Polarized games

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Abstract

We generalize the intuitionistic Hyland–Ong games (and in a second step Abramsky–Jagadeesan–Malacaria games) to a notion of *polarized games* allowing games with plays starting by proponent moves. The usual constructions on games are adjusted to fit this setting yielding game models for both *Intuitionistic Linear Logic* and *Polarized Linear Logic*. We prove a definability result for this polarized model and this gives complete game models for various classical systems: LC, $\lambda\mu$-calculus, ... for both call-by-name and call-by-value evaluations.

Introduction

Game semantics has been used to interpret both logical systems and programming languages. The logical step has often been a preliminary step towards the study of game models for programming languages. Moreover Linear Logic (LL) has taken a very important place in this first step. We can classify these models of linear logic along two main constraints:

- some of them are restricted to linear fragments (without exponential connective) of LL, such as MLL [1] or MALL [2],
- the others are restricted to intuitionistic fragments [3, 4, 5].

In a different spirit, a model of MELL is given in [6] but introduces non-deterministic strategies to model a deterministic language and does not lead to a completeness result.

On the computer science side, games have been developed to model different kinds of languages (PCF [4, 7], μPCF [8], Idealized Algol [9], ...). These games are based on call-by-name computation which corresponds to the technical property that plays only start by opponent moves which is also the constraint appearing in games for Intuitionistic Linear Logic (ILL). The idea of defining games constructions for proponent starting games has been used to build a model of call-by-value computation [10] (another approach for call-by-value games appears in [11]).

One of our goals is to liberalize these starting conditions in order to recover a real symmetry between the two players. This is extremely natural in the spirit of LL, where duality (lost in intuitionistic systems) plays a key role, but it is known to be a difficult problem: in Blass's work [12], composition is not associative, and non-determinism is required in [6]. Our solution is to put together opponent starting and proponent starting games but to refuse plays starting by both players in the same game. The introduction of two families of games: positive (proponent starting) and negative (opponent starting) corresponds to the notion of polarity developed by Girard for his system of classical logic LC [13] and studied by the author in Polarized Linear Logic (LLP).

As is clearly the case for game semantics, full LL is a difficult system to deal with. The problem is to find a more simple fragment of LL which is expressive enough. The main proposition has been ILL, but it refuses the linear negation connective which may be considered as the main connective of LL since it gives duality. From an expressiveness view point, ILL is a good system for the study of intuitionistic logic but the translations of classical logic into ILL are in fact $\neg\neg$-translations. Using Girard's idea of polarization for classical logic, the system LLP [14] gives another possibility. It is obtained from LL by restricting to polarized formulas and by generalizing structural rules to any negative formula (instead of only ?-formulas) to get classical features. The study of this system is easier than for LL (proof-nets, ...), and the current presentation will enforce this view point by giving a game model. Translations of various classical systems into LLP have been developed and LLP appears as the part of LL corresponding to classical logic.

We are going to describe the notion of *polarized games* containing both proponent starting and opponent starting games. They are presented as both a model of ILL and a model of LLP, where ILL has to be considered as the natural linear setting for the study of intuitionistic logic and LLP as the natural one for classical logic.

In order to get a model of these two systems, the key ingredient is to have a good interaction between the numerous constructions required. The model of ILL is based on negative games only whereas LLP also requires positive games. In particular the multiplicative structure of the two models is different since ILL is based on a negative tensor product (denoted by $\odot$ and coming from [5]), and LLP is based on a positive tensor product (denoted by $\otimes$ and coming from [10]). The same story continues with the exponential structure which transforms a negative game into a negative one for ILL (denoted by $\sharp$) and into a positive one for LLP (denoted by !). These two structures are related for example by the fact that $\sharp A \odot \sharp B \twoheadrightarrow C$ and $!A \otimes !B \multimap C$ lead to the same game but correspond to an analysis in ILL for the former and in LLP for the latter.

From the use of polarities in ludics [15], we get the idea of introducing two new *lifting* connectives $\downarrow$ and $\uparrow$ allowing a "linear" change of polarity (they have also been introduced by Lamarche [3], and used recently in [16]). These two connectives act on games by adding a new move at the beginning of each play, in such a way that $\downarrow$ (resp. $\uparrow$) turns a negative (resp. positive) game into a positive (resp. negative) one, and they are responsible for the good cohabitation of positive and negative games.

The introduction of this large collection of connectives allows to go one step further than in LL in the decomposition of the classical connectives and allows to give a precise analysis of the structure of games. In particular, the separation between positive and negative games allows to solve the Blass's problem [12] of composing strategies. The introduction of the lifting connectives gives a solution to McCusker's problem with well-openness for defining the ! construction [5] and leads to a decomposition of the main LL isomorphism $!A \otimes !B \simeq !(A \,\&\, B)$.

As our polarized games approach leads to a model of both ILL and LLP, it can be used to describe models of many other systems. ILL has been used to embed the λ-calculus and linearized variants of the λ-calculus [17, 18], and the author has studied generalizations of Girard's translations of intuitionistic logic into LL to embed many classical systems in LLP [14], using $A \to B \rightsquigarrow !A \multimap B$ (the negative interpretation) for the call-by-name systems: $\lambda\mu$-calculus [19], LKT [20], λc-calculus, ... and $A \to B \rightsquigarrow !(A \multimap ?B)$ (the positive interpretation) for the call-by-value systems: $\lambda\mu_V$-calculus [21], LKQ [20], ... This unification between call-by-name and call-by-value (an idea independently carried out by Levy [22], without linear logic) realized by LLP through the duality positive/negative (or focalization/reversibility), in the spirit of [23, 24], is preserved in our polarized game model. Indeed, this model is definable without any particular choice between call-by-name and call-by-value, and we then get a model of a particular evaluation paradigm by choosing the corresponding interpretation: negative for call-by-name and positive for call-by-value.

Games are not only used because they allow to define models for a large class of systems but also because they lead to *full completeness* results, see [1, 2, 3, 4, 7] for example. We end our study of polarized games with a full completeness (or definability) theorem with respect to LLP (without atom): a strategy on a polarized game is the interpretation of a proof of LLP. And, as a consequence, we get the same result for both call-by-name and call-by-value $\lambda\mu$-calculi.

The first section of the paper is devoted to the definition of polarized games in an HO [7] setting following McCusker's ideas. Section 2 describes the induced model of ILL recalling McCusker's results. In section 3, we present the model of LLP with the corresponding definability result. In a second part of the paper corresponding to section 4, we develop the same ideas in an AJM [4] setting leading to another complete model of LLP. Finally section 5 describes more explicitly the consequences on the denotational semantics of call-by-name and call-by-value $\lambda\mu$-calculi.

1 Polarized HO game semantics

In this section, we recall the key ingredients of HO-style game semantics. Our goal is to give the required definitions with the appropriate extensions to the polarized case. Some of these definitions are quite technical and it is possible to find more explanations in [5, 25].

1.1 Arenas and games

We introduce our notion of *polarized games* by extending the usual definitions of Hyland–Ong games [7] with plays possibly starting by proponent moves. We are following McCusker's presentation [5], but we change the notations for some constructions in order to have a precise correspondence between games constructions and LLP connectives, moreover we remove the Q/A distinction.

Definition 1 (Polarized arena)
A *polarized arena* is a tuple

$$A = (\pi_A, \mathcal{M}_A, \lambda_A, \vdash_A)$$

where:

- $\pi_A \in \{O, P\}$ is the polarity of the arena, an O-arena (resp. P-arena) is also called *negative* (resp. *positive*);
- $\mathcal{M}_A$ is the set of *moves*;
- λ_A is the *labelling function* from $\mathcal{M}_A$ to $\{O, P\}$, we use the notation $m^{\lambda_A(m)}$ to make explicit the label of a move m;
- $\vdash_A$ is the *enabling relation*, that is a subset of $(\{*\} \cup \mathcal{M}_A) \times \mathcal{M}_A$ denoted by $m \vdash_A n$. The moves m such that $* \vdash_A m$ are the *initial moves* of the arena. This relation has to satisfy:
  - $* \vdash_A m \Rightarrow \lambda_A(m) = \pi_A \wedge \forall n \in \mathcal{M}_A, n \nvdash_A m$
  - $m \vdash_A n \Rightarrow \lambda_A(m) \neq \lambda_A(n)$

We denote by $\overline{\pi_A}$ (resp. $\overline{\lambda_A}$) the opposite of π_A (resp. λ_A) and by $\mathcal{M}_A^i$ (resp. $\mathcal{M}_A^{ni}$) the initial (resp. non-initial) moves of A.

Definition 2 (Justified sequence)
A *justified sequence* s on A is a sequence of moves of A with, for each non-initial move n, a pointer to an earlier move m such that $m \vdash_A n$, we say that m *justifies* n in s.

If there exists a sub-sequence $n_0, \ldots, n_k$ of moves of s such that n_i justifies n_{i+1}, we say that n_0 *hereditarily justifies* n_k in s. If n is a move of a justified sequence s, there is a unique initial move m which hereditarily justifies n in s.

Notations and conventions.

- ε is the empty sequence of moves.
- $\leq$ is the prefix order on (justified) sequences of moves.
- The P-prefix order is defined by $s \leq^P t$ if $s \leq t$ and s ends by a P-move (including $s = \varepsilon$).
- $cl^P(.)$ is the P-prefix closure of a set of (justified) sequences.
- If s is a (justified) sequence of moves, *"m is a move of s"* will always mean that m is an *occurrence* of move of s.
- When the context is not ambiguous, we will sometimes say "*move*" instead of "*move with its pointer*".

Definition 3 (View)
Let s be a justified sequence, we define a sub-sequence called the *proponent view* $\ulcorner s \urcorner$ by: $\ulcorner \varepsilon \urcorner = \varepsilon$, $\ulcorner sm^P \urcorner = \ulcorner s \urcorner m^P$, $\ulcorner sm^O \urcorner = m^O$ if m is initial, and $\ulcorner smtn^O \urcorner = \ulcorner s \urcorner mn^O$ if m justifies n. The *opponent view* $\llcorner s \lrcorner$ is defined exactly as the proponent view by exchanging the two players.

The *bi-view* $\llcorner s \urcorner$ is given by: $\llcorner \varepsilon \urcorner = \varepsilon$, $\llcorner sm \urcorner = m$ if m is initial, and $\llcorner smtn \urcorner = \llcorner sm \urcorner n$ if m justifies n.

Definition 4 (Legal position)
A justified sequence s is a *legal position* if:

- *alternation*: $tmn \leq s \Rightarrow \lambda(m) \neq \lambda(n)$
- *proponent visibility*: $tm^P \leq s \Rightarrow m$ points in $\ulcorner t \urcorner$ if m is not initial
- *opponent visibility*: $tm^O \leq s \Rightarrow m$ points in $\llcorner t \lrcorner$ if m is not initial

The set of the legal positions of an arena A is denoted by L_A.

A legal position s is *well opened* if the only initial move in s is the first one.

Lemma 1
If s is a legal position, $\llcorner s \urcorner = \llcorner \ulcorner s \urcorner \lrcorner = \ulcorner \llcorner s \lrcorner \urcorner$.

Proof: We prove the first equality by induction on the length of s:

- $\llcorner \varepsilon \urcorner = \varepsilon = \llcorner \ulcorner \varepsilon \urcorner \lrcorner$.
- If m^O is initial, $\llcorner sm^O \urcorner = m^O$ and $\llcorner \ulcorner sm^O \urcorner \lrcorner = \llcorner m^O \lrcorner = m^O$.
- If m^P is initial, $\llcorner sm^P \urcorner = m^P$ and $\llcorner \ulcorner sm^P \urcorner \lrcorner = \llcorner \ulcorner s \urcorner m^P \lrcorner = m^P$.
- If m justifies n^O, $\llcorner smtn^O \urcorner = \llcorner sm \urcorner n^O$ and $\llcorner \ulcorner smtn^O \urcorner \lrcorner = \llcorner \ulcorner sm \urcorner n^O \lrcorner = \llcorner \ulcorner sm \urcorner \lrcorner n^O$, with $\llcorner sm \urcorner = \llcorner \ulcorner sm \urcorner \lrcorner$ by induction hypothesis.
- If m justifies n^P, $\llcorner smtn^P \urcorner = \llcorner sm \urcorner n^P$ and $\llcorner \ulcorner smtn^P \urcorner \lrcorner = \llcorner \ulcorner smt \urcorner n^P \lrcorner = \llcorner \ulcorner sm \urcorner \lrcorner n^P$ because if $m \in \ulcorner smt \urcorner$ (true by proponent visibility), $\ulcorner sm \urcorner$ is the prefix of $\ulcorner smt \urcorner$ ending by m. Moreover $\llcorner sm \urcorner = \llcorner \ulcorner sm \urcorner \lrcorner$ by induction hypothesis. □

Definition 5 (Projection on initial moves)
Let s be a legal position in A and I be a set of occurrences of initial moves of s, the *projection* $s \upharpoonright_I$ of s on I is the justified sub-sequence of s of the moves hereditarily justified by a move of I.

If m is an occurrence of initial move of s and J is a set of occurrences of moves of s justified by m, the *projection* $s \upharpoonright_{mJ}$ is the justified sub-sequence of s containing m, the moves of J and the moves hereditarily justified by a move of J.

If m is initial and justifies n and if the moves of K are justified by n, we define $s \upharpoonright_{mnK}$ in the same spirit, ...

We can see that with the conditions described above, $s \upharpoonright_I$, $s \upharpoonright_{mJ}$, $s \upharpoonright_{mnK}$, ... are legal positions in A, and $s \upharpoonright_{mJ}$, $s \upharpoonright_{mnK}$, ... are well opened.

The main constructions we need on polarized arenas are the following:

Sum of arenas. Let A and B be two arenas of the same polarity, we define the arena $A + B$ by:

- $\pi_{A+B} = \pi_A = \pi_B$
- $\mathcal{M}_{A+B} = \mathcal{M}_A + \mathcal{M}_B$ (disjoint sum)
- $\lambda_{A+B} = [\lambda_A, \lambda_B]$
- $* \vdash_{A+B} m \iff * \vdash_A m \vee * \vdash_B m$
- $m \vdash_{A+B} n \iff m \vdash_A n \vee m \vdash_B n$

If s is a legal position of $A+B$, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sub-sequence of s containing the moves of A (resp. B).

Product of arenas. If A and B have the same polarity, $A \times B$ is defined by:

- $\pi_{A\times B} = \pi_A = \pi_B$
- $\mathcal{M}_{A\times B} = \mathcal{M}^i_A \times \mathcal{M}^i_B + \mathcal{M}^{ni}_A + \mathcal{M}^{ni}_B$
- $\lambda_{A\times B}(m_1, m_2) = \lambda_A(m_1) = \lambda_B(m_2)$ if $(m_1, m_2) \in \mathcal{M}^i_A \times \mathcal{M}^i_B$
- $\lambda_{A\times B}(m) = [\lambda_A, \lambda_B](m)$ if $m \in \mathcal{M}^{ni}_A + \mathcal{M}^{ni}_B$
- $* \vdash_{A\times B} (m_1, m_2)$
- $(m_1, m_2) \vdash_{A\times B} n \iff m_1 \vdash_A n \vee m_2 \vdash_B n$ if (m_1, m_2) is initial
- $m \vdash_{A\times B} n \iff m \vdash_A n \vee m \vdash_B n$ if m is not initial

If s is a well opened legal position of $A\times B$, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sub-sequence of s containing the moves of A (resp. B) thus the first (resp. second) component of the initial move.

Remark: Defining the notion of projection on a component for a non well opened position of $A \times B$ would be more complex, this is why we will restrict ourselves to this particular case in the definition of the product of arenas. This is sufficient for what we want in this paper since this construction will mainly be used from section 3.3 and only with well opened games.

Exponential of arenas. Let A and B be two arenas of the same polarity, we define the arena B^A by:

- $\pi_{B^A} = \pi_A = \pi_B$
- $\mathcal{M}_{B^A} = \mathcal{M}_A + \mathcal{M}_B$
- $\lambda_{B^A} = [\overline{\lambda_A}, \lambda_B]$
- $* \vdash_{B^A} m \iff * \vdash_B m$
- $m \vdash_{B^A} n \iff m \vdash_A n \vee m \vdash_B n \vee (* \vdash_B m \wedge * \vdash_A n)$

If s is a legal position of B^A, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sub-sequence of s containing the moves of A (resp. B).

Lifting of arenas. Let A be an arena, $\updownarrow A$ is the arena of opposite polarity defined by:

- $\pi_{\updownarrow A} = \overline{\pi_A}$
- $\mathcal{M}_{\updownarrow A} = \{\circ\} + \mathcal{M}_A$ where $\circ$ is a new move not in $\mathcal{M}_A$
- $\lambda_{\updownarrow A} = \lambda_A$ for the moves of $\mathcal{M}_A$
- $\lambda_{\updownarrow A}(\circ) = \overline{\pi_A}$
- $* \vdash_{\updownarrow A} \circ$
- $\circ \vdash_{\updownarrow A} m \iff * \vdash_A m$
- $m \vdash_{\updownarrow A} n \iff m \vdash_A n$ if $m \in \mathcal{M}_A$

Arenas allow to describe the "game board", and to obtain a game we add a set of accepted plays giving a "game rule".

Definition 6 (Polarized game)
A *polarized game* is a tuple

$$A = (\pi_A, \mathcal{M}_A, \lambda_A, \vdash_A, \mathcal{P}_A)$$

where $(\pi_A, \mathcal{M}_A, \lambda_A, \vdash_A)$ is a polarized arena and $\mathcal{P}_A$, called the set of *plays* of A, is a non-empty prefix-closed set of legal positions such that if $s \in \mathcal{P}_A$ and I is a set of occurrences of initial moves of s, $s \upharpoonright_I \in \mathcal{P}_A$.

A game is *well opened* if all its plays are well opened.

The condition $s\restriction_I \in \mathcal{P}_A$ in the previous definition is used to obtain $\mathcal{P}_A \subset \mathcal{P}_{\sharp A}$ and is immediately true for well opened games.

We denote by $\mathcal{P}_A^P$ the plays ending by P-moves (including the empty play).

We turn now to the description of the various constructions of polarized games we are interested in.

In the sequel, we will use N, M, L, ... for negative games (or formulas) and P, Q, R, ... for positive ones. A, B, C, ... denote games (or formulas) of any polarity.

Dual. $A^\perp = (\overline{\pi_A}, \mathcal{M}_A, \overline{\lambda_A}, \vdash_A, \mathcal{P}_A)$

Top. $\top = (O, \emptyset, \emptyset, \emptyset, \{\varepsilon\})$

Bottom. $\perp = (O, \{\circ\}, \lambda_\perp(\circ) = O, \{(*, \circ)\}, \{\varepsilon, \circ\})$

Negative tensor. If M and N are negative, the arena of $M \odot N$ is $M + N$ and $\mathcal{P}_{M\odot N} = \{s \in L_{M+N} \mid s\restriction_M \in \mathcal{P}_M \wedge s\restriction_N \in \mathcal{P}_N\}$.

Implication. If M and N are negative, the arena of $M \rightarrow N$ is N^M and $\mathcal{P}_{M\rightarrow N} = \{s \in L_{N^M} \mid s\restriction_M \in \mathcal{P}_M \wedge s\restriction_N \in \mathcal{P}_N\}$.

With. If M and N are negative, the arena of $M \,\&\, N$ is $M + N$ and $\mathcal{P}_{M\&N} = \mathcal{P}_M \cup \mathcal{P}_N$ (the only common play is ε).

Par. If M and N are *well opened* negative games, the arena of $M \parr N$ is $M \times N$ and $\mathcal{P}_{M\parr N} = \{s \in L_{M\times N} \mid s\restriction_M \in \mathcal{P}_M \wedge s\restriction_N \in \mathcal{P}_N\}$.

Sharp. If N is negative, $\sharp N$ has the same arena as N and $\mathcal{P}_{\sharp N} = \{s \in L_N \mid s\restriction_n \in \mathcal{P}_N, \forall n \text{ initial}\}$.

Lift. If P is positive, the arena of $\uparrow P$ is $\updownarrow P$ and $\mathcal{P}_{\uparrow P} = \circ.\mathcal{P}_P + \{\varepsilon\}$.

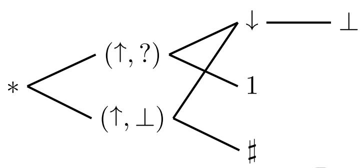
Positive constructions. The positive constructions are obtained by duality: $0 = \top^\perp$, $1 = \perp^\perp$, $P \otimes Q = (P^\perp \parr Q^\perp)^\perp$, $P \oplus Q = (P^\perp \,\&\, Q^\perp)^\perp$, $\flat P = (\sharp P^\perp)^\perp$ and $\downarrow N = (\uparrow N^\perp)^\perp$.

Linear implication. $P \multimap N = P^\perp \parr N$ which is just a notation.

Exponentials. $!N = \downarrow\sharp N$ and $?P = \uparrow\flat P = (!P^\perp)^\perp$ (just notations).

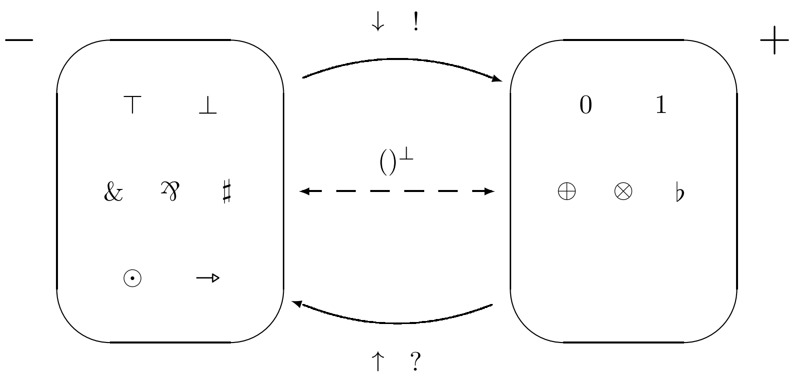
The constraints on the first move and on projections are sufficient to automatically get the usual *switching conditions* for $\odot$, $\parr$, $\sharp$, ... That is, only one player is allowed to switch between the components of a game during a play, which is opponent for a "conjunctive" connective ($\odot$, $\&$, $\sharp$, ...) and proponent for a "disjunctive" connective ($\parr$, $\rightarrow$, ...).

Example 1 (A polarized game)
The arena of the game $\uparrow\downarrow(\top \odot \perp) \parr (?1 \& (\sharp\perp \rightarrow \perp))$ is negative, has 6 moves $\{(\uparrow, ?)^O, (\uparrow, \perp)^O, \downarrow^P, 1^P, \sharp^P, \perp^O\}$, and its enabling relation is:



The following justified sequence is a play in this game: $(\uparrow, ?)\ \downarrow\ \perp\ 1$.

Remark: The $\parr$-construction is a variant of $\otimes$ defined in [10]. The lifting constructions $\downarrow$ and $\uparrow$ already appeared in Lamarche's games [3] but their use here is much in the spirit of Girard's ludics [15]. The novelty is to put these constructions together with an important place given to the lifts:



To compare the numerous constructions, we now introduce a strong notion of isomorphism of games without any reference to strategies (defined later). The idea is to represent the main properties of logical connectives by structural properties of games in a simpler way than the use of categorical isomorphisms.

Definition 7 (Play isomorphism)
A *p-isomorphism* between two games A and B of the same polarity is a bijective function f between $\mathcal{P}_A$ and $\mathcal{P}_B$ which preserves the length and such that f respects:

- *prefixes*: if $s \leq t$ then $f(s) \leq f(t)$,
- *pointers*: if the i^{th} move of s points to the j^{th} move of s then the i^{th} move of $f(s)$ points to the j^{th} move of $f(s)$,
- *bi-views*: if $\llcorner sa \lrcorner = \llcorner ta \lrcorner$ and $f(sa) = s'b$ then $f(ta) = t'b$.

Two games A and B are *p-isomorphic*, denoted by $A \simeq_p B$, if there exists a p-isomorphism between A and B.

Remark: The notion of arena we use is coming from [5, 25], and is slightly more general than the original one in [7] since the enabling relation is not required to correspond to a forest ordering. This makes our presentation more general and helps us to give a simpler definition of the product of arenas.

However this requires to add the bi-view condition in the previous definition, while in the forest case, this condition is just $f(sa) = s'b \Rightarrow f(ta) = t'b$ since the bi-view is determined by the last move. In our case this would be too strong to get $N_0 \parr (M_0 \,\&\, L_0) \simeq_p (N_0 \parr M_0) \,\&\, (N_0 \parr L_0)$ in proposition 1. Here, p-isomorphisms are not necessarily coming from isomorphisms between the underlying arenas.

A game construction has a given p-property if the underlying isomorphism is a p-isomorphism. For example, "*$\odot$ is p-commutative*" means $M \odot N \simeq_p N \odot M$ for any negative games M and N.

Proposition 1 (Structure of constructions)
All the binary constructions are p-commutative and p-associative (except $\rightarrowtriangle$ and $\multimap$), and:

$$
\begin{array}{lrcl}
\bot \textit{ p-unit for } \parr\textit{:} & \bot \parr N_0 & \simeq_p & N_0 \\
\top \textit{ p-unit for } \odot\textit{:} & \top \odot N & \simeq_p & N \\
\top \textit{ p-unit for } \&\textit{:} & \top \,\&\, N & \simeq_p & N \\
\top \textit{ p-left-unit for } \rightarrowtriangle\textit{:} & \top \rightarrowtriangle N & \simeq_p & N \\
\parr \textit{ p-distributive over } \&\textit{:} & N_0 \parr (M_0 \,\&\, L_0) & \simeq_p & (N_0 \parr M_0) \,\&\, (N_0 \parr L_0) \\
\top \textit{ p-zero for } \parr\textit{:} & \top \parr N_0 & \simeq_p & \top \\
\top \textit{ p-right-zero for } \rightarrowtriangle\textit{:} & N \rightarrowtriangle \top & \simeq_p & \top \\
 & M \rightarrowtriangle N_0 & \simeq_p & \uparrow M^{\perp} \parr N_0 \\
 & \downarrow(M \odot N) & \simeq_p & \downarrow M \otimes \downarrow N \\
 & \sharp(M \,\&\, N) & \simeq_p & \sharp M \odot \sharp N \\
 & !(M \,\&\, N) & \simeq_p & !M \otimes !N \\
 & (P_0 \otimes Q_0) \multimap L_0 & \simeq_p & P_0 \multimap (Q_0 \multimap L_0) \\
 & (M \odot N) \rightarrowtriangle L & \simeq_p & M \rightarrowtriangle (N \rightarrowtriangle L) \\
 & \top & = & \sharp\top \\
 & 1 & = & \downarrow\top = !\top
\end{array}
$$

where L, M and N are negative games, L_0*,* M_0 *and* N_0 *are* well opened *negative games and* P_0 *and* Q_0 *are* well opened *positive games.*

PROOF: We prove two key cases $M \multimap N_0 \simeq_p \uparrow M^\perp \parr N_0$ and $N_0 \parr (M_0 \,\&\, L_0) \simeq_p (N_0 \parr M_0) \,\&\, (N_0 \parr L_0)$.

We define the function f from $\mathcal{P}_{M\multimap N_0}$ into $\mathcal{P}_{\uparrow M^\perp \parr N_0}$ by $f(\varepsilon) = \varepsilon$ and $f(ns) = (\circ, n)s$. We show that f is a p-isomorphism:

- f is injective by definition and if t is a non-empty play in $\mathcal{P}_{\uparrow M^\perp \parr N_0}$ it starts by a move $(\circ, n)$ so that it is in the image of f, thus f is bijective.
- f preserves the length, and if $s \leq t$ (with $s \neq \varepsilon$), we can write $s = ns'$ and $t = nt'$ with $s' \leq t'$ and we have $f(s) = (\circ, n)s' \leq (\circ, n)t' = f(t)$.
- f does not modify pointers.
- f only modifies the first move of plays thus the last condition is straightforward.

If L_0, M_0 and N_0 are well opened negative games, we define the function f from $\mathcal{P}_{N_0 \parr (M_0 \& L_0)}$ into $\mathcal{P}_{(N_0 \parr M_0) \& (N_0 \parr L_0)}$ by $f(\varepsilon) = \varepsilon$, $f((n, (1, m))s) = (1, (n, m))(1, s)$ with $m \in \mathcal{M}_{M_0}$ and $f((n, (2, l))s) = (2, (n, l))(2, s)$ with $l \in \mathcal{M}_{L_0}$ (where (i, s) $(i = 1, 2)$ is obtained by replacing any occurrence of move $n \in \mathcal{M}_{N_0}$ in s by (i, n)). We show that f is a p-isomorphism:

- f is injective by definition and the inverse function is easy to define so that f is bijective.
- f preserves the length, and respects prefixes.
- f does not modify pointers.
- If $\ulcorner sa \urcorner = \ulcorner ta \urcorner$, then sa and ta have the same initial move (because the game is well opened) which is of the shape $(n, (i, c))$. If a is in $\mathcal{M}_{M_0}$ or in $\mathcal{M}_{L_0}$, the result is immediate since $b = a$, if a is in $\mathcal{M}_{N_0}$, $b = (i, a)$ and the result also holds.

The other cases are left to the reader. □

Remark:

- The decomposition of the ! connective into two distinct operations gives rise to a decomposition of the main LL isomorphism $!(M \,\&\, N) \simeq_p\, !M \otimes !N$ through $\downarrow(M \odot N) \simeq_p \downarrow M \otimes \downarrow N$ and $\sharp(M \,\&\, N) \simeq_p \sharp M \odot \sharp N$.
- $M \multimap N \simeq_p \uparrow M^\perp \parr N$ can be interpreted as a linear version of Girard's translation of the intuitionistic implication in LL: $M \to N = ?M^\perp \parr N$.
- The introduction of polarities gives the p-associativity of the $\oplus$-construction which is stronger than the result obtained for the corresponding negative connective of [5] and could be related to the fact that this negative connective is a *weak* coproduct. The negative construction can be decomposed by the positive one $\oplus$ into $\uparrow(\downarrow M \oplus \downarrow N)$.

A natural direction in game semantics is to move constraints on plays to constraints on strategies (in order to use arenas instead of games for example). Since we want to build a game model in which it is possible to interpret both ILL and LLP in order to compare them, we must be able to make a difference between the two games $M \,\&\, N$ and $M \odot N$ which are based on the same arena (as for N and $\sharp N$). Without plays, we cannot be precise enough in the description of games and we would lose some equations of proposition 1. In order to build a model of LLP only, it is possible to show that arenas are sufficient and Laird's results for the $\lambda\mu$-calculus [8] can be extended to a $\parr$ connective.

1.2 Strategies

We are going to introduce the notion of strategy. They will be used to interpret proofs and programs. From a categorical view point, strategies correspond to morphisms.

Definition 8 (Strategy)
A *strategy* σ on the game A, denoted by $\sigma : A$, is a non-empty P-prefix-closed subset of $\mathcal{P}_A^P$. Moreover we require some additional properties:

- *determinism*: if $sa^P \in \sigma$ and $sb^P \in \sigma$ then $sa = sb$;
- *innocence*: if $sab^P \in \sigma$, $t \in \sigma$, $ta \in \mathcal{P}_A$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$ then $tab \in \sigma$.

A strategy on a negative (resp. positive) game can be represented as a partial function from proponent views s of odd (resp. even) length to P-moves with pointers into s, called the *view function* (see [25] for example).

We define some properties of strategies:

- A strategy σ is *finite* if the sum of the lengths of the proponent views in the graph of its view function is finite. We define the *size* $|\sigma|$ of a finite strategy σ to be this sum.
- A strategy $\sigma : A$ is *total* if when $s \in \sigma$ and $sa^O \in \mathcal{P}_A$ there exists some b such that $sab \in \sigma$ (moreover, if A is positive $\exists b, b \in \sigma$). This is equivalent to ask the view function to be total.
- A strategy $\sigma : M \multimap N$ is *strict*, denoted by $\sigma : M \overset{\bullet}{\multimap} N$, if either $\mathcal{P}_N = \{\varepsilon\}$ or σ contains a play nm with $m \in \mathcal{M}_M$ for each initial move n of N.

Lemma 2 (Views and p-isomorphisms)
Let f be a p-isomorphism from A to B, $f(\ulcorner s \urcorner) = \ulcorner f(s) \urcorner$.

PROOF: By induction on the length of s:

- $f(\ulcorner \varepsilon \urcorner) = f(\varepsilon) = \varepsilon = \ulcorner \varepsilon \urcorner = \ulcorner f(\varepsilon) \urcorner$.
- $f(\ulcorner sm^P \urcorner) = f(\ulcorner s \urcorner m^P) = f(\ulcorner s \urcorner) m'^P$, and by induction hypothesis we have $f(\ulcorner s \urcorner) m'^P = \ulcorner f(s) \urcorner m'^P = \ulcorner f(s) m'^P \urcorner$, moreover $\llcorner \ulcorner s \urcorner m^P \lrcorner = \llcorner sm^P \lrcorner$ (by lemma 1) so that $f(sm^P) = f(s)m'^P$.
- If m^O is initial, $f(\ulcorner sm^O \urcorner) = f(m^O) = m'^O$, moreover $\llcorner m^O \lrcorner = \llcorner sm^O \lrcorner$ so that $f(sm^O) = f(s)m'^O$ and $\ulcorner f(sm^O) \urcorner = \ulcorner f(s)m'^O \urcorner = m'^O$.
- If m justifies n^O, $f(\ulcorner smtn^O \urcorner) = f(\ulcorner sm \urcorner n^O) = f(\ulcorner sm \urcorner) n'^O = \ulcorner f(sm) \urcorner n'^O$ and $\ulcorner f(smtn^O) \urcorner = \ulcorner f(sm) t'' n''^O \urcorner = \ulcorner f(sm) \urcorner n''^O$, moreover $\llcorner \ulcorner sm \urcorner n^O \lrcorner = \llcorner smtn^O \lrcorner$ so that $n' = n''$. □

Lemma 3 (Strategies and p-isomorphisms)
Let σ be a strategy on a game A and f be a p-isomorphism from A to B, $f(\sigma)$ is a strategy on B.

PROOF: The set of plays $f(\sigma)$ is non-empty because $\varepsilon = f(\varepsilon)$. If $s \in f(\sigma)$ and s' is a P-prefix of s of length k, there exists some t such that $s = f(t)$. Let t' be the P-prefix of t with length k (which belongs to σ), by definition of a p-isomorphism: $f(t')$ is a P-prefix of s of length k thus $s' = f(t')$ and $s' \in f(\sigma)$.

If $sa^P \in f(\sigma)$ and $sb^P \in f(\sigma)$, we have $sa = f(t_1 a')$ and $sb = f(t_2 b')$ with $t_1 a' \in \sigma$ and $t_2 b' \in \sigma$, so that $s = f(t_1) = f(t_2)$ which implies $t_1 = t_2$ by injectivity of f. By determinism of σ, we have $t_1 a' = t_2 b'$ and finally $sa = f(t_1 a') = f(t_2 b') = sb$.

If $sab^P \in f(\sigma)$, $t \in f(\sigma)$, $ta \in \mathcal{P}_B$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$, there exist $s'a'b' \in \sigma$ and $t' \in \sigma$ such that $sab = f(s'a'b')$ and $ta = f(t'a'')$. Then we have, by lemma 2, $f(\ulcorner s'a' \urcorner) = \ulcorner sa \urcorner = \ulcorner ta \urcorner = f(\ulcorner t'a'' \urcorner)$ so that $\ulcorner s'a' \urcorner = \ulcorner t'a'' \urcorner$ (in particular $a' = a''$) which gives $t'a'b' \in \sigma$ by innocence of σ and finally $f(t'a'b') \in f(\sigma)$ but $\ulcorner s'a'b' \urcorner = \ulcorner s'a' \urcorner b' = \ulcorner t'a' \urcorner b' = \ulcorner t'a'b' \urcorner$ entails $\llcorner s'a'b' \lrcorner = \llcorner t'a'b' \lrcorner$ so that $f(t'a'b') = tab \in f(\sigma)$. □

Using this lemma, we will often say that a strategy $\sigma : A$ is a strategy on B if there exists a canonical p-isomorphism f between A and B (in particular for the p-isomorphisms given by proposition 1) identifying σ and $f(\sigma)$.

Definition 9 (Identity)
Let N be a negative game, the *identity* id_N on $N \overset{\bullet}{\multimap} N$ is the strict strategy given by $id_N = \{s \in \mathcal{P}^P_{N_1 \multimap N_2} \mid \forall t \leq^P s, t \upharpoonright_{N_1} = t \upharpoonright_{N_2}\}$ (the indexes are only used to distinguish the two occurrences of N).

Definition 10 (Composition)
Let $\sigma : L \multimap M$ and $\tau : M \multimap N$ be two strategies, the *composition* $\sigma\,;\tau$ is the strategy on $L \multimap N$ defined by:

$$\sigma\,;\tau = \{s\upharpoonright_{L\multimap N} \in \mathcal{P}^P_{L\multimap N} \mid s \in \mathrm{int}(L,M,N) \wedge s\upharpoonright_{L\multimap M} \in \sigma \wedge s\upharpoonright_{M\multimap N} \in \tau\}$$

where $\mathrm{int}(L,M,N)$ is the set of justified sequences s of $(L \multimap M) \multimap N$ such that $s\upharpoonright_{L\multimap M} \in \mathcal{P}_{L\multimap M}$ and $s\upharpoonright_{M\multimap N} \in \mathcal{P}_{M\multimap N}$. $s\upharpoonright_{L\multimap N}$ is obtained by replacing the pointer of the L-moves pointing in M (thus on an initial M-move) by the justifier of the M-move (that must be an initial N-move).

Composition can be generalized to obtain a strategy on N from a strategy $\sigma : M$ and a strategy $\tau : M \multimap N$ since $M \simeq_p \top \multimap M$.

Proposition 2 (HO category)
Negative games with strategies on $M \multimap N$ as morphisms between M and N give a category denoted by $\mathcal{HO}_-$.

Proof: See [5]. □

We can extend the game constructions to strategies (*i.e.* from objects to morphisms). Some associated categorical structures are given in the following sections (propositions 3, 4, 6, 7, 14, lemma 5, ...). Let $\sigma : M_1 \multimap N_1$ and $\tau : M_2 \multimap N_2$ be two strategies:

Tensor product. The strategy $\sigma \odot \tau$ is $\{s \in \mathcal{P}^P_{M_1\odot M_2\multimap N_1\odot N_2} \mid s\upharpoonright_{M_1\multimap N_1} \in \sigma \wedge s\upharpoonright_{M_2\multimap N_2} \in \tau\} : M_1 \odot M_2 \multimap N_1 \odot N_2$.

Cartesian product. The strategy $\sigma \,\&\, \tau$ is $\sigma \cup \tau = \{s \in \mathcal{P}^P_{M_1\&M_2\multimap N_1\&N_2} \mid s\upharpoonright_{M_1\multimap N_1} \in \sigma \wedge s\upharpoonright_{M_2\multimap N_2} \in \tau\} : M_1 \,\&\, M_2 \multimap N_1 \,\&\, N_2$.

Par product. If M_1, N_1, M_2 and N_2 are well opened, we define the set of plays $\sigma \parr \tau$ as $cl^P(\{s \in \mathcal{P}^P_{M_1\parr M_2\multimap N_1\parr N_2} \mid s\upharpoonright_{M_1\multimap N_1} \in \sigma \wedge s\upharpoonright_{M_2\multimap N_2} \in \tau\})$. This set is a strategy on $M_1 \parr M_2 \multimap N_1 \parr N_2$ only if $\sigma : M_1 \overset{\bullet}{\multimap} N_1$ is strict or $\tau : M_2 \overset{\bullet}{\multimap} N_2$ is strict, and $\sigma \parr \tau$ is strict if both σ and τ are strict (see example 2).

We denote $id_N \parr \tau$ by $N \parr \tau$, which is a strategy for any strategy τ since id_N is a strict strategy on $N \overset{\bullet}{\multimap} N$.

Negative lifting. The strict strategy $\uparrow\sigma$ is $cl^P(\{\circ_{M_1}\circ_{N_1}s \mid s \in \sigma\}) : \uparrow N_1^\perp \overset{\bullet}{\multimap} \uparrow M_1^\perp$.

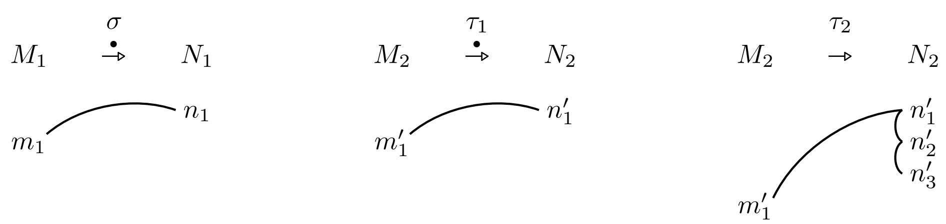
Positive lifting. If M and N are well opened and if $\rho : M \parr N$, the strict strategy $\downarrow\rho$ is $cl^P(\{m\circ ns \mid (m,n)s \in \rho\}) : \uparrow N^\perp \overset{\bullet}{\multimap} M$.

Promotion. If $\rho : \sharp M \multimap N$, the strategy $\rho^\dagger$ is $\{s \in \mathcal{P}_{\sharp M\multimap\sharp N} \mid s\upharpoonright_n \in \rho, \forall n \text{ initial}\} : \sharp M \multimap \sharp N$.

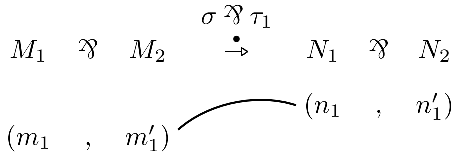
Contraction. Let N be a negative game, if t is a play in $\sharp N_0 \multimap \sharp N_1 \odot \sharp N_2$ (where the indexes are just used to distinguish the occurrences), we denote by t_i $(i = 1,2)$ the sub-sequence of t containing the moves in $\sharp N_i$ and the moves in $\sharp N_0$ hereditarily justified by an initial move in $\sharp N_i$. We define the strict strategy $c_{\sharp N} = \{s \in \mathcal{P}^P_{\sharp N_0\multimap\sharp N_1\odot\sharp N_2} \mid \forall t \leq^P s, t_i \in id_{\sharp N}, i = 1,2\} : \sharp N \overset{\bullet}{\multimap} \sharp N \odot \sharp N$.

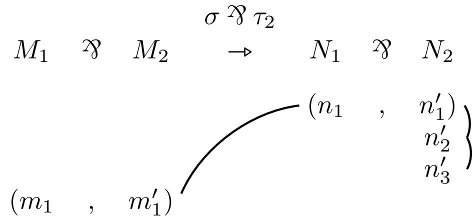
Weakening. w_N is the strict strategy on $N \overset{\bullet}{\multimap} \top$ defined by $w_N = \{\varepsilon\}$.

Example 2 (Par product of strategies)
The construction of the par product of two strategies contains a kind of synchronization scheme between the two strategies. If we consider three strategies $\sigma : M_1 \overset{\bullet}{\multimap} N_1$, $\tau_1 : M_2 \overset{\bullet}{\multimap} N_2$ and $\tau_2 : M_2 \multimap N_2$ containing the following plays (with M_1, N_1, M_2 and N_2 well opened):



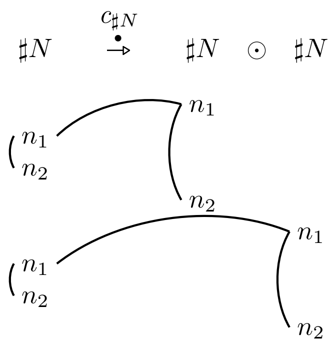
we have the following plays in $\sigma \parr \tau_1$ and $\sigma \parr \tau_2$:





In each case, the two starting moves are synchronized into a unique one in $N_1 \parr N_2$ (as required by the definition of this arena). In the case of two strict strategies, this leads to two moves in M_1 and M_2 which are immediately synchronized (and then the play is given as for a tensor product). In $\sigma \parr \tau_2$, σ is immediately ready to synchronize on the left but has to wait that τ_2 wants to do so. In the case of two non-strict strategies, it is not possible to build such synchronizations as described in remark 3.2.

Example 3 (Contraction)
Let N be a negative game with at least two moves n_1 and n_2 (with $* \vdash_N n_1^O \vdash_N n_2^P$), the following play belongs to $c_{\sharp N}$:



2 Intuitionistic Linear Logic

We are now able to recall McCusker's results [5] about the relation between negative games and Intuitionistic Linear Logic (ILL). Due to the "opponent starts" constraint appearing in many game models, ILL has been the natural linear setting to define linear game models (for example [3]).

2.1 Linear part: IMALL

As we will do for LLP, we first consider the linear (without exponential) fragment of ILL.

Linear intuitionistic formulas.

$$A \quad ::= \quad X \quad | \quad \mathtt{I} \quad | \quad A \odot A \quad | \quad A \multimap A \quad | \quad \top \quad | \quad A \,\&\, A$$

Sequents are of the shape $\Gamma \vdash A$ where Γ is a multiset of formulas.

Rules.

$$\frac{}{A \vdash A}\,ax \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,cut$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \odot B}\,\odot_R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \odot B \vdash C}\,\odot_L$$

$$\frac{}{\vdash \mathtt{I}}\,\mathtt{I}_R \qquad \frac{\Gamma \vdash C}{\Gamma, \mathtt{I} \vdash C}\,\mathtt{I}_L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \multimap_R \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \multimap_L$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \&_R \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C} \&_L^1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C} \&_L^2$$

$$\frac{}{\Gamma \vdash \top} \top$$

Proposition 3 (McCusker)
$(\mathcal{HO}_-, \odot, \top, \multimap)$ is a symmetric monoidal closed category with finite products $(\&, \top)$.

From a denotational model point of view, this means:

Corollary 3.1
*If we interpret atoms by negative games and each connective by the corresponding construction of games (*I *is interpreted by $\top$), $\mathcal{HO}_-$ is a denotational model of* IMALL.

2.2 ILL

Adding exponentials leads to the ILL system which is expressive enough to embed the λ-calculus [26].

Intuitionistic formulas.

$$A \quad ::= \quad X \quad | \quad \mathtt{I} \quad | \quad A \odot A \quad | \quad A \multimap A \quad | \quad \top \quad | \quad A \,\&\, A \quad | \quad \sharp A$$

Rules. We add the rules for the $\sharp$-connective:

$$\frac{\sharp\Gamma \vdash A}{\sharp\Gamma \vdash \sharp A} \sharp \qquad \frac{\Gamma, A \vdash C}{\Gamma, \sharp A \vdash C} \sharp d \qquad \frac{\Gamma, \sharp A, \sharp A \vdash C}{\Gamma, \sharp A \vdash C} \sharp c \qquad \frac{\Gamma \vdash C}{\Gamma, \sharp A \vdash C} \sharp w$$

Proposition 4 ($\odot$-comonoid $\sharp N$)
If N is a negative game, the triple $(\sharp N, c_{\sharp N}, w_{\sharp N})$ is a $\odot$-comonoid in $\mathcal{HO}_-$.

Proof: Since associativity and commutativity are straightforward, we only prove the unit property: we show that $c_{\sharp N}\,;(w_{\sharp N} \odot id_{\sharp N}) : \sharp N \multimap \top \odot \sharp N$ is the canonical strategy on $\sharp N \overset{\bullet}{\multimap} \top \odot \sharp N$ coming from $id_{\sharp N} : \sharp N \overset{\bullet}{\multimap} \sharp N$ with $\sharp N \multimap \sharp N \simeq_p \sharp N \multimap \top \odot \sharp N$. If $s \in c_{\sharp N};(w_{\sharp N} \odot id_{\sharp N})$, by definition of composition, s is the projection on $\sharp N_0 \multimap \top \odot \sharp N_3$ of a sequence s_0 in $(\sharp N_0 \multimap \sharp N_1 \odot \sharp N_2) \multimap \top \odot \sharp N_3$ such that $s_0 \upharpoonright_{\sharp N_0 \multimap \sharp N_1 \odot \sharp N_2} \in c_{\sharp N}$ and $s_0 \upharpoonright_{\sharp N_1 \odot \sharp N_2 \multimap \top \odot \sharp N_3} \in w_{\sharp N} \odot id_{\sharp N}$ (the indexes are used to distinguish the occurrences of N). This entails that $s_0 \upharpoonright_{\sharp N_1 \multimap \top} = \varepsilon$ and $s_0 \upharpoonright_{\sharp N_2} = s_0 \upharpoonright_{\sharp N_3}$, moreover since $s_0 \upharpoonright_{\sharp N_1} = \varepsilon$ we have $s_0 \upharpoonright_{\sharp N_0} = s_0 \upharpoonright_{\sharp N_2}$ and finally $s \upharpoonright_{\sharp N_0} = s \upharpoonright_{\sharp N_3}$. □

Together with the definition of $\sigma^\dagger$, this proposition allows to give an interpretation of the $\sharp$, $\sharp c$ and $\sharp w$ rules but not of the $\sharp d$ rule! It is *not* possible to define a strategy on $\sharp N \multimap N$ adequate to interpret the $\sharp d$ rule, except if N is a well opened game (see [5] for more details).

Dereliction. Let N be a *well opened* negative game, the plays of id_N are plays in $\sharp N \multimap N$ since any play of N is a play of $\sharp N$ by definition 6. We define $d_N = id_N : \sharp N \overset{\bullet}{\multimap} N$.

This constraint of well openness required for the definition of d_N is the reason why $\sharp$ is not a comonad on $\mathcal{HO}_-$ and why $\mathcal{HO}_-$ is *not* a model of ILL. However this is sufficient to give a model of the λ-calculus or PCF since any simple type is interpreted as a well opened negative game.

The use of $\otimes$ instead of $\odot$ and ! instead of $\sharp$ in LLP can be seen as a way of using only well opened games and being able to define d_N for any required N.

3 Polarized linear logic

Polarized Linear Logic has been introduced as a subsystem of Linear Logic with more structure. It is easier to study but expressive enough to interpret classical logic. The main deterministic classical systems have translations into LLP (see in particular appendixes C and D). Moreover LLP allows to interpret both *call-by-name* and *call-by-value* classical logics by pointing out negative or positive formulas.

Our main goal is to move from the intuitionistic setting described above which is appropriate for the study of the λ-calculus to a more classical setting as realized by LLP, with moreover the possibility of using well opened games only. As a first step, we will consider only the fragment MALLP of LLP without exponential connective.

3.1 Linear part: MALLP

This calculus is a linear fragment (without contraction and weakening) of polarized linear logic, the full system will be studied in section 3.3. In this linear setting, exponentials are replaced by *lifting* operators used to change the polarity (with rules corresponding to promotion and dereliction).

Linear polarized formulas. Formulas are split into two parts: positive and negative ones which interact through $\downarrow$ and $\uparrow$.

$$\begin{array}{lcccccccccccc} P & ::= & X^{\perp} & \mid & 1 & \mid & 0 & \mid & P \otimes P & \mid & P \oplus P & \mid & \downarrow N \\ N & ::= & X & \mid & \perp & \mid & \top & \mid & N \parr N & \mid & N \,\&\, N & \mid & \uparrow P \end{array}$$

Rules.

$$\frac{}{\vdash N, N^{\perp}}\,ax \qquad \frac{\vdash \Gamma, N \qquad \vdash \Delta, N^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\,\otimes \qquad \frac{\vdash \Gamma, M, N}{\vdash \Gamma, M \parr N}\,\parr$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\,\oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\,\oplus_2 \qquad \frac{\vdash \Gamma, M \qquad \vdash \Gamma, N}{\vdash \Gamma, M \,\&\, N}\,\&$$

$$\frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash \Gamma, \top}\,\top$$

with at most one positive formula in Γ for the $\top$-rule

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\,\downarrow \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, \uparrow P}\,\uparrow$$

where $\mathcal{N}$ contains only negative formulas

Lemma 4 (Positive formula)
If $\vdash \Gamma$ is provable in MALLP*, Γ contains at most one positive formula.*

Proof: By induction on the size of the proof, with the two key constraints of the rules in the $\top$-case and in the $\downarrow$-case. □

3.2 Linear HO model

The game interpretation of ILL is mainly based on the constructions $\odot$, $\multimap$, & and $\sharp$, we are going to use $\parr$, $\uparrow$, & and ! for LLP as constructions of well opened games.

Remark: $\perp$, $\top$ and $\uparrow P$ are well opened and if M and N are well opened then $N^{\perp}$, $M \multimap N$, $M \,\&\, N$, $M \parr N$ are well opened.

Definition 11 (Well opened HO category)
The category $wo\mathcal{HO}_-$ is the full subcategory of $\mathcal{HO}_-$ of *well opened* negative games, and $wo\mathcal{HO}_-^\bullet$ is the subcategory of $wo\mathcal{HO}_-$ containing only *strict* strategies.

Lemma 5
$\parr$ *is a bifunctor in* $wo\mathcal{HO}_-^\bullet$.

Proof: The key point is to verify that, if $\sigma : M_1 \overset{\bullet}{\multimap} N_1$ and $\tau : M_2 \overset{\bullet}{\multimap} N_2$ are two strict strategies, we have $(\sigma \parr id_{M_2})\,;(id_{N_1} \parr \tau) = \sigma \parr \tau = (id_{M_1} \parr \tau)\,;(\sigma \parr id_{N_2}) : M_1 \parr M_2 \overset{\bullet}{\multimap} N_1 \parr N_2$. □

Remark: This result is false for general strategies because $\parr$ is only defined if at least one of the two strategies is strict. The fact that $\parr$ is not bifunctorial in the full category corresponds to the premonoidal structure of control categories of P. Selinger [23] (see appendix B). All this has also to be linked with the problem of constructions on strategies for Blass games [12], solved here by adding the strictness constraint.

Lemma 6
Let P be a positive game, there is a one to one correspondence between strategies on P and strategies on $\uparrow P$.

If N_1, ..., N_k are well opened negative games, a strategy σ on $N_1 \parr \dots \parr N_k$ is said to be *strict* in N_i if either one of the games has a set of plays reduced to $\{\varepsilon\}$ or if σ contains a play nn_i with $n_i \in \mathcal{M}_{N_i}$ for each initial move n of $N_1 \parr \dots \parr N_k$. A strategy $\sigma : P^\perp \overset{\bullet}{\multimap} N_1 \parr \dots \parr N_k$ is strict according to definition 8 if and only if the corresponding strategy on $\uparrow P \parr N_1 \parr \dots \parr N_k$ is strict in $\uparrow P$ according to this new definition.

Formulas are interpreted as well opened games of the corresponding polarity. Given an interpretation of atoms X (resp. $X^\perp$), ... as negative (resp. positive) well opened games, the various connectives are interpreted by the corresponding game constructions. Since we have already used the appropriate notations, we will use the same notation for a formula and for the corresponding game. In particular, the interpretation of the sequent $\vdash \Gamma$ will be denoted by Γ and has to be understood as $N_1 \parr \dots \parr N_k$ if $\Gamma = N_1, \dots, N_k$ and as $\uparrow P \parr N_1 \parr \dots \parr N_k$ if $\Gamma = P, N_1, \dots, N_k$ (according to lemma 4).

A proof π of the sequent $\vdash N_1, \dots, N_k$ is interpreted by a *strategy* σ_π on $N_1 \parr \dots \parr N_k$ (with the particular case $N_1 \parr \dots \parr N_k = \perp$ if $k = 0$), and a proof π of $\vdash P, N_1, \dots, N_k$ by a strategy σ_π on $\uparrow P \parr N_1 \parr \dots \parr N_k$ strict in $\uparrow P$ which can also be seen as a strict strategy on $P^\perp \overset{\bullet}{\multimap} N_1 \parr \dots \parr N_k$ (if $k = 0$, σ_π is a strategy on $\uparrow P$ or equivalently on P according to lemma 6). We make a strong use of lemma 3 (in particular for $\uparrow M^\perp \parr N \simeq_p M \multimap N$).

Axioms.

- The *ax*-rule introducing $\vdash N, N^\perp$ is interpreted by the strategy on $\uparrow N^\perp \parr N$ strict in $\uparrow N^\perp$ corresponding to $id_N : N \overset{\bullet}{\multimap} N$.
- The 1-rule is interpreted by the strategy $\{\varepsilon, \circ\}$ on 1.
- The $\top$-rule is interpreted by the strategy $\{\varepsilon\}$ (strict in any component).

Cut rule. The interpretation of the two premises gives a strategy $\sigma : \Gamma \parr N$ and a strict strategy $\tau : N \overset{\bullet}{\multimap} \Delta$. The *cut*-rule is interpreted by the composition $\sigma\,;(\Gamma \parr \tau) : \Gamma \parr \Delta$ which is strict in $\uparrow P$ if Γ contains P.

Multiplicatives.

- $\perp$: by lemma 3, a strategy on Γ gives us a strategy on $\Gamma \parr \perp$.
- $\parr$: this rule does not modify the interpretation.
- $\otimes$: if $\sigma : \uparrow P \parr \Gamma$ strict in $\uparrow P$ and $\tau : \uparrow Q \parr \Delta$ strict in $\uparrow Q$ are the interpretations of the two premises, we obtain the strategy $\sigma \parr \tau : \uparrow(P \otimes Q) \parr \Gamma \parr \Delta$ strict in $\uparrow(P \otimes Q)$.

Additives.

- $\&$: if σ is the strategy on $\Gamma \parr M$ and τ is the strategy on $\Gamma \parr N$, we use the strategy $\sigma \cup \tau$ on $\Gamma \parr (M \,\&\, N)$.
- $\oplus_i$: if $\sigma : \uparrow P_i \parr \Gamma$, we obtain the strategy $\sigma : \uparrow(P_1 \oplus P_2) \parr \Gamma$.

Lifts.

- $\uparrow$: this rule does not modify the interpretation.
- $\downarrow$: if σ is a strategy on $\mathcal{N} \parr N$, we obtain the strategy on $\mathcal{N} \parr \uparrow\downarrow N$ strict in $\uparrow\downarrow N$ which corresponds to $\downarrow\sigma : \uparrow N^\perp \stackrel{\bullet}{\to} \mathcal{N}$.

Remark: The MIX-rule (sometimes added to linear logic, see for example [1]) cannot be interpreted in a natural way:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\,\mathrm{MIX}$$

if $\gamma_1\gamma_2 \in \sigma : \Gamma$ and $\delta_1\delta_2 \in \tau : \Delta$ we want to build plays in $\Gamma \parr \Delta$ in a symmetric way but after the move (γ_1, δ_1) we have to make a choice between γ_2 and δ_2 and we cannot choose the two moves if we want a deterministic strategy. This corresponds again to the non-bifunctoriality of $\parr$ in $wo\mathcal{HO}_-$.

The cut elimination procedure $\pi \to \pi'$ for MALLP is simply defined in the natural way coming from the LL cut elimination procedure [27] (see appendix A).

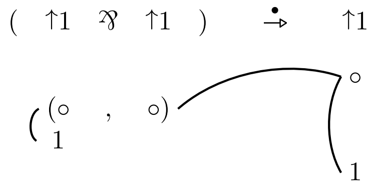
Theorem 1 (Correctness)
If $\pi \to \pi'$ then $\sigma_\pi = \sigma_{\pi'}$.

Proof: Since they are easy to reconstruct, we omit the pointers in the plays.

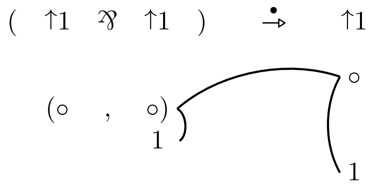
- Axiom cut: If the cut-formula is negative in the axiom, it is just composition with the identity; if this formula is positive, we use $id_N \parr \Gamma = id_{N\parr\Gamma}$ by lemma 5.
- $\downarrow - \uparrow$: let $\sigma : N \parr \mathcal{N}$ and $\tau : N \stackrel{\bullet}{\to} \Gamma$ be two strategies, we have to prove $\sigma\,;(\tau \parr \mathcal{N}) = \tau\,;(\Gamma \parr \downarrow\sigma) : \Gamma \parr \mathcal{N}$. Let s be a play in $\sigma;(\tau \parr \mathcal{N})$, it is the projection on $\Gamma \parr \mathcal{N}$ of a sequence $s_1 = (\gamma, \nu)(n, \nu)s'_1$ in $\mathcal{P}_{N\parr\mathcal{N}\to\Gamma\parr\mathcal{N}}$. We define $s_2 = (\gamma, \nu)(\gamma, \circ)ns'_1$ in $\mathcal{P}_{\Gamma\parr\uparrow N^\perp\to\Gamma\parr\mathcal{N}}$, we have $s_2 \upharpoonright_{\Gamma\parr\uparrow N^\perp} \in \tau$ by definition of $\tau \parr \mathcal{N}$ and $s_2 \in \Gamma \parr \downarrow\sigma$ because $s_1 \upharpoonright_{N\parr\mathcal{N}} \in \sigma$, moreover $s_2 \upharpoonright_{\Gamma\parr\mathcal{N}} = s_1 \upharpoonright_{\Gamma\parr\mathcal{N}} = s$ thus $s \in \tau\,;(\Gamma \parr \downarrow\sigma)$. We prove the converse in the same way.
- $\downarrow - *$: Let $\sigma : N \parr \mathcal{N} \parr M$ and $\tau : N \stackrel{\bullet}{\to} \Gamma$ be two strategies, we have to show that $\downarrow(\sigma\,;(\tau \parr \mathcal{N} \parr M)) = \downarrow\sigma\,;(\tau \parr \mathcal{N} \parr \uparrow\downarrow M)$. If s is a play of $\downarrow(\sigma\,;(\tau \parr \mathcal{N} \parr M))$, by definition of $\downarrow$, $s = (\gamma, \nu, \circ)\circ ms'$ with $s_1 = (\gamma, \nu, m)s' \in \sigma;(\tau \parr \mathcal{N} \parr M)$. By definition of composition, s_1 is the projection on $\Gamma \parr \mathcal{N} \parr M$ of a sequence $s_0 = (\gamma, \nu, m)(n, \nu, m)s'_0$ in $\mathcal{P}_{N\parr\mathcal{N}\parr M\to\Gamma\parr\mathcal{N}\parr M}$, let $s_2 = (\gamma, \nu, \circ)(n, \nu, \circ)\circ ms'_0$ in $\mathcal{P}_{N\parr\mathcal{N}\parr\uparrow\downarrow M\to\Gamma\parr\mathcal{N}\parr\uparrow\downarrow M}$, we have $s_2 \upharpoonright_{N\parr\mathcal{N}\parr\uparrow\downarrow M} \in \downarrow\sigma$ because $s_0 \upharpoonright_{N\parr\mathcal{N}\parr M} \in \sigma$ and we have $s_2 \upharpoonright_{N\to\Gamma} \in \tau$. Moreover $s = s_2 \upharpoonright_{\Gamma\parr\mathcal{N}\parr\uparrow\downarrow M}$ thus $s \in \downarrow\sigma\,;(\tau \parr \mathcal{N} \parr \uparrow\downarrow M)$. We prove the converse in the same way.
- $\otimes - \parr$: By lemma 5, if $\sigma : M \stackrel{\bullet}{\to} \Gamma$ and $\tau : N \stackrel{\bullet}{\to} \Delta$, we have $\sigma \parr \tau \parr \Xi = \sigma \parr N \parr \Xi\,; \Gamma \parr \tau \parr \Xi = M \parr \tau \parr \Xi\,; \sigma \parr \Delta \parr \Xi$.
- Additive steps are basically proved as in [7].
- $1 - \perp$: Straightforward.
- $\top$: The strategy $\{\varepsilon\}$ composed with any strategy gives the strategy $\{\varepsilon\}$ (because strategies are never empty). $\square$

In fact this result may be extended to a focalized calculus for MALL (see [15] for example) by replacing the constraint of a negative context in the $\downarrow$-rule by a *focalization* constraint (stoup [13] or η-constraint [28] for example), even if provable sequents in these systems may contain several positive formulas (see [14]). These polarized games are a good candidate to establish a precise link between ludics and games semantics (in the spirit of [29]) which is not surprising since they have been developed by an introduction of ludics ideas in a more traditional game setting.

The HO game model we have obtained for MALLP is not fully complete as shown by the following example:



and



This pseudo-contraction, coming from $\uparrow 1 \simeq_p ?1$, is not definable by a proof in MALLP, otherwise the last rule of the corresponding proof would be:

$$\frac{\vdash \downarrow\perp, \uparrow 1 \qquad \vdash \downarrow\perp, \uparrow 1}{\vdash \downarrow\perp \otimes \downarrow\perp, \uparrow 1} \otimes$$

and the problem appears in the splitting of the context for $\otimes$-rules.

Nevertheless, using a propagation condition as in ludics [15, 29] or using the recent work of Laird on coherent games [18], it should be possible to add constraints to the model in order to go closer towards completeness.

Remark: The notion of categorical models of MALLP corresponding to theorem 1 has not been clearly described yet (Melliès and Selinger are working on such questions [30]). This is why we just mention here the known categorical structures (lemma 5 for example) and we give the model of MALLP directly through its concrete description.

3.3 LLP

To get a really expressive system corresponding to classical logic, we go from MALLP to the full system LLP. In this way we will moreover get the definability property.

Polarized formulas. We replace the lifted formulas of MALLP by the corresponding exponential versions.

$$\begin{array}{lclcccccccccc} P & ::= & X^\perp & \mid & 1 & \mid & 0 & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \\ N & ::= & X & \mid & \perp & \mid & \top & \mid & N \parr N & \mid & N \,\&\, N & \mid & ?P \end{array}$$

Rules. The two lifting rules are replaced by *promotion* and *dereliction*:

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\, ! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\, ?d$$

where $\mathcal{N}$ contains only negative formulas

And we add structural rules on negative formulas.

$$\frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\, ?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N}\, ?w$$

Instead of the usual LL structural rules on ?-formulas, LLP allows structural rules on any negative formula N. So that LLP is obtained from LL by first restricting linear formulas to polarized ones and then by using the properties of the induced system to generalize structural rules.

3.4 HO model

In order to interpret classical logic, we have to restrict to some particular games allowing us to define structural rules: contraction and weakening. These *multiple games* are closed under the constructions required to interpret polarized formulas and have the required structure (⅋-monoid) for the interpretation of the structural rules.

Definition 12 (Multiple game)
A game A is a *multiple game* if it is *well opened* and:

- *restriction*: if $s \in \mathcal{P}_A$, $* \vdash_A m$ and $m \vdash_A n$ then $s \upharpoonright_{mn} \in \mathcal{P}_A$.
- *interleaving*: if $s \in L_A$ is a well opened position with an initial move m, if $I + J$ is a partition of the occurrences of moves justified by m in s and if $s \upharpoonright_{mI}, s \upharpoonright_{mJ} \in \mathcal{P}_A$ then $s \in \mathcal{P}_A$.

Proposition 5 (Multiple constructions)
Multiplicity is closed under the following constructions:

- 1, 0, $\top$ *and* $\bot$ *are multiple games.*
- *If* N *is negative and* P *is positive,* $!N$ *and* $?P$ *are multiple games.*
- *If* P, Q, M *and* N *are* multiple *then* $P \otimes Q$, $P \oplus Q$, $M \parr N$ *and* $M \,\&\, N$ *are multiple games.*

PROOF: The cases of $\top$, $\bot$ and $M \,\&\, N$ are straightforward. Up to duality, we only look at the two remaining negative cases (since A is multiple if and only if $A^\perp$ is multiple).

- $?P$ case:
  - The only initial move is $\circ$, thus a well opened legal position in $L_{?P}$ is $\circ s$ with $s \in L_P$.
  - $?P = \uparrow\flat P$ is well opened as mentioned in remark 3.2.
  - If $\circ s \in \mathcal{P}_{?P}$ and $\circ \vdash_{?P} n$ then $* \vdash_P n$, by definition of $\flat P$ we have $s \upharpoonright_n \in \mathcal{P}_P$ thus $\circ s \upharpoonright_{\circ n} \in \mathcal{P}_{?P}$.
  - Let $I + J$ be a partition of the occurrences of initial moves of s such that $\circ s \upharpoonright_{\circ I}, \circ s \upharpoonright_{\circ J} \in \mathcal{P}_{?P}$, that is $s \upharpoonright_I, s \upharpoonright_J \in \mathcal{P}_{\flat P}$ thus $s \in \mathcal{P}_{\flat P}$ and $\circ s \in \mathcal{P}_{?P}$.
- $M \parr N$ case:
  - If M and N are well opened then $M \parr N$ is well opened as mentioned in remark 3.2.
  - Let s be a play in $\mathcal{P}_{M \parr N}$ and (m, n) and a be two occurrences of moves of s such that $* \vdash_{M \parr N} (m, n)$ and $(m, n) \vdash_{M \parr N} a$. Let assume $a \in M$, we have $m \vdash_M a$ and by definition of $M \parr N$ all the moves of $s \upharpoonright_{(m,n)a}$ (except the first one) are in M which is multiple so that $s \upharpoonright_{(m,n)a} \upharpoonright_M \in \mathcal{P}_M$ and $s \upharpoonright_{(m,n)a} \in \mathcal{P}_{M \parr N}$.
  - If $s \in L_{M \parr N}$ is well opened, (m, n) is its initial move and $I + J$ is a partition of the occurrences of moves justified by (m, n) in s such that $s \upharpoonright_{(m,n)I}, s \upharpoonright_{(m,n)J} \in \mathcal{P}_{M \parr N}$ then we have $s \upharpoonright_M \upharpoonright_{mI} = s \upharpoonright_{(m,n)I} \upharpoonright_M \in \mathcal{P}_M$ and $s \upharpoonright_M \upharpoonright_{mJ} = s \upharpoonright_{(m,n)J} \upharpoonright_M \in \mathcal{P}_M$ thus $s \upharpoonright_M \in \mathcal{P}_M$ since M is multiple. In the same way, $s \upharpoonright_N \in \mathcal{P}_N$ so that $s \in \mathcal{P}_{M \parr N}$. □

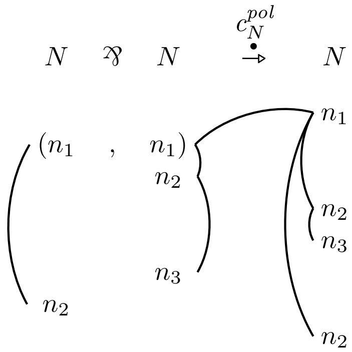
Polarized contraction. Let N be a multiple negative game, if t is a play in $N_1 \parr N_2 \multimap N_0$ (where the indexes are just used to distinguish the occurrences), we can decompose it into $t = t_0 t'$ where t_0 contains only moves in N_0 and t' starts by a move in $N_1 \parr N_2$ (this entails that any move of t' in N_0 comes after at least one move in $N_1 \parr N_2$). We denote by t_i ($i = 1, 2$) the sub-sequence of t containing t_0, all the moves in N_i, and the moves of t' in N_0 before which the last move in $N_1 \parr N_2$ is in N_i.

We define the strategy $c_N^{pol} = \{s \in \mathcal{P}^P_{N_1 \parr N_2 \multimap N_0} \mid \forall t \leq^P s, t_i \in id_N, i = 1, 2\} : N \parr N \overset{\bullet}{\multimap} N$.

Polarized weakening. w_N^{pol} is the strategy on $\bot \overset{\bullet}{\multimap} N$ defined by $w_N^{pol} = \{\varepsilon\} \cup \{n\circ \mid n \in \mathcal{M}_N^i\}$.

Example 4 (Polarized contraction)
Let N be a multiple negative game with at least three moves n_1, n_2 and n_3 (with $* \vdash_N n_1^O \vdash_N n_2^P \vdash_N n_3^O$),

the following play belongs to c_N^{pol}:



Proposition 6 ($\parr$-monoid N)
If N is a multiple negative game, the triple $(N, c_N^{pol}, w_N^{pol})$ is a $\parr$-monoid in $wo\mathcal{HO}_-^\bullet$.

Proof: Since associativity and commutativity are straightforward, we only prove the unit property: we show that $(w_N^{pol} \parr id_N) \,;\, c_N^{pol} : \bot \parr N \overset{\bullet}{\multimap} N$ is the canonical strategy on $\bot \parr N \overset{\bullet}{\multimap} N$. If $s \in (w_N^{pol} \parr id_N) \,;\, c_N^{pol}$, by definition of composition, s is the projection on $\bot \parr N_0 \multimap N_3$ of a sequence s_0 such that $s_0 \upharpoonright_{\bot \parr N_0 \multimap N_1 \parr N_2} \in w_N^{pol} \parr id_N$ and $s_0 \upharpoonright_{N_1 \parr N_2 \multimap N_3} \in c_N^{pol}$ (the indexes are used to distinguish the occurrences of N).

If $s_0 \upharpoonright_{\bot \multimap N_1} = \varepsilon$, we have $s_0 = \varepsilon$ and $s = \varepsilon$. If $s_0 \upharpoonright_{\bot \multimap N_1} = n\circ$ for some initial move n of N, we know that $s_0 \upharpoonright_{N_0} = s_0 \upharpoonright_{N_2}$ and, since $s_0 \upharpoonright_{N_1} = n$, $s_0 \upharpoonright_{N_2} = s_0 \upharpoonright_{N_3}$ (by definition of c_N^{pol}) so that $s \upharpoonright_{N_0} = s \upharpoonright_{N_3}$ with $s \upharpoonright_{\bot} = \circ$. ☐

Given an interpretation of atoms as multiple games of the corresponding polarity, we now interpret formulas as multiple games and proofs as strategies. The interpretation of axioms, cuts, multiplicatives and additives is the same as for the linear case (section 3.2). For the exponential rules, we have:

- $!$: If $\sigma : \mathcal{N} \parr N$, we define $!\sigma = \{s \in \mathcal{P}^P_{?N^\perp \multimap \mathcal{N}} \mid \forall m \text{ initial and } * \vdash_N n, s \upharpoonright_{mon} \in \downarrow\sigma\} : ?N^\perp \overset{\bullet}{\multimap} \mathcal{N}$ and the interpretation of the proof is the corresponding strategy on $\mathcal{N} \parr \uparrow !N$ strict in $\uparrow !N$.
- $?d$: If $\sigma : P^\perp \overset{\bullet}{\multimap} \Gamma$ then we obtain a strategy on $?P \parr \Gamma$ from $d_{P^\perp} \,;\, \sigma : \sharp P^\perp \overset{\bullet}{\multimap} \Gamma$.
- $?c$: If $\sigma : \Gamma \parr N \parr N$, we compose it with $\Gamma \parr c_N^{pol}$ to obtain a strategy on $\Gamma \parr N$.
- $?w$: If $\sigma : \Gamma$, we compose it with $\Gamma \parr w_N^{pol}$ to obtain a strategy on $\Gamma \parr N$, using $\Gamma \simeq_p \Gamma \parr \bot$.

Theorem 2 (Correctness)
If $\pi \to \pi'$ then $\sigma_\pi = \sigma_{\pi'}$.

Proof: This is an adaptation of the corresponding result for Hyland–Ong games [7, 5, 25] where the linear connectives are treated like in the proof of theorem 1. The cut elimination steps are given in appendix A. ☐

Corollary 2.1 (Finiteness and totality)
The interpretation σ_π of a proof π is a finite total *strategy.*

Proof: Let π' be a normal form of π, by theorem 2, we have $\sigma_\pi = \sigma_{\pi'}$. It is then easy to verify that the interpretation of a proof without cuts is a finite total strategy. ☐

Remark: As given by proposition 2, the composition of two strategies is a strategy. Since our model of LLP is based on finite total strategies, it would be natural to wonder if finite total strategies also compose. It is possible to prove it from the definability theorem of the next section. However it would be nice to have a more semantical proof. Such a proof can certainly be obtained from Abramsky's notion of *winning strategies* [31].

3.5 HO definability

As a converse of theorem 2 and corollary 2.1, we prove a definability result for LLP *without atom*, showing that every finite total strategy is the interpretation of an LLP proof. We first show how it is possible to get rid of the multiplicative connectives.

Lemma 7 (Additive type)
Let N be a game corresponding to a negative formula of LLP *without atom, there exist some negative formulas $N_1, \dots, N_n$ such that $N \simeq_p \&_{1 \leq i \leq n} ?N_i^\perp$. Moreover this isomorphism is definable.*

Proof: By induction on the size of N with a strong use of proposition 1:

- If $N = \top$, we have $n = 0$.
- If $N = \bot$, we use $\bot \simeq_p ?0 = ?\top^\perp$.
- If $N = ?P$, we have the result with $n = 1$.
- If $N = M_1 \& M_2$, it is straightforward by induction hypothesis for M_1 and M_2.
- If $N = M_1 \parr M_2$, by induction hypothesis, we have $M_1 \simeq_p \&_{1 \leq i \leq p} ?M_i'^\perp$ and $M_2 \simeq_p \&_{1 \leq j \leq q} ?M_j''^\perp$. By p-distributivity of $\parr$ over $\&$ and with $?P \parr ?Q \simeq_p ?(P \oplus Q)$ we get:

$$\begin{aligned} N &\simeq_p (\bigwith_{1 \leq i \leq p} ?M_i'^\perp) \parr (\bigwith_{1 \leq j \leq q} ?M_j''^\perp) \\ &\simeq_p \bigwith_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} (?M_i'^\perp \parr ?M_j''^\perp) \\ &\simeq_p \bigwith_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} ?(M_i' \& M_j'')^\perp \end{aligned}$$

These isomorphisms are known to be provable in LL thus in LLP. □

Before going into the proof of the definability theorem, we first prove some "reversibility" lemmas, showing that in some particular cases, it is always possible to extract a last rule from a strategy.

Lemma 8 (Bang lemma)
If $\sigma : ?M^\perp \overset{\bullet}{\multimap} \mathcal{N}$ is a strict strategy then $\sigma = !(d_M ; \sigma \parr M)$ where we use $d_M : ?M^\perp \parr M$.

Proof: We use the notation $\sigma_0 = d_M ; \sigma \parr M : \mathcal{N} \parr M$. The empty play ε is both in σ and $!\sigma_0$. A play of length 2 in σ is $n\circ$ where n is an initial move of $\mathcal{N}$ and $\circ$ is the first move of $?M^\perp$. Such a play $n\circ$ is also in $!\sigma_0$ and moreover any play of length 2 in $!\sigma_0$ is a play $n\circ$.

Since σ and $!\sigma_0$ are innocent, they are characterized by their proponent views. If s is a proponent view of σ of length > 2, it has the shape $s = n\circ mt$ where m is the only initial move of M in s. We easily see that $(n, m)t \in \sigma_0$ so that $s \in !\sigma_0$. Conversely, a proponent view of $!\sigma_0$ is also of the shape $s = n\circ mt$ with $(n, m)t \in \sigma_0$ and it entails $s = n\circ mt \in \sigma$ by definition of the composition. □

Lemma 9 (Plus lemma)
If $\sigma : (\&_{1 \leq j \leq q} ?M_j^\perp) \overset{\bullet}{\multimap} ?N^\perp$ then there exists $1 \leq j_0 \leq q$ such that $\sigma : ?M_{j_0}^\perp \overset{\bullet}{\multimap} ?N^\perp$.

Proof: The game $?N^\perp$ has a unique initial move $\circ$, this entails that all the non-empty plays of σ have the shape $\circ m_0 s$ with the same m_0 (by determinism). Let j_0 be the index such that $m_0 \in \mathcal{M}_{?M_{j_0}^\perp}$, a play in $(\&_{1 \leq j \leq q} ?M_j^\perp) \multimap ?N^\perp$ containing the move m_0 is entirely contained in $?M_{j_0}^\perp \multimap ?N^\perp$, so that σ is a strategy on $?M_{j_0}^\perp \overset{\bullet}{\multimap} ?N^\perp$. □

Theorem 3 (Definability)
Let A be a polarized formula without atom, if σ is a finite total strategy on A, σ is the interpretation of a proof of $\vdash A$ in LLP.

Proof: We will in fact prove that, moreover, if σ is a strategy on $\uparrow P \parr \mathcal{N}$ strict in $\uparrow P$ (*i.e.* on $P^\perp \overset{\bullet}{\multimap} \mathcal{N}$), σ is the interpretation of a proof of $\vdash P, \mathcal{N}$ in LLP.

By lemma 6, we can assume that A is negative, and by lemma 7, we can restrict ourselves to the case of types $\&_{1\leq i\leq p} ?N_i^\perp$ and $(\&_{1\leq j\leq q} ?M_j^\perp) \overset{\bullet}{\multimap} \&_{1\leq i\leq p} ?N_i^\perp$. We prove the result by induction on the pair $(|\sigma|, |A|)$ where the size $|.|$ of a formula is its number of symbols (and the size of a finite strategy has been defined page 9). We first reduce the cases $p \neq 1$ or $q \neq 1$ to the case $p = 1$ and $q = 1$:

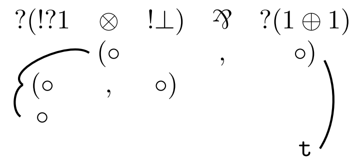
- If $p = 0$, the game is empty and σ is $\{\varepsilon\}$, that corresponds to a $\top$-rule.
- If $p > 1$, then $\sigma_i = \sigma \upharpoonright_{?N_i^\perp}$ (resp. $\sigma \upharpoonright_{(\&_{1\leq j\leq q} ?M_j^\perp) \multimap ?N_i^\perp}$) is a definable strategy by induction hypothesis with $\sigma = \bigcup_{1\leq i\leq p} \sigma_i$, which corresponds to &-rules.
- If $p = 1$ and $q = 0$, σ cannot be strict on $\top \overset{\bullet}{\multimap} ?N_1^\perp$.
- If $p = 1$ and $q > 1$, by the plus lemma (lemma 9), σ is a strategy on $?M_{j_0}^\perp \overset{\bullet}{\multimap} ?N_1^\perp$ and is definable by induction hypothesis. The strategy σ is obtained on $(\&_{1\leq j\leq q} ?M_j^\perp) \overset{\bullet}{\multimap} ?N_1^\perp$ by $\oplus$-rules.

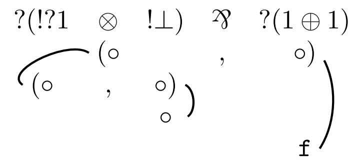
We now prove the cases of formulas $?N^\perp$ or $?M^\perp \overset{\bullet}{\multimap} ?N^\perp$. For the second one, by the bang lemma (lemma 8), we just have to prove the definability of $d_M ; \sigma \parr M$. This is a smaller strategy on $?N^\perp \parr M$ thus definable by induction hypothesis.

If σ is a strategy on $?N^\perp$, either there is only one move justified by the initial one in each play and $\sigma = d_N ; \sigma'$, this corresponds to a dereliction rule on a strategy of the same size on a smaller formula (thus definable). Or there exists a play with two occurrences of moves justified by the initial one. We define the strategy σ_1 on $?N_1^\perp \parr ?N_2^\perp$ (the indexes are just used to distinguish the occurrences) by: if s is a play in σ, the play in $?N_1^\perp \parr ?N_2^\perp$, obtained by putting the first proponent move and the moves justified by it in $?N_1^\perp$ and the other ones in $?N_2^\perp$, is a play in σ_1. We have $\sigma = \sigma_1 ; c^{pol}_{?N^\perp}$. It is easy to see that $\sigma_1 = d_N ; \sigma_2$ where σ_2 is a strategy on $N \overset{\bullet}{\multimap} ?N^\perp$. By applying the p-isomorphism of lemma 7 to N and the plus lemma, we get a strategy σ_3 on a game $?M^\perp \overset{\bullet}{\multimap} ?N^\perp$. Finally, we apply the bang lemma and we obtain a strategy σ_4 on $?N^\perp \parr M$ which is smaller than σ. This last step is a bit complicated because if $N = ?N'^\perp$ we may have $|\sigma| = |\sigma_1| = |\sigma_2| = |\sigma_3|$, but we always have $|\sigma_4| < |\sigma_3|$. □

Using the usual techniques of game semantics and the notion of uniform families of strategies, dinatural transformations, ... the definability result can certainly be extended to formulas with atoms.

Example 5 (Catch)
We consider the finite total strategy on $?(!?1 \otimes !\bot) \parr ?(1 \oplus 1)$ ($\simeq_p !(!?1 \multimap ?1) \multimap ?(1 \oplus 1)$) containing the P-prefixes of the following two plays:





where t and f correspond to the moves coming from the two non-initial moves of $?(1 \oplus 1)$.

This strategy corresponds to the `catch` function $(A \to A) \to \mathbb{B}$ which tells if its argument is strict or not, using the fact that $?(1 \oplus 1)$ corresponds to the usual interpretation of booleans $\mathbb{B}$ in game models [7].

If we apply our definability theorem to this strategy, we build a proof in the following bottom-up way:

- we first move the arena to its corresponding additive form:

$$
\begin{aligned}
?(!?1 \otimes !\bot) \parr ?(1 \oplus 1) &\simeq_p ?(!?!\top \otimes !?0) \parr ?(!\top \oplus !\top) \\
&\simeq_p ?(!(?!\top \,\&\, ?0) \oplus (!\top \oplus !\top))
\end{aligned}
$$

- there are several moves justified by the initial one in the plays thus we isolate the first one corresponding to the initial move of $!(?!\top \,\&\, ?0)$, this means that the proof ends by:

$$
\dfrac{\dfrac{\vdots}{\vdash\, !(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top), ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}}{\dfrac{\vdash ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top)), ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}{\vdash ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}\, ?c}\, ?d
$$

- we obtain a central strategy which always plays in the left-hand side of the $\oplus$, by application of the $\oplus$-lemma and of the !-lemma, the proof contains:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\vdash ?!\top \mathbin{\&} ?0, ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}}{\vdash\, !(?!\top \mathbin{\&} ?0), ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}\, !}{\dfrac{\vdash\, !(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top), ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}{\vdots}}\, \oplus_1
$$

- up to a p-isomorphism, we have a strategy on:

$$
\begin{aligned}
&(?!\top \mathbin{\&} ?0) \parr (?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top)))\\
\simeq_p &(?!\top \parr ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))) \mathbin{\&} (?0 \parr ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top)))
\end{aligned}
$$

which means that the proof contains a &-rule:

$$
\dfrac{\dfrac{\vdots}{\vdash ?!\top, ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))} \qquad \dfrac{\vdots}{\vdash ?0, ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}}{\dfrac{\vdash ?!\top \mathbin{\&} ?0, ?(!(?!\top \mathbin{\&} ?0) \oplus (!\top \oplus !\top))}{\vdots}}\, \&
$$

- and so on ...
- if we move back to the original formula, we eventually get the following LLP proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1_2}\,1}{\vdash 1_2 \oplus 1_3}\,\oplus_1}{\vdash ?1_1, 1_2 \oplus 1_3}\,?w}{\vdash ?1_1, ?(1_2 \oplus 1_3)}\,?d}{\vdash\, !?1_1, ?(1_2 \oplus 1_3)}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1_3}\,1}{\vdash 1_2 \oplus 1_3}\,\oplus_2}{\vdash \bot, 1_2 \oplus 1_3}\,\bot}{\vdash \bot, ?(1_2 \oplus 1_3)}\,?d}{\vdash\, !\bot, ?(1_2 \oplus 1_3)}\,!}{\vdash\, !?1_1 \otimes !\bot, ?(1_2 \oplus 1_3), ?(1_2 \oplus 1_3)}\,\otimes}{\vdash ?(!?1_1 \otimes !\bot), ?(1_2 \oplus 1_3), ?(1_2 \oplus 1_3)}\,?d}{\dfrac{\vdash ?(!?1_1 \otimes !\bot), ?(1_2 \oplus 1_3)}{\vdash ?(!?1_1 \otimes !\bot) \parr ?(1_2 \oplus 1_3)}\,\parr}\,?c
$$

where the indexes are used to clarify the structure of the proof.

3.6 Comparison with ILL

We have extracted two particular game models from our general framework of polarized games: one for ILL and one for LLP. We are going to show how it is possible to reconstruct some of the constructions used in the first one from the second one.

Definition 13 (Lifting functor)
Let M and N be two negative games and σ be a strategy on $M \multimap N$, we define $\mathcal{L}N = \uparrow N^\perp$ and $\mathcal{L}\sigma = \uparrow\sigma : \uparrow N^\perp \overset{\bullet}{\multimap} \uparrow M^\perp$.

A similar functor has also been considered in [16].

Lemma 10 (Lift lemma)
If $\sigma : \uparrow N^\perp \overset{\bullet}{\multimap} \uparrow M^\perp$ is a strict strategy, there exists a unique strategy $\tau : M \multimap N$ such that $\sigma = \mathcal{L}\tau$.

Proposition 7
$\mathcal{L}$ is a full and faithful strong symmetric monoidal functor from $(\mathcal{HO}_-, \odot, \mathtt{I})$ to $(wo\mathcal{HO}_-^{\bullet\ \mathrm{op}}, \parr, \perp)$.

Proof: Straightforward verifications. □

Since the functor $\mathcal{L}$ is full and faithful, we can give a correspondence between strategies from M to N and the associated strict strategies from $\uparrow N^\perp$ to $\uparrow M^\perp$. In particular, using also $\mathcal{L}\sharp N = ?N^\perp$ (by definition of ?):

$$
\begin{array}{ccc}
\sigma \odot \tau : M_1 \odot M_2 \multimap N_1 \odot N_2 & \overset{\mathcal{L}}{\longleftrightarrow} & \sigma \parr \tau : \uparrow N_1^\perp \parr \uparrow N_2^\perp \overset{\bullet}{\multimap} \uparrow M_1^\perp \parr \uparrow M_2^\perp \\
\sigma^\dagger : \sharp M \multimap \sharp N & \overset{\mathcal{L}}{\longleftrightarrow} & !\sigma : ?N^\perp \overset{\bullet}{\multimap} ?M^\perp \\
c_{\sharp N} : \sharp N \multimap \sharp N \odot \sharp N & \overset{\mathcal{L}}{\longleftrightarrow} & c^{pol}_{?N^\perp} : ?N^\perp \parr ?N^\perp \overset{\bullet}{\multimap} ?N^\perp \\
w_{\sharp N} : \sharp N \multimap \mathtt{I} & \overset{\mathcal{L}}{\longleftrightarrow} & w^{pol}_{?N^\perp} : \perp \overset{\bullet}{\multimap} ?N^\perp \\
\sharp N \odot \sharp M \simeq \sharp(N \,\&\, M) & \overset{\mathcal{L}}{\longleftrightarrow} & ?N^\perp \parr ?M^\perp \overset{\bullet}{\simeq} ?(N \,\&\, M)^\perp
\end{array}
$$

Notice that the right constructions are more general than the left ones since they have been defined for more general games than only those in the image of $\mathcal{L}$.

Another structure of $(\mathcal{HO}_-, \odot, \mathtt{I})$ that can be explained by $(wo\mathcal{HO}_-^{\bullet\ \mathrm{op}}, \parr, \perp)$ but not directly with the functor $\mathcal{L}$ is the curryfication isomorphism for $(\odot, \multimap)$ which is a consequence of the corresponding one for $(\otimes, \multimap)$:

$$
\begin{aligned}
(M \odot N) \multimap L &\simeq_p \downarrow(M \odot N) \multimap L \\
&\simeq_p (\downarrow M \otimes \downarrow N) \multimap L \\
&\simeq_p \downarrow M \multimap \downarrow N \multimap L \\
&\simeq_p M \multimap N \multimap L
\end{aligned}
$$

Remark: Some of these results can be interpreted as a partial embedding of ILL in an extension of LLP with liftings, following the way polarized games are able to describe both systems together. Without being completely formal, the key ideas of the syntactical counterpart of this section are the following.

We restrict intuitionistic formulas to the sub-grammar:

$$
\begin{array}{rcccccccc}
A & ::= & X & | & M \multimap A & | & \top & | & A \,\&\, A \\
M & ::= & A & | & \sharp A & | & \mathtt{I} & | & M \odot M
\end{array}
$$

We can show, up to the equations of proposition 1, that a formula A is a negative formula in the variant of LLP using both liftings and exponentials, and that a formula M is such that $\downarrow M$ gives a positive formula.

A proof of $M_1, \ldots, M_k \vdash M$ in ILL is then interpreted as a proof of the sequent $\vdash \uparrow M_1^\perp, \ldots, \uparrow M_k^\perp, \downarrow M$ in LLP with liftings with a strong use of the derivable rule:

$$
\frac{\vdash \Gamma, \downarrow N}{\vdash \Gamma, N}\,\mathrm{REV} \quad = \quad \frac{\vdash \Gamma, \downarrow N \qquad \dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash \uparrow N^\perp, N}\,\uparrow}{\vdash \Gamma, N}\,cut
$$

Left rules of ILL are in this way encoded with the corresponding right rules of LLP: $\odot_L$ with $\parr$, $\sharp c$ with $?c$, ... in the spirit of the contravariant functor $\mathcal{L}$ of proposition 7.

4 Polarized AJM model

The first part of the paper has used the HO game setting to describe models of ILL and LLP. We want to show now that all this work is possible in an AJM setting. Instead of replaying all the previous results, we will only concentrate on the description of the AJM polarized model of LLP (see [4] for the description of the corresponding model of ILL).

4.1 Games

Starting from the definitions of [4], we introduce the required extensions to get a notion of polarized game and the constructions we need.

Definition 14 (Polarized game)
A *polarized game* is a tuple

$$A = (\pi_A, \mathcal{M}_A, \lambda_A, \mathcal{P}_A, \equiv_A)$$

where:

- $\pi_A \in \{O, P\}$ is the polarity of the game;
- $\mathcal{M}_A$ is the set of moves;
- λ_A is the labelling function from $\mathcal{M}_A$ to $\{O, P\}$;
- $\mathcal{P}_A$ is a non-empty prefix-closed set of alternated sequences of moves starting by moves of polarity π_A, called the set of *plays*;
- $\equiv_A$ is an equivalence relation on plays such that:
  - it respects the length: $s \equiv_A t \Rightarrow |s| = |t|$
  - it is prefix-closed: if $s \equiv_A t$ and $s' \leq s$, $t' \leq t$ with $|s'| = |t'|$ then $s' \equiv_A t'$
  - it is extensible: if $s \equiv_A t$ and $sa \in \mathcal{P}_A$ then there exists some $tb \in \mathcal{P}_A$ such that $sa \equiv_A tb$

A move is *initial* in A if it appears as the first move of a play of A. We will only consider *well opened* games, that is such that if a is an initial move of A, it never appears as a non-initial move (*i.e.* $sa \in \mathcal{P}_A \Rightarrow s = \varepsilon$).

There are two main differences between these games and the HO games: we do not have an enabling relation anymore, so that plays are just sequences of moves (no pointers), and we add the equivalence relation on the set of plays.

The "linear" game constructions $\top$, $\bot$, $\multimap$ and $\&$ are the same as the HO ones for π_A, $\mathcal{M}_A$, λ_A and $\mathcal{P}_A$ and they can be found in [4]. We just give the precise definitions of the $⅋$ and $\uparrow$ constructions:

Par. If M and N are negative games, $M ⅋ N$ is the negative game defined by:

- $\mathcal{M}_{M⅋N} = \mathcal{M}^i_M \times \mathcal{M}^i_N + \mathcal{M}^{ni}_M + \mathcal{M}^{ni}_N$
- $\lambda_{M⅋N}(m_1, m_2) = \lambda_M(m_1) = \lambda_N(m_2)$ if $(m_1, m_2) \in \mathcal{M}^i_M \times \mathcal{M}^i_N$
- $\lambda_{M⅋N}(m) = [\lambda_M, \lambda_N](m)$ if $m \in \mathcal{M}^{ni}_M + \mathcal{M}^{ni}_N$
- $\mathcal{P}_{M⅋N} = \{s \mid s \upharpoonright_M \in \mathcal{P}_M \wedge s \upharpoonright_N \in \mathcal{P}_N\}$
- $s \equiv_{M⅋N} t$ if $s \upharpoonright_M \equiv_M t \upharpoonright_M$, $s \upharpoonright_N \equiv_N t \upharpoonright_N$, and s and t have the *same interleaving*.

Two plays s and t of $M ⅋ N$ are said to have the *same interleaving* if whenever the k^{th} move ($k \geq 2$) of s is in M (resp. N), the k^{th} move of t is also in M (resp. N).

Lift. If P is a positive game, $\uparrow P$ is the negative game defined by:

- $\mathcal{M}_{\uparrow P} = \{\circ\} + \mathcal{M}_P$ where $\circ$ is a new move not in $\mathcal{M}_P$
- $\lambda_{\uparrow P} = \lambda_P$ for the moves of $\mathcal{M}_P$

- $\lambda_{\uparrow P}(\circ) = O$
- $\mathcal{P}_{\uparrow P} = \circ.\mathcal{P}_P + \{\varepsilon\}$
- $\circ s \equiv_{\uparrow P} \circ t$ if $s \equiv_P t$

For the other linear constructions we give the description of $\equiv_A$ only:

$$\begin{array}{rcl}
s \equiv_{A^\perp} t & \iff & s \equiv_A t \\
s \equiv_\top t & \iff & s = t \\
s \equiv_\bot t & \iff & s = t \\
s \equiv_{M \multimap N} t & \iff & s \restriction_M \equiv_M t \restriction_M \text{ and } s \restriction_N \equiv_N t \restriction_N \\
& & \quad \text{and } s,\ t \text{ have the same interleaving} \\
s \equiv_{M \& N} t & \iff & s, t \in \mathcal{P}_M \wedge s \equiv_M t \text{ or } s, t \in \mathcal{P}_N \wedge s \equiv_N t
\end{array}$$

The positive constructions are obtained using the definition of $A^\perp$ and the corresponding negative construction.

The true difference with HO games is in the definition of $\sharp$ which replaces the use of pointers by indexes:

Sharp. If N is a negative game, $\sharp N$ is the negative game defined by:

- $\mathcal{M}_{\sharp N} = \mathbb{N} \times \mathcal{M}_N$
- $\lambda_{\sharp N}(i, n) = \lambda_N(n)$
- $\mathcal{P}_{\sharp N} = \{s \mid \forall i \in \mathbb{N},\ s \restriction_i \in \mathcal{P}_N\}$ where $s \restriction_i$ is the sub-sequence of s obtained by replacing any move of the shape (i, n) by n and by removing the other moves.
- $s \equiv_{\sharp N} t$ if there exists a permutation θ of $\mathbb{N}$ such that for each i, $s \restriction_i \equiv_N \theta(t) \restriction_i$ and s and $\theta(t)$ have the same interleaving (where $\theta(t)$ is obtained by replacing any move (i, n) of t by $(\theta(i), n)$).

Two plays s and t of $\sharp N$ are said to have the *same interleaving* if whenever the k^{th} move of s has index i, the k^{th} move of t has also index i.

The main novelty required to define a model of LLP with AJM games is the appropriate notion of multiple games.

Definition 15 (Multiple game)
A game A is a *multiple game* if the non-initial moves of any play are pairs starting with an integer (called the *index*) and:

- *restriction*: if $s \in \mathcal{P}_A$ and $i \in \mathbb{N}$, $s_{(i)} \in \mathcal{P}_A$ where $s_{(i)}$ is the sub-sequence of s containing the first move and the moves with index i.
- *renaming*: if φ is an injective function $\mathbb{N} \to \mathbb{N}$, $\varphi(s) \in \mathcal{P}_A$ and $s \equiv_A \varphi(s)$ where $\varphi(s)$ is obtained from s by replacing any index i by $\varphi(i)$.
- *interleaving*: if $s_1, s_2 \in \mathcal{P}_A$ with the same initial move and disjoint sets of indexes, and s is an interleaving of s_1 and s_2, we have $s \in \mathcal{P}_A$. Moreover, in the same conditions, if s is an interleaving of s_1 and s_2 and t is an interleaving of t_1 and t_2 with the same interleaving, $s_1 \equiv_A t_1 \wedge s_2 \equiv_A t_2 \iff s \equiv_A t$.

An interleaving of two plays s and t of a multiple game with the same initial move m is an alternated sequence u starting by m, such that there exists a partition of the non-initial moves of u into two sub-sequences s' and t' with $s = ms'$ and $t = mt'$.

Remark: As a consequence of the definition, we have the *switching condition*: if two successive moves of a play have different indexes then the polarity of the second one is $\overline{\pi_A}$.

Proposition 8 (Multiple constructions)
Multiplicity is closed under the following constructions:

- 1, 0, $\top$ *and* $\bot$ *are multiple games.*

- *If N is negative and P is positive, $!N$ and $?P$ are multiple games.*
- *If P, Q, M and N are* multiple *then $P \otimes Q$, $P \oplus Q$, $M \parr N$ and $M \& N$ are multiple games.*

Proof: The positive cases are deduced from the negative ones, the result for $\top$, $\bot$ and $M \& N$ is straightforward, we concentrate on the two other cases $?P$ and $M \parr N$:

- $?P$ case:
  - If $s \in \mathcal{P}_{?P}$ then the play obtained from $s_{(i)}$ by erasing the indexes is in $\mathcal{P}_{\uparrow P}$ thus $s_{(i)} \in \mathcal{P}_{?P}$.
  - The renaming by an injective function is easy to check.
  - If $s_1, s_2 \in \mathcal{P}_{?P}$ and s is an interleaving of s_1 and s_2, let i be an index, since s_1 and s_2 have disjoint indexes, $s_{(i)} = s_{1(i)}$ or $s_{(i)} = s_{2(i)}$, thus by removing the indexes of $s_{(i)}$ we get a play in $\mathcal{P}_{\uparrow P}$ and, by definition, $s \in \mathcal{P}_{?P}$. If t is an interleaving of t_1 and t_2 with the same interleaving as s, if $s_1 \equiv_{?P} t_1$ and $s_2 \equiv_{?P} t_2$ then there exist two permutations θ_1 and θ_2 such that $s_k \upharpoonright_i \equiv_{\uparrow P} \theta_k(t_k) \upharpoonright_i$. Using the fact that s_1 and s_2 (resp. t_1 and t_2) have disjoint indexes, we can build a permutation θ such that $s \upharpoonright_i \equiv_{\uparrow P} \theta(t) \upharpoonright_i$ thus $s \equiv_{?P} t$. For the converse, if $s \equiv_{?P} t$, we have θ such that $s \upharpoonright_i \equiv_{\uparrow P} \theta(t) \upharpoonright_i$ and we just use $\theta_1 = \theta_2 = \theta$ to get $s_k \upharpoonright_i \equiv_{\uparrow P} \theta_k(t_k) \upharpoonright_i$.
- $M \parr N$ case:
  - Since $s_{(i)} \upharpoonright_M = (s \upharpoonright_M)_{(i)}$, we have $s_{(i)} \upharpoonright_M \in \mathcal{P}_M$ and $s_{(i)} \upharpoonright_N \in \mathcal{P}_N$ by hypothesis on M and N, thus $s_{(i)} \in \mathcal{P}_{M \parr N}$.
  - Let φ be in injection, since $\varphi(s) \upharpoonright_M = \varphi(s \upharpoonright_M)$ we have the result.
  - If s is an interleaving of s_1 and s_2, $s \upharpoonright_M$ is an interleaving of $s_1 \upharpoonright_M$ and $s_2 \upharpoonright_M$ (and the same for $s \upharpoonright_N$) so that $s \in \mathcal{P}_{M \parr N}$. If t is an interleaving of t_1 and t_2 with the same interleaving as s, if $s_1 \equiv_{M \parr N} t_1$ and $s_2 \equiv_{M \parr N} t_2$ then $s_k \upharpoonright_M \equiv_M t_k \upharpoonright_M$ and $s \upharpoonright_M$ (resp. $t \upharpoonright_M$) is an interleaving of $s_1 \upharpoonright_M$ (resp. $t_1 \upharpoonright_M$) and $s_2 \upharpoonright_M$ (resp. $t_2 \upharpoonright_M$) so that, by hypothesis on M, $s \upharpoonright_M \equiv_M t \upharpoonright_M$, and the same for N. The converse is proved in the same spirit. □

4.2 Strategies

In order to get a denotational model, we have to introduce the notion of equivalence of strategies.

Definition 16 (Strategy and partial equivalence)
A *strategy* σ on the game A, denoted by $\sigma : A$, is a non-empty P-prefix-closed subset of sequences in $\mathcal{P}_A^P$ such that if $sa^P \in \sigma$ and $sb^P \in \sigma$ then $a = b$.

Two strategies σ and τ on the game A are equivalent ($\sigma \approx \tau$) if:

$$sab \in \sigma, t \in \tau, sa \equiv_A ta' \Rightarrow \exists ta'b' \in \tau, sab \equiv_A ta'b'$$

together with the symmetric condition.

Proposition 9 (AJM category)
Negative games with equivalence classes of strategies on $M \multimap N$ as morphisms give a category denoted by $\mathcal{AJM}_-$.

Proof: See [4]. □

A p-isomorphism $f : A \to B$ should now preserve the equivalence relation: $s \equiv_A s' \iff f(s) \equiv_B f(s')$. It is easy to check that properties of proposition 1 are still correct.

Lemma 11
Let σ be a strategy on a game A such that $\sigma \approx \sigma$ and f be a p-isomorphism from A to B, $f(\sigma)$ is a strategy on B and $f(\sigma) \approx f(\sigma)$.

Proof: By lemma 3, $f(\sigma)$ is a strategy on B. If $ta_2b_2 = f(sa_1b_1) \in f(\sigma)$ and $t'a'_2 = f(s'a'_1)$ with $s' \in \sigma$ are such that $ta_2 \equiv_B t'a'_2$ hence $sa_1 \equiv_A s'a'_1$, since $\sigma \approx \sigma$ there exists b'_1 such that $s'a'_1b'_1 \in \sigma$ and $sa_1b_1 \equiv_A s'a'_1b'_1$ thus $t'a'_2b'_2 = f(sa'_1b'_1) \in f(\sigma)$ and $ta_2b_2 \equiv_B t'a'_2b'_2$. □

Promotion. If c is an injection $\mathbb{N}^2 \to \mathbb{N}$, let t be a play on $?N^\perp \multimap M$ (with M multiple) we denote by t_i the sub-sequence of t containing the first moves in M and $?N^\perp$, every move $n \in \mathcal{M}_N$ which appears in $?N^\perp$ with index i and for every move in M of the shape $(c(i,j),m)$, the move (j,m). If σ is a strategy on $M \parr N$, we define $!^c\sigma : ?N^\perp \overset{\bullet}{\multimap} M$ to be the strategy $\{s \in \mathcal{P}^P_{?N^\perp \multimap M} \mid s \subset \bigcup_{i\in\mathbb{N}} s_i \wedge \forall i \in \mathbb{N}, s_i \in {\downarrow}\sigma\}$ where $s \subset \bigcup_{i\in\mathbb{N}} s_i$ means that each move of s must appear in some s_i.

Dereliction. If i is an integer, der^i_N is the strategy on $\uparrow N^\perp \overset{\bullet}{\multimap} \uparrow\flat N^\perp$ defined by $der^i_N = \{s \in \mathcal{P}^P_{\uparrow N^\perp \multimap \uparrow\flat N^\perp} \mid \forall t \leq^P s, t\restriction_{\uparrow\flat N^\perp} = i.(t\restriction_{\uparrow N^\perp})\}$.

Contraction. Let N be a multiple negative game, if t is a play of $N_1 \parr N_2 \multimap N_0$ (where the indexes are just used to distinguish occurrences), we can decompose it into $t = t_0t'$ where t_0 contains only moves in N_0 and t' starts by a move in $N_1 \parr N_2$ (this entails that any move of t' in N_0 comes after at least one move in $N_1 \parr N_2$). We denote by t_i $(i = 1,2)$ the sub-sequence of t containing t_0, all the moves in N_i, and the moves of t' in N_0 before which the last move in $N_1 \parr N_2$ is in N_i.

If l and r are two injective functions $\mathbb{N} \to \mathbb{N}$ with disjoint codomains, $c^{l,r}_N$ is the strategy on $N \parr N \overset{\bullet}{\multimap} N$ defined by $c^{l,r}_N = \{s \in \mathcal{P}^P_{N_1 \parr N_2 \multimap N_0} \mid \forall t \leq^P s, t_1\restriction_{N_1} = l(t_1\restriction_{N_0}) \wedge t_2\restriction_{N_2} = r(t_2\restriction_{N_0})\}$.

Weakening. w_N is the strategy on $\perp \overset{\bullet}{\multimap} N$ defined by $w_N = \{\varepsilon\} \cup \{n\circ \mid n \in \mathcal{M}^i_N\}$.

Lemma 12
If l, r and l', r' are two pairs of injective functions $\mathbb{N} \to \mathbb{N}$ with disjoint codomains, $c^{l,r}_N \approx c^{l',r'}_N$.

Proof: Let sab be a play in $c^{l,r}_N : N_1 \parr N_2 \overset{\bullet}{\multimap} N_0$, t be a play in $c^{l',r'}_N : N_1 \parr N_2 \overset{\bullet}{\multimap} N_0$, if $sa \equiv_{N_1 \parr N_2 \multimap N_0} ta'$ we have to find b' such that $ta'b' \in c^{l',r'}_N$ and $sab \equiv_{N_1 \parr N_2 \multimap N_0} ta'b'$.

- If $s = \varepsilon$, then $t = \varepsilon$ so that $b = (a,a)$ and by taking $b' = (a',a')$ we have the result.
- If $sab = s(i,n)(l(i),n)$ with $(i,n) \in N_1$ (the case N_2 is proved in the same way), then $a' = (j,m)$. Since $s(i,n) \equiv t(j,m)$, we have $(j,m) \in N_1$. We choose $b' = (l'(j),m)$ so that $t(j,m)(l'(j),m) \in c^{l',r'}_N$ and we have to show that $s(i,n)(l(i),n) \equiv t(j,m)(l'(j),m)$:
  - $s(i,n)(l(i),n)$ and $t(j,m)(l'(j),m)$ have the same interleaving (of moves of N_1, N_2 and N_0) since it is true for $s(i,n)$ and $t(j,m)$, and $(l(i),n)$ and $(l'(j),m)$ are both in N_0.
  - $s(i,n)(l(i),n)\restriction_{N_k} \equiv_N t(j,m)(l'(j),m)\restriction_{N_k}$ since $s(i,n)\restriction_{N_k} \equiv_N t(j,m)\restriction_{N_k}$ for $k = 1,2$.
  - It remains to prove $s(i,n)(l(i),n)\restriction_{N_0} \equiv_N t(j,m)(l'(j),m)\restriction_{N_0}$, that is $s\restriction_{N_0}(l(i),n) \equiv_N t\restriction_{N_0}(l'(j),m)$. We know that $s\restriction_{N_0}(l(i),n)$ is an interleaving of $l(s(i,n)\restriction_{N_1})$ and $r(s\restriction_{N_2})$, and this holds also for t with the same interleaving because a P-move (resp. O-move) in $s\restriction_{N_0}$ comes from a move in $s\restriction_{N_1}$ if (in s) it is after (resp. before) a move in N_1 and moreover the p^{th} move of s is in N_k $(k = 1,2)$ if and only if the p^{th} move of t is also in N_k by $s(i,n) \equiv t(j,m)$. We also have $s(i,n)\restriction_{N_1} \equiv_N t(j,m)\restriction_{N_1}$ so that $l(s(i,n)\restriction_{N_1}) \equiv_N l'(t(j,m)\restriction_{N_1})$ by definition of a multiple game (renaming condition) since l and l' are injective (idem with r, r' and N_2). Finally $l(s(i,n)\restriction_{N_1})$ and $r(s(i,n)\restriction_{N_2})$ have disjoint sets of indexes (idem for t) so that $s\restriction_{N_0}(l(i),n) \equiv_N t\restriction_{N_0}(l'(j),m)$ by the interleaving condition for multiple games.
- If $sab = s(l(i),n)(i,n)$ (the case with r is similar), by the switching property of multiple games, the last move of s is of the shape $(l(i),x)$ so that the previous move of s is in N_1 and by an argument similar to what we have done just before, since $s(l(i),n) \equiv ta'$, it must be the case that $a' = (l'(j),m)$ for some j and some m. We choose $b' = (j,m) \in N_1$ and we immediately have $t(l'(j),m)(j,m) \in c^{l',r'}_N$. We now show that $s(l(i),n)(i,n) \equiv t(l'(j),m)(j,m)$. The interleavings are the same and the projections on N_2 and N_0 are in relation by $s(l(i),n) \equiv t(l'(j),m)$. Concerning N_1, $s(l(i),n)(i,n)\restriction_{N_0}$ is an interleaving of $l(s(l(i),n)(i,n)\restriction_{N_1})$ and $r(s(l(i),n)(i,n)\restriction_{N_2})$ (and also for t with the same interleaving) so that, by the interleaving condition, $s(l(i),n)(i,n)\restriction_{N_0} = s(l(i),n)\restriction_{N_0} \equiv_N t(l'(j),m)\restriction_{N_0} = t(l'(j),m)(j,m)\restriction_{N_0}$ implies $l(s(l(i),n)(i,n)\restriction_{N_1}) \equiv_N l'(t(l'(j),m)(j,m)\restriction_{N_1})$ and, by the renaming condition, $s(l(i),n)(i,n)\restriction_{N_1} \equiv_N t(l'(j),m)(j,m)\restriction_{N_1}$. □

Proposition 10 ($\parr$-monoid N)
If N is a multiple negative game, the triple $(N, c^{l,r}_N, w_N)$ is a $\parr$-monoid in $\mathcal{AJM}_-$.

PROOF: As for proposition 6. □

Proposition 11 (AJM co-monad)
$\sharp$ is a co-monad in the category $\mathcal{AJM}_-$ and $\sharp N \odot \sharp M \simeq \sharp(N \,\&\, M)$.

PROOF: See [4]. □

In the sequel we will say "*strategy σ*" instead of "*strategy σ such that $\sigma \approx \sigma$*".

4.3 AJM models

Given an interpretation of atoms as multiple games of the corresponding polarity, we now interpret formulas as multiple games and proofs as equivalence classes of strategies σ (with $\sigma \approx \sigma$). The interpretation of axioms, cuts, multiplicatives and additives is the same as for the linear HO case (section 3.2). We just have to verify that the strategies σ obtained from proofs verify $\sigma \approx \sigma$.

We consider an arbitrary choice of an integer $i \in \mathbb{N}$, of a pair l, r of injections $\mathbb{N} \to \mathbb{N}$ with disjoint codomains and of an injection $c : \mathbb{N}^2 \to \mathbb{N}$.

4.3.1 First model.

We give the interpretation of the exponential rules:

- $!$: If $\sigma : \mathcal{N} \parr N$, then $!^c\sigma$ is the strategy on $?N^\perp \overset{\bullet}{\multimap} \mathcal{N}$.
- $?d$: If $\sigma : N \overset{\bullet}{\multimap} \Gamma$, composing σ (considered on $\Gamma \parr \uparrow N^\perp$) with the strategy $\Gamma \parr der^i_N$ gives the strategy on $\Gamma \parr ?N^\perp$.
- $?c$: If $\sigma : \Gamma \parr N \parr N$, we compose it with $\Gamma \parr c^{l,r}_N$ to obtain a strategy on $\Gamma \parr N$.
- $?w$: If $\sigma : \Gamma$, we compose it with $\Gamma \parr w_N$ to obtain a strategy on $\Gamma \parr N$ (using $\Gamma \simeq_p \Gamma \parr \perp$).

Proposition 12 (Correctness)
Multiple polarized AJM games are a denotational model of LLP.

4.3.2 Second model.

In order to get a completeness result for an AJM model, strategies have to respect the "history free" condition but this condition is lost in our first model in the $!^c$ construction. This is why we are going to refine the model.

Definition 17 (History free strategy)
A strategy $\sigma : A$ is *history free* if $sab \in \sigma$ and $tac \in \sigma$ implies $b = c$.

This difficulty with the AJM definability already appears in the usual linear AJM setting since the required "history free" condition is not respected by the product (the natural strategy $A \multimap A \,\&\, A$ is not history free but the one on $\sharp A \multimap A \,\&\, A$ is), this is why we have to move to the co-Kleisli category with respect to $\sharp$. This is known to work for the product and will also solve our problem with promotion.

Definition 18 (Linear strategy)
A strategy $\sigma : \sharp N \overset{\bullet}{\multimap} M$ is *linear* if, in each play of σ, all the moves in $\sharp N$ have the same index.

In our modified model, proofs of $\vdash P, N_1, \ldots, N_k$ are interpreted by (equivalence classes of) *linear history free* strategies on $\sharp P^\perp \overset{\bullet}{\multimap} N_1 \parr \ldots \parr N_k$.

History free promotion. Let c be an injection $\mathbb{N}^2 \to \mathbb{N}$ and d be an injection from the initial moves of M to $\mathbb{N}$, if σ is a strategy on $M \parr N$, we define $!^{c,d}_{\mathrm{hf}}\sigma : \sharp ?N^\perp \overset{\bullet}{\multimap} M$ to be the linear strategy $\{d(s) \mid s \in {!^c}\sigma\}$ where $d(s)$ is obtained from s by replacing any move n in $?N^\perp$ by $(d(m), n)$ with m the initial move of s.

Lemma 13
If $\sigma : M \parr N$ is an history free strategy, $!^{c,d}_{\mathrm{hf}}\sigma : \sharp ?N^\perp \stackrel{\bullet}{\multimap} M$ is an history free strategy.

Proof: If $sab \in !^{c,d}_{\mathrm{hf}}\sigma$ and $tab' \in !^{c,d}_{\mathrm{hf}}\sigma$, we look at the lengths of s and t:

- If $s = \varepsilon$, a is an initial move m of M. This implies $t = \varepsilon$ and by definition of $!^{c,d}_{\mathrm{hf}}\sigma$, $b = b' = (d(m), \circ)$.
- If $s = m(d(m), \circ)$, we have $a = (d(m), i, n)$ with n initial in N which implies $t = m'(d(m'), \circ)$ and $m' = m$ because d is injective. Since σ is deterministic it answers a move (j, b_1) to the play (m, n) so that $b = b' = (d(m), i, j, b_1) \in \sharp ?N^\perp$ if $(j, b_1) \in N$ or $b = b' = (c(i, j), b_1) \in M$ if $(j, b_1) \in M$.
- If the length of s is at least 4, it is also the case for t as we have seen. In this case $a = (d(m_0), i, j, n) \in \sharp ?N^\perp$ (resp. $a = (c(i, j), m) \in M$) and the answer of the history free strategy σ to (j, n) (resp. (j, m)) is a move $(k, n') \in N$ or $(k, m') \in M$ and we have $b = b' = (d(m_0), i, k, n') \in \sharp ?N^\perp$ or $b = b' = (c(i, k), m') \in M$ (note that i is computable from $c(i, j)$ because c is injective).

□

The interpretation of the exponential rules is not very different from the first model:

- $!$: If $\sigma : \mathcal{N} \parr N$, then $!^{c,d}_{\mathrm{hf}}\sigma$ is the strategy on $\sharp ?N^\perp \stackrel{\bullet}{\multimap} \mathcal{N}$.
- $?d$: If $\sigma : \sharp N \stackrel{\bullet}{\multimap} \Gamma$, the rule does not modify the interpretation, using $\sharp N \multimap \Gamma \simeq_p ?N^\perp \parr \Gamma$.
- $?c$: If $\sigma : \Gamma \parr N \parr N$, we compose it with $\Gamma \parr c^{l,r}_N$ to obtain a strategy on $\Gamma \parr N$.
- $?w$: If $\sigma : \Gamma$, we compose it with $\Gamma \parr w_N$ to obtain a strategy on $\Gamma \parr N$.

Proposition 13 (Correctness)
Through the interpretation given by the second model, multiple polarized AJM games (with equivalence classes of history free strategies) are a denotational model of LLP.

It is now possible to state a definability result for this second model. Since we do not want to address the question of compact strategies (corresponding to finite strategies in the HO setting), we will only state a *local definability* result.

Theorem 4 (Local definability)
Let A be a polarized formula without atom, if σ is a total history free strategy on A, there exists a rule R of LLP *with conclusion $\vdash A$ and arity n and there exist n total strategies σ_1, ..., σ_n such that σ is the interpretation of R applied to σ_1, ..., σ_n.*

Proof: Very similar to what we have done for theorem 3, together with the AJM definability proof for PCF [4]. □

Using the local definability, we can immediately extract a, possibly infinite, proof of LLP from any total history free strategy.

Example 6 (Infinite proof)
We consider the smallest total history free strategy on $?!\perp$ containing:

$$\begin{array}{c} ?!\perp \\ \\ ? \\ (1, !) \\ (1, (i, \perp)) \\ (2, !) \\ (2, (j, \perp)) \\ \vdots \end{array}$$

where $?$ is the first move of $?!\perp$, $!$ is the first move of $!\perp$ and $\perp$ is the unique move of $\perp$. This strategy is the interpretation of the *infinite* proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash ?!\bot\end{array}}{\vdash \bot, ?!\bot}\,\bot}{\vdash !\bot, ?!\bot}\,!}{\vdash ?!\bot, ?!\bot}\,?d}{\vdash ?!\bot}\,?c$$

while $\vdash ?!\bot$ is not provable in LL and LLP.

Remark: We have described two polarized game models based on the HO and on the AJM exponentials. It should be possible to do the same kind of work in the framework of *sequential algorithms* [32, 33] with the introduction of the corresponding notion of multiple games.

5 Applications to the $\lambda\mu$-calculus

We have described a game model for Polarized Linear Logic claiming that it gives models for many other systems by translation in LLP. We will focus on Parigot's $\lambda\mu$-calculus [19, 21, 23] for both call-by-name and call-by-value evaluations.

Some details about the translations of the $\lambda\mu$-calculus into LLP are given in appendixes C and D and we are going to describe some consequences of our results for the semantics of the $\lambda\mu$-calculus. The following results strongly rely on the results sketched in appendixes in order to directly transfer the results from LLP to the $\lambda\mu$-calculus. However it is also possible to replay the corresponding proofs given previously for LLP and to get a direct proof for the $\lambda\mu$-calculus.

By proposition 18 and theorem 2, HO negative games give a denotational model of the call-by-name $\lambda\mu$-calculus. This can also be expressed through Selinger's control categories [23] (see appendix B for some elements about the relation between control categories and LLP) for which the call-by-name $\lambda\mu$-calculus forms an internal language.

Proposition 14 (Control category of HO games)
The category of HO multiple negative games with morphisms given by strategies on $!M \multimap N$ ($\simeq_p \sharp M \rightarrowtriangle N$ used in intuitionistic games) is a control category.

We can apply the definability result to the $\lambda\mu$-calculus:

Proposition 15 (Call-by-name full completeness)
Let A be a type without variable and σ be a finite total strategy on A^-, there exists a $\lambda\mu$-term u of type A such that σ is the call-by-name interpretation of u.

Proof: By theorem 3, there exists a proof π of A^- in LLP such that σ is the interpretation of π. Using [14, 34], we can associate with π a proof-net $\mathcal{R}$ which has the same game interpretation as π and given such a proof-net, there exists a $\lambda\mu$-term u such that $u^- = \mathcal{R}$, thus σ is the call-by-name interpretation of u. □

In the same way, we obtain a model of the call-by-value $\lambda\mu$-calculus from positive HO games (by proposition 19 and theorem 2):

Proposition 16 (Co-control category of HO games)
The category of HO multiple positive games with morphisms from P to Q given by strategies on $P^\perp \parr ?Q$ is a co-control category.

Proof: This category is the opposite of the control category of proposition 14. □

Proposition 17 (Call-by-value full completeness)
Let A be a type without variable and σ be a finite total strategy on $?A^+$, there exists a $\lambda\mu$-term u of type A such that σ is the call-by-value interpretation of u.

Proof: By proposition 15, there exists a $\lambda\mu$-term t such that the call-by-name interpretation of t is σ. Let $\tilde{t}$ be Selinger's syntactical dual [23] of t, the call-by-value interpretation of $\tilde{t}$ is σ. □

This shows that polarized games give a tool for building models of call-by-name and call-by-value programming languages with control operators. In particular,we can easily interpret `call/cc`, `catch` (see example 5), ...

We also get models of both call-by-name and call-by-value $\lambda\mu$-calculi from the AJM polarized game model of LLP, except for completeness which is reduced to its local version.

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A Cut elimination for LLP

The definition of the cut-elimination procedure for LLP in sequent calculus requires a lot of commutative steps. However it is possible to define a proof-net syntax for LLP [14] showing that almost all these commutative steps are innocuous. We only give here the sequent calculus steps which are shown to be meaningful by the proof-net syntax.

The reader can verify either that the game interpretation of a sequent calculus proof is definable on the corresponding proof-net or that the "innocuous" commutative steps do not modify the interpretation of proofs.

MALLP

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash \Gamma, A}\,cut \quad \overset{ax}{\longrightarrow} \quad \vdash \Gamma, A$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\,\downarrow \qquad \dfrac{\vdash N^{\perp}, \Gamma}{\vdash \uparrow N^{\perp}, \Gamma}\,\uparrow}{\vdash \mathcal{N}, \Gamma}\,cut \quad \overset{\downarrow-\uparrow}{\longrightarrow} \quad \dfrac{\vdash \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash \mathcal{N}, \Gamma}\,cut$$

$$\dfrac{\dfrac{\vdash M, \mathcal{N}, N}{\vdash \downarrow M, \mathcal{N}, N}\,\downarrow \qquad \vdash N^{\perp}, \Gamma}{\vdash \downarrow M, \mathcal{N}, \Gamma}\,cut \quad \overset{\downarrow-*}{\longrightarrow} \quad \dfrac{\dfrac{\vdash M, \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash M, \mathcal{N}, \Gamma}\,cut}{\vdash \downarrow M, \mathcal{N}, \Gamma}\,\downarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, M^{\perp} \qquad \vdash \Delta, N^{\perp}}{\vdash \Gamma, \Delta, M^{\perp} \otimes N^{\perp}}\,\otimes \qquad \dfrac{\vdash M, N, \Xi}{\vdash M \parr N, \Xi}\,\parr}{\vdash \Gamma, \Delta, \Xi}\,cut$$

$$\overset{\otimes-\parr}{\longrightarrow} \quad \dfrac{\vdash \Gamma, M^{\perp} \qquad \dfrac{\vdash \Delta, N^{\perp} \qquad \vdash M, N, \Xi}{\vdash M, \Delta, \Xi}\,cut}{\vdash \Gamma, \Delta, \Xi}\,cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N_i^{\perp}}{\vdash \Gamma, N_1^{\perp} \oplus N_2^{\perp}}\,\oplus_i \qquad \dfrac{\vdash N_1, \Delta \qquad \vdash N_2, \Delta}{\vdash N_1 \,\&\, N_2, \Delta}\,\&}{\vdash \Gamma, \Delta}\,cut$$

$$\overset{\oplus-\&}{\longrightarrow} \quad \dfrac{\vdash \Gamma, N_i^{\perp} \qquad \vdash N_i, \Delta}{\vdash \Gamma, \Delta}\,cut$$

$$\dfrac{\dfrac{\vdash M_1, \Gamma, A \qquad \vdash M_2, \Gamma, A}{\vdash M_1 \,\&\, M_2, \Gamma, A}\,\& \qquad \vdash A^{\perp}, \Delta}{\vdash M_1 \,\&\, M_2, \Gamma, \Delta}\,cut$$

$$\overset{\&-*}{\longrightarrow} \quad \dfrac{\dfrac{\vdash M_1, \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash M_1, \Gamma, \Delta}\,cut \qquad \dfrac{\vdash M_2, \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash M_2, \Gamma, \Delta}\,cut}{\vdash M_1 \,\&\, M_2, \Gamma, \Delta}\,\&$$

$$\dfrac{\dfrac{}{\vdash 1}\,1 \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp}{\vdash \Gamma}\,cut \quad \overset{1-\perp}{\longrightarrow} \quad \vdash \Gamma$$

$$\dfrac{\dfrac{}{\vdash \top, \Gamma, A}\,\top \qquad \vdash A^{\perp}, \Delta}{\vdash \top, \Gamma, \Delta}\,cut \quad \overset{\top}{\longrightarrow} \quad \dfrac{}{\vdash \top, \Gamma, \Delta}\,\top$$

LLP

$$\dfrac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \dfrac{\vdash N^{\perp}, \Gamma}{\vdash ?N^{\perp}, \Gamma}\,?d}{\vdash \mathcal{N}, \Gamma}\,cut \quad \xrightarrow{!-?d} \quad \dfrac{\vdash \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash \mathcal{N}, \Gamma}\,cut$$

$$\dfrac{\dfrac{\vdash M, \mathcal{N}, N}{\vdash !M, \mathcal{N}, N}\,! \qquad \vdash N^{\perp}, \Gamma}{\vdash !M, \mathcal{N}, \Gamma}\,cut \quad \xrightarrow{!-*} \quad \dfrac{\dfrac{\vdash M, \mathcal{N}, N \qquad \vdash N^{\perp}, \Gamma}{\vdash M, \mathcal{N}, \Gamma}\,cut}{\vdash !M, \mathcal{N}, \Gamma}\,!$$

$$\dfrac{\dfrac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\,?c \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,cut$$

$$\xrightarrow{?c-*} \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, N, N \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta, N}\,cut \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta, \Delta}\,cut}{\vdash \Gamma, \Delta}\,?c$$

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, N}\,?w \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,cut \quad \xrightarrow{?w-*} \quad \dfrac{\vdash \Gamma}{\vdash \Gamma, \Delta}\,?w$$

B Categorical interpretation of LLP

P. Selinger introduced the notion of *control categories* [23] as models of both call-by-name and call-by-value $\lambda\mu$-calculi.

We recall the main ingredients of control categories:

Definition 19 (Control category)
A category $(\mathcal{C}, \&, \top, \rightarrow, \parr, \bot)$ is a *control category* if:

- $(\mathcal{C}, \&, \top, \rightarrow)$ is a cartesian closed category
- $(\mathcal{C}, \parr, \bot)$ is a symmetric premonoidal category (see [35]) with codiagonals (that is with a $\parr$-monoidal structure for each object)
- $\parr$ distributes over &
- there is a natural isomorphism between $A \rightarrow (B \parr C)$ and $(A \rightarrow B) \parr C$

with some more commutative diagrams.

$\mathcal{C}$ is a *co-control category* if $\mathcal{C}^{\text{op}}$ is a control category.

Theorem 5 (Selinger)
Control categories are denotational models of the call-by-name $\lambda\mu$-calculus and co-control categories are denotational models of the call-by-value $\lambda\mu$-calculus.

It is possible to interpret LLP into any control category $\mathcal{C}$. Moreover Selinger's interpretations of the call-by-name and the call-by-value $\lambda\mu$-calculi into a (co)-control category $\mathcal{C}$ factorize through the translations $(.)^-$ and $(.)^+$ of the $\lambda\mu$-calculi in LLP (see appendixes C and D) and this interpretation $(.)^\star$ of LLP in $\mathcal{C}$.

The interpretation of formulas as objects is straightforward:

$$
\begin{aligned}
(N \parr M)^\star &= N^\star \parr M^\star \\
(N \,\&\, M)^\star &= N^\star \,\&\, M^\star \\
\bot^\star &= \bot \\
\top^\star &= \top \\
(?N^\perp)^\star &= N^\star \rightarrow \bot
\end{aligned}
$$

A proof π of $\vdash \mathcal{N}$ is interpreted as a morphism $\pi^\star$ from $\top$ to $\mathcal{N}^\star$ and a proof π of $\vdash \mathcal{N}, P$ is interpreted as a *central* morphism $\pi^\star$ from $(P^\perp)^\star$ to $\mathcal{N}^\star$. The details of this interpretation are given in [14].

C Call by name $\lambda\mu$-calculus and LLP

The translation $(.)^-$ of the call-by-name $\lambda\mu$-calculus into LLP is obtained by translating types as negative formulas:

$$
\begin{array}{ccc}
X & \rightsquigarrow & X \\
\mathrm{T} & \rightsquigarrow & \top \\
\mathrm{F} & \rightsquigarrow & \perp \\
A \wedge B & \rightsquigarrow & A^- \,\&\, B^- \\
A \vee B & \rightsquigarrow & A^- \parr B^- \\
A \to B & \rightsquigarrow & !A^- \multimap B^-
\end{array}
$$

the judgment $\Gamma \vdash t : A \mid \Delta$ is translated as $\vdash ?(\Gamma^-)^\perp, A^-, \Delta^-$.

The translation of terms is given in the following way:

$$
(x^A : A)^- = \dfrac{\dfrac{}{\vdash A^{-\perp}, A^-}}{\vdash ?A^{-\perp}, A^-}
$$

$$
(\lambda x^A.t^B : A \to B)^- = \dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, B^-, \Delta^-\end{array}}{\vdash ?\Gamma^{-\perp} \setminus \{?A^{-\perp}\}, ?A^{-\perp} \parr B^-, \Delta^-}
$$

$$
((t^{A\to B})u^A : B)^- =
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, ?A^{-\perp} \parr B^-, \Delta^-\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma'^{-\perp}, A^-, \Delta'^-\end{array}}{\vdash ?\Gamma'^{-\perp}, !A^-, \Delta'^-} \qquad \dfrac{}{\vdash B^{-\perp}, B^-}}{\vdash ?\Gamma'^{-\perp}, !A^- \otimes B^{-\perp}, B^-, \Delta'^-}}{\vdash ?\Gamma^{-\perp}, ?\Gamma'^{-\perp}, B^-, \Delta^-, \Delta'^-}}{\vdash ?\Gamma^{-\perp} \cup ?\Gamma'^{-\perp}, B^-, \Delta^- \cup \Delta'^-}
$$

$$
([\alpha^A]t^A : \mathrm{F})^- = \dfrac{\dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, A^-, \Delta^-\end{array}}{\vdash ?\Gamma^{-\perp}, A^-, \Delta^- \setminus \{A^-\}}}{\vdash ?\Gamma^{-\perp}, \perp, A^-, \Delta^- \setminus \{A^-\}}
$$

$$
(\mu\alpha^A.t^{\mathrm{F}} : A)^- = \dfrac{\dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, \perp, \Delta^-\end{array} \qquad \dfrac{}{\vdash 1}}{\vdash ?\Gamma^{-\perp}, \Delta^-}}{\vdash ?\Gamma^{-\perp}, A^-, \Delta^- \setminus \{A^-\}}
$$

$$
(\langle t^A, u^B\rangle : A \wedge B)^- = \dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, A^-, \Delta^-\end{array} \qquad \begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, B^-, \Delta^-\end{array}}{\vdash ?\Gamma^{-\perp}, A^- \,\&\, B^-, \Delta^-}
$$

$$
(\pi_1 t^{A\wedge B} : A)^- = \dfrac{\begin{array}{c}\vdots \\ \vdash ?\Gamma^{-\perp}, A^- \,\&\, B^-, \Delta^-\end{array} \qquad \dfrac{\dfrac{}{\vdash A^{-\perp}, A^-}}{\vdash A^{-\perp} \oplus B^{-\perp}, A^-}}{\vdash ?\Gamma^{-\perp}, A^-, \Delta^-}
$$

$$(\pi_2 t^{A\wedge B} : B)^- = \dfrac{\dfrac{\vdots}{\vdash ?\Gamma^{-\perp}, A^- \,\&\, B^-, \Delta^-} \qquad \dfrac{\dfrac{}{\vdash B^{-\perp}, B^-}}{\vdash A^{-\perp} \oplus B^{-\perp}, B^-}}{\vdash ?\Gamma^{-\perp}, B^-, \Delta^-}$$

$$(\star^{\mathrm{T}} : \mathrm{T})^- = \dfrac{}{\vdash ?\Gamma^{-\perp}, \top, \Delta^-}$$

$$([\alpha^A, \beta^B] t^{A\vee B} : \mathrm{F})^- = \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash ?\Gamma^{-\perp}, A^- \parr B^-, \Delta^-} \qquad \dfrac{\dfrac{}{\vdash A^{-\perp}, A^-} \quad \dfrac{}{\vdash B^{-\perp}, B^-}}{\vdash A^{-\perp} \otimes B^{-\perp}, A^-, B^-}}{\vdash ?\Gamma^{-\perp}, A^-, B^-, \Delta^-}}{\vdash ?\Gamma^{-\perp}, A^-, B^-, \Delta^- \setminus \{A^-, B^-\}}}{\vdash ?\Gamma^{-\perp}, \perp, A^-, B^-, \Delta^- \setminus \{A^-, B^-\}}$$

$$(\mu(\alpha^A, \beta^B).t^{\mathrm{F}} : A \vee B)^- = \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash ?\Gamma^{-\perp}, \perp, \Delta^-} \quad \dfrac{}{\vdash 1}}{\vdash ?\Gamma^{-\perp}, \Delta^-}}{\vdash ?\Gamma^{-\perp}, A^-, B^-, \Delta^- \setminus \{A^-, B^-\}}}{\vdash ?\Gamma^{-\perp}, A^- \parr B^-, \Delta^- \setminus \{A^-, B^-\}}$$

Proposition 18 (Simulation)
The translation $(.)^-$ allows to simulate the reduction of the call-by-name $\lambda\mu$-calculus by the cut-elimination of LLP.

PROOF: This result is proved in [34] for the case of simple types and is easy to extend to T, F, $\wedge$ and $\vee$. □

D Call by value $\lambda\mu$-calculus and LLP

The translation $(.)^+$ of the call-by-value $\lambda\mu$-calculus into LLP is obtained by translating types as positive formulas:

$$\begin{array}{ccc} X & \rightsquigarrow & X^{\perp} \\ \mathrm{T} & \rightsquigarrow & 1 \\ \mathrm{F} & \rightsquigarrow & 0 \\ A \wedge B & \rightsquigarrow & A^+ \otimes B^+ \\ A \vee B & \rightsquigarrow & A^+ \oplus B^+ \\ A \to B & \rightsquigarrow & !(A^+ \multimap ?B^+) \end{array}$$

the judgment $\Gamma \vdash t : A \mid \Delta$ is translated as $\vdash (\Gamma^+)^{\perp}, ?A^+, ?\Delta^+$.

The translation of terms is given in the following way:

$$(x^A : A)^+ = \dfrac{\dfrac{}{\vdash A^{+\perp}, A^+}}{\vdash A^{+\perp}, ?A^+}$$

$$(\lambda x^A.t^B : A \to B)^+ = \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma^{+\perp}, ?B^+, ?\Delta^+\end{matrix}}{\vdash \Gamma^{+\perp} \setminus \{A^{+\perp}\}, A^{+\perp} \parr ?B^+, ?\Delta^+}}{\vdash \Gamma^{+\perp} \setminus \{A^{+\perp}\}, !(A^{+\perp} \parr ?B^+), ?\Delta^+}}{\vdash \Gamma^{+\perp} \setminus \{A^{+\perp}\}, ?!(A^{+\perp} \parr ?B^+), ?\Delta^+}$$

$$((t^{A\to B})u^A : B)^+ = \dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma^{+\perp}, ?!(A^{+\perp} \parr ?B^+), ?\Delta^+\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma'^{+\perp}, ?A^+, ?\Delta'^+\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{+\perp}, A^+} \qquad \dfrac{}{\vdash !B^{+\perp}, ?B^+}}{\vdash A^{+\perp}, A^+ \otimes !B^{+\perp}, ?B^+}}{\vdash A^{+\perp}, ?(A^+ \otimes !B^{+\perp}), ?B^+}}{\vdash !A^{+\perp}, ?(A^+ \otimes !B^{+\perp}), ?B^+}}{\vdash \Gamma'^{+\perp}, ?(A^+ \otimes !B^{+\perp}), ?B^+, ?\Delta'^+}}{\vdash \Gamma'^{+\perp}, !?(A^+ \otimes !B^{+\perp}), ?B^+, ?\Delta'^+}}{\vdash \Gamma^{+\perp}, \Gamma'^{+\perp}, ?B^+, ?\Delta^+, ?\Delta'^+}}{\vdash \Gamma^{+\perp} \cup \Gamma'^{+\perp}, ?B^+, ?\Delta^+ \cup ?\Delta'^+}$$

$$([\alpha^A]t^A : \mathrm{F})^+ = \dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma^{+\perp}, ?A^+, ?\Delta^+\end{matrix}}{\vdash \Gamma^{+\perp}, ?A^+, ?\Delta^+ \setminus \{?A^+\}}}{\vdash \Gamma^{+\perp}, ?0, ?A^+, ?\Delta^+ \setminus \{?A^+\}}$$

$$(\mu\alpha^A.t^{\mathrm{F}} : A)^+ = \dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma^{+\perp}, ?0, ?\Delta^+\end{matrix} \qquad \dfrac{\dfrac{}{\vdash \top}}{\vdash !\top}}{\vdash \Gamma^{+\perp}, ?\Delta^+}}{\vdash \Gamma^{+\perp}, ?A^+, ?\Delta^+ \setminus \{?A^+\}}$$

$$(\langle t^A, u^B\rangle : A \wedge B)^+ = \dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma^{+\perp}, ?A^+, ?\Delta^+\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma'^{+\perp}, ?B^+, ?\Delta'^+\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{+\perp}, A^+} \qquad \dfrac{}{\vdash B^+, B^{+\perp}}}{\vdash A^{+\perp}, A^+ \otimes B^+, B^{+\perp}}}{\vdash A^{+\perp}, ?(A^+ \otimes B^+), B^{+\perp}}}{\vdash A^{+\perp}, ?(A^+ \otimes B^+), !B^{+\perp}}}{\vdash \Gamma'^{+\perp}, A^{+\perp}, ?(A^+ \otimes B^+), ?\Delta'^+}}{\vdash \Gamma'^{+\perp}, !A^{+\perp}, ?(A^+ \otimes B^+), ?\Delta'^+}}{\dfrac{\vdash \Gamma^{+\perp}, \Gamma'^{+\perp}, ?(A^+ \otimes B^+), ?\Delta^+, ?\Delta'^+}{\vdash \Gamma^{+\perp} \cup \Gamma'^{+\perp}, ?(A^+ \otimes B^+), ?\Delta^+ \cup ?\Delta'^+}}$$

$$(\pi_1 t^{A\wedge B} : A)^+ = \dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma^{+\perp}, ?(A^+ \otimes B^+), ?\Delta^+\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{+\perp}, A^+}}{\vdash A^{+\perp}, B^{+\perp}, A^+}}{\vdash A^{+\perp} \parr B^{+\perp}, A^+}}{\vdash A^{+\perp} \parr B^{+\perp}, ?A^+}}{\vdash !(A^{+\perp} \parr B^{+\perp}), ?A^+}}{\vdash \Gamma^{+\perp}, ?A^+, ?\Delta^+}$$

$$(\pi_2 t^{A\wedge B} : B)^+ = \dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma^{+\perp}, ?(A^+ \otimes B^+), ?\Delta^+\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^{+\perp}, B^+}}{\vdash A^{+\perp}, B^{+\perp}, B^+}}{\vdash A^{+\perp} \parr B^{+\perp}, B^+}}{\vdash A^{+\perp} \parr B^{+\perp}, ?B^+}}{\vdash !(A^{+\perp} \parr B^{+\perp}), ?B^+}}{\vdash \Gamma^{+\perp}, ?B^+, ?\Delta^+}$$

$$(\star^{\mathrm{T}} : \mathrm{T})^+ = \dfrac{\dfrac{}{\vdash 1}}{\vdash ?1}$$

$$([\alpha^A, \beta^B] t^{A\vee B} : \mathrm{F})^+ = \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma^{+\perp}, ?(A^+ \oplus B^+), ?\Delta^+\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{+\perp}, A^+}}{\vdash A^{+\perp}, ?A^+}}{\vdash A^{+\perp}, ?A^+, ?B^+} \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{+\perp}, B^+}}{\vdash B^{+\perp}, ?B^+}}{\vdash B^{+\perp}, ?A^+, ?B^+}}{\vdash A^{+\perp} \,\&\, B^{+\perp}, ?A^+, ?B^+}}{\vdash !(A^{+\perp} \,\&\, B^{+\perp}), ?A^+, ?B^+}}{\vdash \Gamma^{+\perp}, ?A^+, ?B^+, ?\Delta^+}}{\dfrac{\vdash \Gamma^{+\perp}, ?A^+, ?B^+, ?\Delta^+ \setminus \{?A^+, ?B^+\}}{\vdash \Gamma^{+\perp}, ?0, ?A^+, ?B^+, ?\Delta^+ \setminus \{?A^+, ?B^+\}}}$$

$(\mu(\alpha^A,\beta^B).t^{\mathrm{F}} : A \vee B)^+ =$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\vdots \quad \dfrac{\dfrac{}{\vdash \top}}{\vdash\, !\top}}{\vdash \Gamma^{+\perp}, ?0, ?\Delta^+ \qquad \vdash\, !\top}
}{\vdash \Gamma^{+\perp}, ?\Delta^+}
}{\vdash \Gamma^{+\perp}, ?A^+, ?B^+, ?\Delta^+ \setminus \{?A^+, ?B^+\}}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^{+\perp}, B^+}}{\vdash B^{+\perp}, A^+ \oplus B^+}}{\vdash B^{+\perp}, ?(A^+ \oplus B^+)}}{\vdash\, !B^{+\perp}, ?(A^+ \oplus B^+)}
}{\vdash \Gamma^{+\perp}, ?A^+, ?(A^+ \oplus B^+), ?\Delta^+ \setminus \{?A^+, ?B^+\}}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{+\perp}, A^+}}{\vdash A^{+\perp}, A^+ \oplus B^+}}{\vdash A^{+\perp}, ?(A^+ \oplus B^+)}}{\vdash\, !A^{+\perp}, ?(A^+ \oplus B^+)}
}{
\dfrac{\vdash \Gamma^{+\perp}, ?(A^+ \oplus B^+), ?(A^+ \oplus B^+), ?\Delta^+ \setminus \{?A^+, ?B^+\}}{\vdash \Gamma^{+\perp}, ?(A^+ \oplus B^+), ?\Delta^+ \setminus \{?A^+, ?B^+\}}
}
$$

Proposition 19 (Simulation)
The translation $(.)^+$ allows to simulate the reduction of the call-by-value $\lambda\mu$-calculus by the cut-elimination of LLP.

Proof: We can extend the proof given in [14] to all the connectives or use Selinger's syntactical duality [23]: if $t \to u$ in call-by-value, let $\tilde{t}$ and $\tilde{u}$ be their call-by-name dual terms, we have $\tilde{t} \to \tilde{u}$ in call-by-name thus $(\tilde{t})^- \to^* (\tilde{u})^-$ by proposition 18, and finally $t^+ \to^* u^+$ since $(\tilde{t})^- = t^+$. □

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