf{Cat}$.

1.5 Equivalences in a 2-category

In a category, we have a notion of isomorphism of objects but not of arrows. In a 2-category, we say that a 2-cell $\alpha : f_1 \to f_2$ is a *2-isomorphism* [or just an isomorphism when we can get away with it] of the 1-cells $f_1, f_2 : c \to c'$ iff there exists a 2-cell $\alpha' : f_2 \to f_1$ such that $\alpha' \bullet \alpha = 1_{f_1}$ and $\alpha \bullet \alpha' = 1_{f_2}$. As α and α' are just isomorphisms in a category, *i.e.* $\hom(c, c')$, α' is unique and we define $\alpha^{-1} := \alpha'$.

If $\alpha_1 : f_1 \to f_2$ and $\alpha_2 : f_2 \to f_3$ are 2-isomorphisms for $f_1, f_2, f_3 : c \to c'$ then $(\alpha_2 \bullet \alpha_1)^{-1} := \alpha_1^{-1} \bullet \alpha_2^{-1}$.

If $\alpha : f_1 \to f_2$ and $\alpha' : f'_1 \to f'_2$ are 2-isomorphisms for $f_1, f_2 : c \to c'$ and $f'_1, f'_2 : c' \to c''$ then $(\alpha'^{-1} \circ \alpha^{-1}) \bullet (\alpha' \circ \alpha) = (\alpha'^{-1} \bullet \alpha') \circ (\alpha^{-1} \bullet \alpha) = 1_{f'_1} \circ 1_{f_1} = 1_{f'_1 \circ f_1}$ and $(\alpha' \circ \alpha) \bullet (\alpha'^{-1} \circ \alpha^{-1}) = (\alpha' \bullet \alpha'^{-1}) \circ (\alpha \bullet \alpha^{-1}) = 1_{f'_2} \circ 1_{f_2} = 1_{f'_2 \circ f_2}$; so we can define $(\alpha' \circ \alpha)^{-1} := \alpha'^{-1} \circ \alpha^{-1}$ [beware the subtle trap].

The 2-isomorphisms of **Cat** are precisely natural isomorphisms: clearly, any 2-isomorphism defines a natural isomorphism; conversely, each $\alpha_c : F_1 c \to F_2 c$ of a natural isomorphism $\alpha : F_1 \to F_2$ [of functors $F_1, F_2 : \mathbf{C} \to \mathbf{C}'$] is invertible, so $\alpha_c^{-1} \circ' \alpha_c = 1'_{F_1 c}$ and $\alpha_c \circ' \alpha_c^{-1} = 1'_{F_2 c}$, *i.e.* $\alpha' \bullet \alpha = 1_{F_1}$ and $\alpha \bullet \alpha' = 1_{F_2}$ as required.

An *equivalence* in a 2-category consists of 1-cells $f : c \to c'$ and $f' : c' \to c$ and 2-isomorphisms $\eta : 1_c \to f' \circ f$ and $\varepsilon : f \circ f' \to 1_{c'}$. An equivalence in **Cat** is precisely an equivalence of categories as defined previously.

An equivalence is *adjoint* iff the so-called 'triangle identities' hold:

$$\begin{array}{ccc} f & \xrightarrow{1_f \circ \eta} & f \circ f' \circ f \\ & {\scriptstyle 1_f} \searrow & \downarrow {\scriptstyle \varepsilon \circ 1_f} \\ & & f \end{array} \qquad \qquad \begin{array}{ccc} f' \circ f \circ f' & \xleftarrow{\eta \circ 1_{f'}} & f' \\ {\scriptstyle 1_{f'} \circ \varepsilon} \downarrow & \swarrow {\scriptstyle 1_{f'}} & \\ f' & & \end{array}$$

If $(f, f', \eta, \varepsilon)$ is an adjoint equivalence then so is $(f', f, \varepsilon^{-1}, \eta^{-1})$: $1_{f'} = 1_{f'}^{-1} = ((\eta \circ 1_{f'}) \bullet (1_{f'} \circ \varepsilon))^{-1} = (\eta \circ 1_{f'})^{-1} \bullet (1_{f'} \circ \varepsilon)^{-1} = (\eta^{-1} \circ 1_{f'}) \bullet (1_{f'} \circ \varepsilon^{-1})$; and $1_f = (1_f \circ \eta)^{-1} \bullet (\varepsilon \circ 1_f)^{-1} = (1_f \circ \eta^{-1}) \bullet (\varepsilon^{-1} \circ 1_f)$.

If $(f, f', \eta, \varepsilon)$ is an equivalence then either triangle identity holds if, and only if, the other one does: ...

If $(f, f', \eta, \varepsilon)$ is an equivalence then there exist $f'' : c' \to c$ and $\varepsilon' : f \circ f'' \to 1_{c'}$ such that $(f, f'', \eta, \varepsilon')$ is an adjoint equivalence: ...

Chapter 2

Category theory II

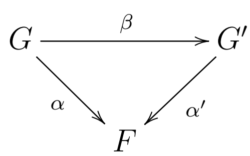
Diagrams, limits, comma categories, universal arrows.

2.1 Categories of diagrams

A *diagram* [more properly, a $\mathbf{J}$-*diagram*] in $\mathbf{C}$ is a functor $F : \mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ is a small, often even finite, category.

A *cone* to F is an object c of $\mathbf{C}$ together with arrows $\alpha_j : c \to Fj$ of $\mathbf{C}$, where j ranges over the objects of $\mathbf{J}$, such that $Ff \circ \alpha_j = \alpha_{j'}$ for all arrows $f : j \to j'$ of $\mathbf{J}$. A cone is thus a natural transformation from the constant functor $\Delta_c : \mathbf{J} \to \mathbf{C}$ [defined by $\Delta_c(j) := c$ for all objects j of $\mathbf{J}$; and $\Delta_c(f : j \to j') := 1_c$ for all arrows f of $\mathbf{J}$] to F.

We call the functor category $\mathbf{C}^{\mathbf{J}}$ the *category of* $\mathbf{J}$*-diagrams in* $\mathbf{C}$; a cone to F is thus an arrow of $\mathbf{C}^{\mathbf{J}}$ of the form $\alpha : \Delta_c \to F$. The category $\mathbf{C}^{\mathbf{J}}/F$ of $\mathbf{J}$-diagrams *over* F is defined to have arrows of $\mathbf{C}^{\mathbf{J}}$ of the form $\alpha : G \to F$ [any G] as objects; and arrows of $\mathbf{C}^{\mathbf{J}}$ of the form $\beta : G \to G'$, such that



commutes, as arrows.

A cone $\upsilon : \Delta_u \to F$ to F is *universal* iff, for any cone $\alpha : \Delta_c \to F$, there is a unique arrow $f : c \to u$ of $\mathbf{C}$ such that $\upsilon_j \circ f = \alpha_j$ for all objects j of $\mathbf{J}$.

A universal cone to F, if it exists, is called a *limit* of [the diagram] F and is *unique up to unique isomorphism.*

2.2 Comma categories

If $F_1 : \mathbf{C}_1 \to \mathbf{C}$ and $F_2 : \mathbf{C}_2 \to \mathbf{C}$ are functors, the *comma category* $F_1 \downarrow F_2$ has, as objects, all triples $(c_1, c_2, f : F_1c_1 \to F_2c_2)$ where c_1 and c_2 are objects of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively and f is an arrow of $\mathbf{C}$; and, as arrows from (c_1, c_2, f) to (c_1', c_2', f'), all pairs $(g_1 : c_1 \to c_1', g_2 : c_2 \to c_2')$, where g_1 and g_2 are arrows of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively, such that

$$\begin{array}{ccc} F_1c_1 & \xrightarrow{F_1g_1} & F_1c_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ F_2c_2 & \xrightarrow[F_2g_2]{} & F_2c_2' \end{array}$$

commutes. Given $(g_1 : c_1 \to c_1', g_2 : c_2 \to c_2')$ and $(g_1' : c_1' \to c_1'', g_2' : c_2' \to c_2'')$, their composite is $(g_1' \circ_1 g_1, g_2' \circ_2 g_2)$; this is well-defined since F_1 and F_2 are functors and associative because $\mathbf{C}$ is a category. The identity arrow for $(c_1, c_2, f : F_1c_1 \to F_2c_2)$ is $(1_{c_1}, 1_{c_2})$; this indeed satisfies the identity property since F_1 and F_2 are functors and $\mathbf{C}$ is a category.

Comma categories are a very general concept that enable a unification of many otherwise seemingly *ad hoc* concepts: in the above discussion of limits, we had to define a notion of category of 'arrows to F' and, moreover, restrict to 'arrows from objects of the form Δ_c'. This can be elegantly presented using comma categories:

If $c : \mathbf{1} \to \mathbf{C}$ is the constant functor selecting the object c in $\mathbf{C}$ then $1_{\mathbf{C}} \downarrow c$ is the *slice category over* c, written $\mathbf{C}/c$, of arrows into c. More generally, $F_1 \downarrow c$ is the *category of arrows from* F_1 *to* c. The *category of cones to [the diagram]* F can therefore be expressed as $\Delta \downarrow F$ where $\Delta_{\mathbf{J}} : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ sends c to Δ_c and $f : c \to c'$ to the natural transformation $\Delta f : \Delta_c \to \Delta_{c'}$ whose components are all f; and $F : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ is the constant functor selecting F.

2.3 Universal arrows

An object $\mathbf{1}$ of $\mathbf{C}$ is *terminal* iff, for all objects c of $\mathbf{C}$, there is exactly one arrow from c to $\mathbf{1}$. Dually, an object $\mathbf{0}$ is *initial* in $\mathbf{C}$ iff, for all objects c of $\mathbf{C}$, there is exactly one arrow from $\mathbf{0}$ to c.

Any singleton set is terminal in **Set**; the category $\mathbf{1}$ is terminal in **Cat**. The empty set is initial in **Set**; the empty category $\mathbf{0}$, with no objects, is initial in **Cat**.

Initial and terminal objects need not be unique but they are always unique *up to isomorphism*: if t and t' are both terminal objects in $\mathbf{C}$, there must be an arrow $f' : t' \to t$ from t' to t and an arrow $f : t \to t'$ from t to t'; so $f' \circ f = 1_t$, the unique arrow from t to itself, and $f \circ f' = 1_{t'}$, the unique arrow from t' to itself. Furthermore, t and t' are isomorphic up to a *unique* isomorphism: f' and f are themselves unique since t and t' are terminal.

A *terminal arrow* from a functor $F : \mathbf{C} \to \mathbf{C}'$ to an object c' of $\mathbf{C}'$ is a terminal object in $F \downarrow c'$. In other words, a terminal arrow is an object c_t of $\mathbf{C}$ and an arrow $f'_t : Fc_t \to c'$ of $\mathbf{C}'$ such that, for any arrow $f' : Fc \to c'$ of $\mathbf{C}'$, there is a *unique* arrow $f^\flat : c \to c_t$ of $\mathbf{C}$ such that

$$\begin{array}{ccc} Fc & \xrightarrow{Ff^\flat} & Fc_t \\ & {\scriptstyle f'} \searrow & \downarrow {\scriptstyle f'_t} \\ & & c' \end{array}$$

commutes.

An *initial arrow* from c' to F is defined dually. We speak of a *universal* arrow when we do not care to stress whether it is initial or terminal.

A universal cone [limit] is therefore the particular case of a terminal arrow from a diagonal functor $\Delta_{\mathbf{J}} : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$. An initial arrow to a diagonal functor is called a *co-limit*.

Products A terminal arrow from $\Delta_{\mathbf{2}} : \mathbf{C} \to \mathbf{C}^{\mathbf{2}}$ to the object* (c_1, c_2) consists of an object, that we write as $c_1 \times c_2$, of $\mathbf{C}$ and an arrow $(\pi_1, \pi_2) : (c_1 \times c_2, c_1 \times c_2) \to (c_1, c_2)$ of $\mathbf{C}^{\mathbf{2}}$ such that, for any arrow $(f_1, f_2) : (c, c) \to (c_1, c_2)$ of $\mathbf{C}^{\mathbf{2}}$, there is a *unique* arrow $f : c \to c_1 \times c_2$ such that

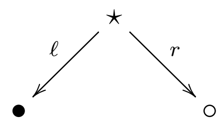
$$\begin{array}{ccc} (c, c) & \xrightarrow{(f,f)} & (c_1 \times c_2, c_1 \times c_2) \\ & {\scriptstyle (f_1,f_2)} \searrow & \downarrow {\scriptstyle (\pi_1,\pi_2)} \\ & & (c_1, c_2) \end{array}$$

commutes.

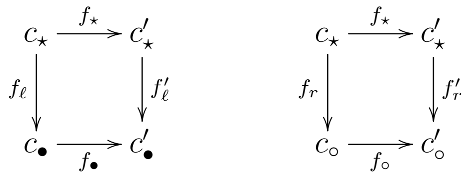
If the object $c_1 \times c_2$ and the arrows $\pi_1 : c_1 \times c_2 \to c_1$ and $\pi_2 : c_1 \times c_2 \to c_2$ exist in $\mathbf{C}$ then we say that $c_1 \times c_2$ is the *product* of c_1 and c_2; π_1 and π_2 are known as the *projections* (from $c_1 \times c_2$) and f as the *pairing* of f_1 and f_2.

*We have exploited the isomorphism $\mathbf{C} \times \mathbf{C} \cong \mathbf{C}^{\mathbf{2}}$, between the product of $\mathbf{C}$ with itself and the functor category from the discrete category $\mathbf{2}$, in order to have a more elementary description of the objects and arrows of $\mathbf{C}^{\mathbf{2}}$ as pairs of objects and pairs of arrows of $\mathbf{C}$.

Pull-backs and push-outs More generally, consider the category $\wedge$ with three objects and two non-identity arrows:



The category $\mathbf{C}^{\wedge}$ has *spans*, *i.e.* diagrams of the form $c_\bullet \xleftarrow{f_\ell} c_\star \xrightarrow{f_r} c_\circ$ in $\mathbf{C}$, as objects and triples $\langle f_\star, f_\bullet, f_\circ \rangle$ of arrows of $\mathbf{C}$ satisfying



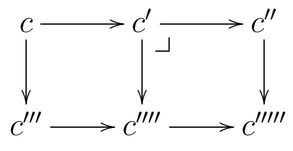
as arrows.

The category of *co-spans* of $\mathbf{C}$, *i.e.* diagrams of the form $c_\bullet \xrightarrow{f_\ell{}^{op}} c_\star \xleftarrow{f_r{}^{op}} c_\circ$ in $\mathbf{C}$, is defined as $\mathbf{C}^{\vee}$ where $\vee := \wedge^{op}$.

A terminal arrow from $\Delta_\vee : \mathbf{C} \to \mathbf{C}^{\vee}$ to the co-span $c_2 \xrightarrow{f_{24}} c_4 \xleftarrow{f_{34}} c_3$ is a *pull-back* of the co-span. Concretely, this consists of an object c_1 and arrows $f_{1i} : c_1 \to c_i$ [for $i = 2, 3, 4$] such that $f_{24} \circ f_{12} = f_{14} = f_{34} \circ f_{13}$, *i.e.* a span making the resulting square commute which additionally satisfies the universal property that any other span making the square commute factors uniquely through it. If c_4 is a terminal object, this degenerates to the product of c_2 and c_3.

Dually, an initial arrow from the span $c_2 \xleftarrow{f_{12}} c_1 \xrightarrow{f_{13}} c_3$ to $\Delta_\wedge$ is a *push-out* from the span. If c_1 is an initial object, this defines a *co-product* of c_2 and c_3: an object $c_2 + c_3$ of $\mathbf{C}$ and *injections* $\iota_2 : c_2 \to c_2 + c_3$ and $\iota_3 : c_3 \to c_2 + c_3$ in $\mathbf{C}$ such that any pair of arrows $f_2 : c_2 \to c$ and $f_3 : c_3 \to c$ factorizes uniquely through the injections via their *co-pairing* $[f_2, f_3] : c_2 + c_3 \to c$.

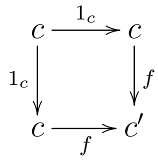
Suppose we have commuting squares



where, as indicated, the right-hand inner square is a pull-back. It follows that the left-hand inner square is a pull-back if, and only if, the outer rectangle is a pull-back. This is called the *pasting lemma* for pull-backs.

2.4 Monos and pull-backs

An arrow $f : c \to c'$ is a *mono* iff, for any pair of parallel arrows $g_1, g_2 : c'' \to c$, if $f \circ g_1 = f \circ g_2$ then $g_1 = g_2$, *i.e.* f is post-cancellable. We write $f : c \rightarrowtail c'$ to specify that f is a mono. The arrow $f : c \to c'$ is a mono if, and only if,



is a pull-back: given $f_1, f_2 : c'' \to c$ where $f \circ f_1 = f \circ f_2$, we have a unique $f' : c'' \to c$ such that $f_1 = 1_c \circ f' = f' = 1_c \circ f' = f_2$; and, for any $f_1, f_2 : c'' \to c$ such that $f \circ f_1 = f \circ f_2$, we have that $f_1 = f_2$ which defines the unique arrow that makes the commuting square $f \circ 1_c = f \circ 1_c$ a pull-back.

If $f : c \rightarrowtail c'$ and $f' : c' \rightarrowtail c''$ then $f' \circ f$ is a mono: if $g_1, g_2 : c''' \to c$ satisfy $(f' \circ f) \circ g_1 = (f' \circ f) \circ g_2$ then $f \circ g_1, f \circ g_2 : c''' \to c'$ and $f \circ g_1 = f \circ g_2$, since f' is a mono, whereupon $g_1 = g_2$ since f is a mono.

If $f : c \rightarrowtail c'$, $f_1 : c \to c''$ and $f_2 : c'' \to c'$ satisfy $f = f_2 \circ f_1$ then f_1 is a mono: if $g_1, g_2 : c''' \to c$ satisfy $f_1 \circ g_1 = f_1 \circ g_2$ then $f_2 \circ (f_1 \circ g_1) = f_2 \circ (f_1 \circ g_2)$, whereupon $g_1 = g_2$ since f is a mono.

Monos are preserved by pull-backs in the following sense: given a co-span $f : c' \to c$ and $g : c'' \rightarrowtail c$ such that the span $f' : c''' \to c''$ and $g' : c''' \to c'$ is a pull-back thereof, it follows that g' is a mono. To see this, suppose that $h_1, h_2 : c'''' \to c'''$ such that $g' \circ h_1 = g' \circ h_2$; then $f \circ (g' \circ h_1) = f \circ (g' \circ h_2)$ and so $f' \circ h_1 = f' \circ h_2$ since the square commutes and g is a mono. Moreover, the span $g' \circ h_1 : c'''' \to c'$ and $f' \circ h_2 : c'''' \to c''$ makes the square commute; so there is a unique $h : c'''' \to c'''$ such that $g' \circ h = g' \circ h_1$ and $f' \circ h = f' \circ h_2$. But both h_1 and h_2 satisfy these conditions on h, so $h_1 = h_2$.

An arrow $f : c' \to c$ of $\mathbf{C}$ is an *epi* iff $f^{op} : c \to c'$ is a mono in $\mathbf{C}^{op}$. An epi is thus pre-cancellable. We write $f : c' \twoheadrightarrow c$ to specify that f is an epi. By definition, epis are preserved by push-outs in the dual of the preceding sense.

Chapter 3

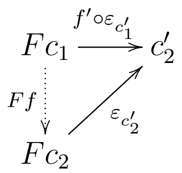
Category theory III

Adjoint functors, ...

3.1 Adjoint functors

Let $F : \mathbf{C} \to \mathbf{C}'$ be a functor and suppose that, for *every* object c' of $\mathbf{C}'$, we have an object $c := Gc'$ and a given terminal arrow $\varepsilon_{c'} : Fc \to c'$.

We extend the object mapping G to a functor $G : \mathbf{C}' \to \mathbf{C}$ by sending each arrow $f' : c'_1 \to c'_2$ of $\mathbf{C}'$ to the unique arrow $f : c_1 \to c_2$ of $\mathbf{C}$ [where $c_1 := Gc'_1$ and $c_2 := Gc'_2$] such that



commutes, *i.e.* $Gf' := (f' \circ \varepsilon_{c'_1})^\flat$.

This is indeed a functor since (i) $1'_{c'} : c' \to c'$ is sent to the unique arrow $f : c \to c$ such that $\varepsilon_{c'} = \varepsilon_{c'} \circ Ff$, so $f = 1_c$ as F is a functor; and (ii) for $f'_1 : c'_1 \to c'_2$ and $f'_2 : c'_2 \to c'_3$, there is a unique arrow $G(f'_2 \circ' f'_1) := g : c_1 \to c_3$ such that $(f'_2 \circ' f'_1) \circ' \varepsilon_{c'_1} = \varepsilon_{c'_3} \circ' Fg$; and unique arrows $Gf'_1 := g_1 : c_1 \to c_2$ and $Gf'_2 := g_2 : c_2 \to c_3$ such that $f'_1 \circ' \varepsilon_{c'_1} = \varepsilon_{c'_2} \circ' Fg_1$ and $f'_2 \circ' \varepsilon_{c'_2} = \varepsilon_{c'_3} \circ' Fg_2$; so $\varepsilon_{c'_3} \circ' F(g_2 \circ g_1) = \varepsilon_{c'_3} \circ' Fg_2 \circ' Fg_1 = f'_2 \circ' \varepsilon_{c'_2} \circ' Fg_1 = f'_2 \circ' f'_1 \circ' \varepsilon_{c'_1}$, *i.e.* $g = g_2 \circ g_1$ as required. [Draw the diagrams!]

We say that F is *left adjoint* to G; note that F does not determine G without the additional data of the terminal arrows $\varepsilon_{c'}$.

The fact that G is a functor means that the terminal arrows $\varepsilon_{c'}$ are in fact the co-ordinates of a natural transformation $\varepsilon : F \circ G \to 1_{\mathbf{C}'}$: given an arrow $f' : c'_1 \to c'_2$ of $\mathbf{C}'$, the required naturality square

$$\begin{array}{ccc} (F \circ G)c'_1 & \xrightarrow{\varepsilon_{c'_1}} & c'_1 \\ {\scriptstyle (F\circ G)f'}\downarrow & & \downarrow{\scriptstyle f'} \\ (F \circ G)c'_2 & \xrightarrow[\varepsilon_{c'_2}]{} & c'_2 \end{array}$$

is simply the above triangle. The natural transformation ε is called the *co-unit* of the adjunction, a remarkably confusing terminology [from universal algebra] since it is induced by *terminal*, not *initial*, properties.

Given an object c of $\mathbf{C}$, define $\eta_c : c \to (G \circ F)c$ to be the unique arrow of $\mathbf{C}$ such that

$$\begin{array}{ccc} Fc & \xrightarrow{1'_{Fc}} & Fc \\ {\scriptstyle F\eta_c}\downarrow & \nearrow{\scriptstyle \varepsilon_{Fc}} & \\ (F \circ G \circ F)c & & \end{array}$$

commutes, *i.e.* $\eta_c := 1'^{\flat}_{Fc}$. Given $g' : Fc \to c'$, we have that $F(Gg' \circ \eta_c) = FGg' \circ F\eta_c$, since F is a functor, and $\varepsilon_{c'} \circ FGg' = g' \circ \varepsilon_{Fc}$, since ε is a natural transformation; therefore $\varepsilon_{c'} \circ F(Gg' \circ \eta_c) = g' \circ (\varepsilon_{Fc} \circ F\eta_c) = g'$ which we can rephrase as $g'^{\flat} = Gg' \circ \eta_c$.

If $f : c \to Gc'$ is an arrow of $\mathbf{C}$, its left adjunct $f^{\sharp} := \varepsilon_{c'} \circ Ff$ factors, by definition, through Ff so that $f = f^{\sharp\flat} = Gf^{\sharp} \circ \eta_c$, *i.e.* f factors through $Gf^{\sharp}$. If another $g' : Fc \to c'$ satisfies $f = Gg' \circ \eta_c$ then $g'^{\flat} = Gg' \circ \eta_c = f = f^{\sharp\flat}$ and so $f^{\sharp} = g'$. This establishes that η_c is an initial arrow from c to G.

Moreover, given an arrow $f : c_1 \to c_2$ of $\mathbf{C}$, the left adjunct $(\eta_{c_2} \circ f)^{\sharp} := \varepsilon_{Fc_2} \circ F(\eta_{c_2} \circ f) = \varepsilon_{Fc_2} \circ F\eta_{c_2} \circ Ff = Ff$ so that $Ff \circ \eta_{c_1} = \eta_{c_2} \circ f$, *i.e.* η is a natural transformation.

Finally, the left adjunct $1^{\sharp}_{Gc'} = \varepsilon_{c'}$ so that $G\varepsilon_{c'} \circ \eta_{Gc'} = 1_{Gc'}$; this is the so-called *triangle identity* for η:

$$\begin{array}{ccc} Gc' & \xrightarrow{1_{Gc'}} & Gc' \\ {\scriptstyle \eta_{Gc'}}\downarrow & \nearrow{\scriptstyle G\varepsilon_{c'}} & \\ (G \circ F \circ G)c' & & \end{array}$$

Let us recap: starting from a functor $F : \mathbf{C} \to \mathbf{C}'$ and a family of terminal arrows $\varepsilon_{c'}$ (in $\mathbf{C}'$, indexed by the objects of $\mathbf{C}'$), we can define (i) a functor $G : \mathbf{C}' \to \mathbf{C}$ for which $\varepsilon : F \circ G \to 1_{\mathbf{C}'}$ becomes a natural transformation; and (ii) a family of initial arrows η_c (in $\mathbf{C}$, indexed by the objects of $\mathbf{C}$) that form a natural transformation $\eta : 1_{\mathbf{C}} \to G \circ F$ that satisfies the triangle identities: one by definition; and the other as shown just above.

Recall that the left adjunct $f^\sharp : Fc \to c'$ of an arrow $f : c \to Gc'$ of $\mathbf{C}$ is defined to be $f^\sharp := \varepsilon_{c'} \circ Ff$. The induced mapping from $\hom_{\mathbf{C}'}(Fc, c')$ to $\hom_{\mathbf{C}}(c, Gc')$ is (i) surjective, since every $f : c \to Gc'$ gives rise to some left adjunct; and (ii) injective, since $\varepsilon_{c'}$ being terminal means that $f^\sharp$ is f's unique left adjunct.

This bijection $\phi^{-1}_{c,c'} : \hom_{\mathbf{C}}(c, Gc') \cong \mathit{hom}_{\mathbf{C}'}(Fc, c')$ is 'natural' in the sense that, given $g : c_0 \to c$, $\phi^{-1}_{c_0,c'}(f \circ g) := \varepsilon_{c'} \circ F(f \circ g) = \varepsilon_{c'} \circ Ff \circ Fg =: \phi^{-1}_{c,c'}(f) \circ Fg$; and, given $g' : c' \to c'_0$, $\phi^{-1}_{c,c'_0}(Gg' \circ f) := \varepsilon_{c'_0} \circ F(Gg' \circ f) = \varepsilon_{c'_0} \circ FGg' \circ Ff = g \circ \varepsilon_{c'} \circ Ff =: g \circ \phi^{-1}(f)$.

If $\mathbf{C}$ and $\mathbf{C}'$ are locally small categories, this gives us a *bona fide* natural isomorphism $\phi_{c,c'} : \hom_{\mathbf{C}'}(Fc, c') \cong \hom_{\mathbf{C}}(c, Gc')$ in **Set** with naturality in c [with respect to $g : c_0 \to c$]

$$\begin{array}{ccc}
\hom_{\mathbf{C}'}(Fc, c') & \xrightarrow{\phi_{c,c'}} & \hom_{\mathbf{C}}(c, Gc') \\
{\scriptstyle \lambda f'.f' \circ Fg}\big\downarrow & & \big\downarrow{\scriptstyle \lambda f.f \circ g} \\
\hom_{\mathbf{C}'}(Fc_0, c') & \xrightarrow[\phi_{c_0,c'}]{} & \hom_{\mathbf{C}}(c_0, Gc')
\end{array}$$

and naturality in c' [with respect to $g' : c' \to c'_0$]

$$\begin{array}{ccc}
\hom_{\mathbf{C}'}(Fc, c') & \xrightarrow{\phi_{c,c'}} & \hom_{\mathbf{C}}(c, Gc') \\
{\scriptstyle \lambda f'.g \circ f'}\big\downarrow & & \big\downarrow{\scriptstyle \lambda f.Gg' \circ f} \\
\hom_{\mathbf{C}'}(Fc, c'_0) & \xrightarrow[\phi_{c,c'_0}]{} & \hom_{\mathbf{C}}(c, Gc'_0)
\end{array}$$

dually to the above.

Note how the terminal arrows $\varepsilon_{c'}$ allow a generalization of the case, found in *equivalences* of categories, where F being a fully faithful functor induces a bijection $\hom_{\mathbf{C}}(c_1, c_2) \cong \hom_{\mathbf{C}'}(Fc_1, Fc_2)$. In the case of an adjunction, despite the lack of the assumption that F be full and faithful, we obtain our bijection by virtue of the universal property of $\varepsilon_{c'}$.