Is $\neg\neg A$ equal to A?

J.-Y. Girard's 60th birthday – Paris

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Understanding the question

- In which logic?
  - classical logic
  - intuitionistic logic
  - minimal logic
  - linear logic
  - constructive classical logic
- With which negation?
  - answer type is **F** or not ($\neg A = A \to \mathbf{F}$ or $A \to R$)
  - linear / non linear
- For which equality?
  - equiprovability (A provable $\iff$ $\neg\neg A$ provable)
  - equivalence ($A \leftrightarrow \neg\neg A$ provable)
  - isomorphism ($A \simeq \neg\neg A$)

Related questions

- If $\neg\neg A \neq A$, what remains?
- What about:

$$\neg(A \vee B) = \neg A \wedge \neg B$$
$$\neg(A \wedge B) = \neg A \vee \neg B$$

- What about:

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$
$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

- What about:

$$\neg(\forall x A) = \exists x \neg A$$
$$\neg(\exists x A) = \forall x \neg A$$

Syntax vs. Semantics

We consider:

- a (syntactically given) logic $\mathcal{L}$
  with derivable formulas: $\vdash_{\mathcal{L}} A$
- a corresponding (semantic) notion of $\mathcal{L}$-model
  with valid formulas: $\mathcal{M} \models A$

We look for:

- Soundness
  $\vdash_{\mathcal{L}} A \quad \Longrightarrow \quad \forall \mathcal{M},\ \mathcal{M} \models A$
- Completeness
  $\forall \mathcal{M},\ \mathcal{M} \models A \quad \Longrightarrow \quad \vdash_{\mathcal{L}} A$

Classical logic (1)

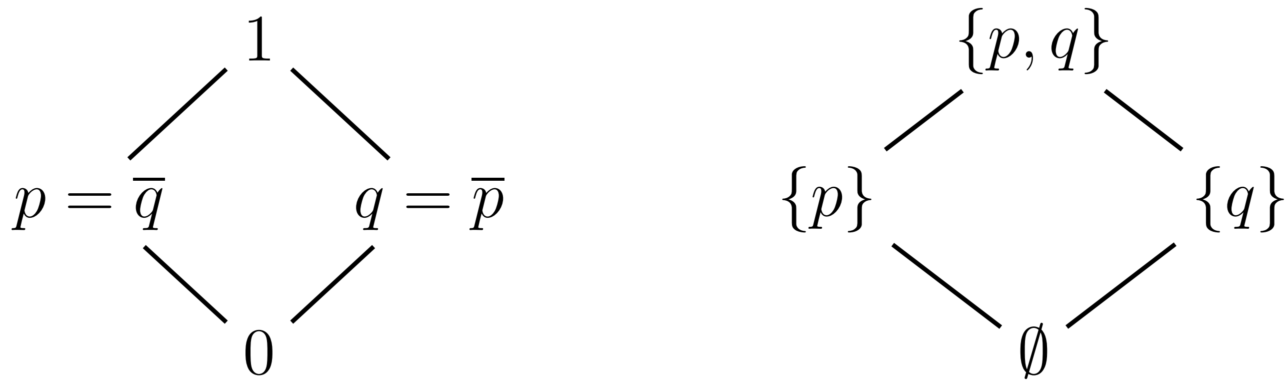
- Syntactically
  $\neg\neg A \leftrightarrow A$ is derivable
- Semantically
  - Truth tables

$[\![A]\!]$	$[\![\neg A]\!]$	$[\![\neg\neg A]\!]$
0	1	0
1	0	1

$[\![A]\!] = 1$ if and only if $[\![\neg\neg A]\!] = 1$

Classical logic (2)

- Semantically
  - Boolean algebras *complemented distributive lattice*
    partial order with finite infs and sups ($\vee$, $\wedge$, 0, 1)
    and complement $\overline{x}$ such that $x \wedge \overline{x} = 0$ and $x \vee \overline{x} = 1$
    typical example: $\mathcal{P}(E)$



    unicity of $\overline{x}$ thus $\overline{\overline{[\![A]\!]}} = [\![A]\!]$
  - Boolean rings *ring with* $x^2 = x$
    (equivalent to Boolean algebras)
    $\overline{x} = 1 - x$ and $1 - (1 - [\![A]\!]) = [\![A]\!]$

Intuitionistic logic (1)

- Syntactically

$$\vdash A \rightarrow \neg\neg A$$

$$\nvdash \neg\neg A \rightarrow A$$

- Semantically
  - Topological semantics / Heyting algebras
    open sets of a topological space $\mathcal{S}$

    $[\![A \rightarrow B]\!] = ([\![B]\!] \cup \mathcal{S} \setminus [\![A]\!])^\circ$
    $[\![\mathbf{F}]\!] = \emptyset$
    $[\![\neg A]\!] = (\mathcal{S} \setminus [\![A]\!])^\circ$

    $[\![A]\!]$ is open thus $[\![A]\!] \subseteq \overset{\circ}{\overline{[\![A]\!]}}$ but in general $\overset{\circ}{\overline{[\![A]\!]}} \not\subseteq [\![A]\!]$

Intuitionistic logic (2)

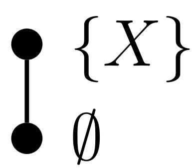
- Semantically
  - Kripke models
    partial order with a morphism $\mathcal{I}$ *into* $(\mathcal{P}(\texttt{Var}), \subseteq)$

$$
\begin{array}{lll}
x \Vdash X & \text{if} & X \in \mathcal{I}(x) \\
x \Vdash A \to B & \text{if} & \forall y \geq x,\ y \Vdash A \Longrightarrow y \Vdash B \\
x \nVdash \mathbf{F} & & \\
x \Vdash \neg A & \text{if} & \forall y \geq x,\ y \nVdash A
\end{array}
$$

Lemma: $x \Vdash A$ and $y \geq x \Longrightarrow y \Vdash A$

$\mathcal{K} \models A \quad \Longleftrightarrow \quad \forall x,\ x \Vdash A$

$$\mathcal{K} \models \neg\neg A \quad \Longleftrightarrow \quad \forall x,\ \exists y \geq x,\ y \Vdash A \quad \Longleftarrow \quad \mathcal{K} \models A$$

Counter model: 

$\not\models \neg\neg X \to X$

Minimal logic

and parametrized negation

$$\neg_R A \equiv A \rightarrow R \qquad \text{(particular case } \neg A \equiv \neg_{\mathbf{F}} A)$$

Intuitionistic logic: $\vdash_I \mathbf{F} \rightarrow A$ and $\nvdash_I R \rightarrow A$

Minimal logic: $\nvdash_M \mathbf{F} \rightarrow A$ and $\nvdash_M R \rightarrow A$

- Syntactically

$$\vdash A \rightarrow \neg_R \neg_R A$$

$$\nvdash \neg_R \neg_R A \rightarrow A$$

- Semantically
  results given for the particular case $\mathbf{F}$ hold in general
  (*ex.*: Kripke models with arbitrary interpretation for R)

Linear logic

- Syntactically

$$A \multimap\!\circ\, A^{\perp\perp}$$

- Semantically   Phase spaces

commutative monoid with a distinguished subset $\perp$

$$[\![A \multimap B]\!] = \{m \mid \forall n \in [\![A]\!],\; mn \in [\![B]\!]\}$$
$$[\![\mathbf{F}]\!] = \perp$$
$$[\![A^{\perp}]\!] = \{m \mid \forall n \in [\![A]\!],\; mn \in \perp\} = [\![A]\!]^{\perp}$$

$$[\![A^{\perp\perp}]\!] = [\![A]\!]^{\perp\perp} \triangleq [\![A]\!]$$

Core proof system

- Sequents $\Gamma \vdash \Delta$

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

$$(n = 0 \quad \Gamma = \mathbf{T}) \qquad \bigwedge_{1 \le i \le n} A_i \to \bigvee_{1 \le j \le m} B_j \qquad (m = 0 \quad \Delta = \mathbf{F})$$

- Rules

$$\frac{}{A \vdash A}\ \mathit{ax} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathit{cut}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg R/\neg\text{intro}$$

$$\frac{\Gamma \vdash \neg A, \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \mathbf{F}, \Delta, \Delta'}\ \neg\text{elim}$$

Variations

- Structural rules

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

- Restrictions
  - Classical logic: all structural rules
  - Intuitionnistic logic: at most one formula on the right (no contraction)
  - Minimal logic: exactly one formula on the right (no weakening either)
  - Linear logic: no structural rules

A and $\neg\neg A$

- $A \rightarrow \neg\neg A$ and $\neg\neg A \rightarrow A$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, \neg A \vdash}\,\neg L}{A \vdash \neg\neg A}\,\neg R \qquad\qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A, \neg A}\,\neg R}{\neg\neg A \vdash A}\,\neg L$$

$A = \neg\neg A$ really means perfect symmetry between left and right

- $\neg\neg A \rightarrow \neg\neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, \neg A \vdash}\,\neg L}{\neg A \vdash \neg A}\,\neg R}{\neg A, \neg\neg A \vdash}\,\neg L}{\neg\neg A \vdash \neg\neg A}\,\neg R \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A, \neg A}\,\neg R}{\neg\neg A \vdash A}\,\neg L}{\neg A, \neg\neg A \vdash}\,\neg L}{\neg\neg A \vdash \neg\neg A}\,\neg R$$

Expressiveness of Linear Logic

- Structural rules under the control of connectives

$$\frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}$$

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

- Translations
  - Minimal Logic
    $(A \to B)^\star = !A^\star \multimap B^\star = ?A^{\star\perp} \parr B^\star$
  - Intuitionistic Logic
    $\mathbf{F}^\star = 0$
  - Classical Logic
    $(A \to B)^\star = ?(!A^\star \multimap B^\star) = ?(?A^{\star\perp} \parr B^\star)$
    $\mathbf{F}^\star = ?0$

Equiprovability

equivalence $\Longrightarrow$ equiprovability

$$\dfrac{\dfrac{\vdots}{\vdash A} \qquad A \vdash \neg\neg A}{\vdash \neg\neg A}\ \mathit{cut} \qquad\qquad \dfrac{\dfrac{\vdots}{\vdash \neg\neg A} \qquad \neg\neg A \vdash A}{\vdash A}\ \mathit{cut}$$

- Classical logic
  immediate from equivalence
- Intuitionnistic logic
  If $\vdash_C A$ then $\vdash_I \neg\neg A$ (for A propositionnal) [GLIVENKO]
- Linear logic
  for $(.)^\perp$: immediate from equivalence
  $\vdash 1$ and $\vdash ?!1$ but $?!1 \not\multimap 1$
  $\vdash A \Longrightarrow \vdash ?!A$ but $\nvdash ?X^\perp \oplus X$ and $\vdash ?!(?X^\perp \oplus X)$

Proof transformations

- Cut elimination

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array}}{\Gamma \vdash \neg A, \Delta}\neg R \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma', \neg A \vdash \Delta'}\neg L}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathit{cut}$$

- Axiom expansion

$$\dfrac{}{\neg A \vdash \neg A}\mathit{ax} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\mathit{ax}}{A, \neg A \vdash}\neg L}{\neg A \vdash \neg A}\neg R$$

Examples

- Cut elimination

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, \neg A \vdash}\,\neg L}{A \vdash \neg\neg A}\,\neg R \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A, \neg A}\,\neg R}{\neg\neg A \vdash A}\,\neg L}{A \vdash A}\,cut
$$

Examples

- Cut elimination

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, \neg A \vdash}\,\neg L \qquad \dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A, \neg A}\,\neg R}{A \vdash A}\,cut$$

Examples

- Cut elimination

$$\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{A \vdash A}\,ax}{A \vdash A}\,cut$$

Examples

- Cut elimination

$$\frac{}{A \vdash A}\, ax$$

Examples

- Cut elimination

$$\frac{}{A \vdash A}\,ax$$

- Axiom expansion

$$\frac{}{\neg\neg A \vdash \neg\neg A}\,ax$$

Examples

- Cut elimination

$$\frac{}{A \vdash A}\,ax$$

- Axiom expansion

$$\cfrac{\cfrac{\cfrac{}{\neg A \vdash \neg A}\,ax}{\neg A, \neg\neg A \vdash}\,\neg L}{\neg\neg A \vdash \neg\neg A}\,\neg R$$

Examples

- Cut elimination

$$\frac{}{A \vdash A}\,\textit{ax}$$

- Axiom expansion

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\textit{ax}}{A, \neg A \vdash}\,\neg L}{\neg A \vdash \neg A}\,\neg R}{\neg A, \neg\neg A \vdash}\,\neg L}{\neg\neg A \vdash \neg\neg A}\,\neg R$$

Proofs as morphisms

A proof of $A \vdash B$ is a morphism from A to B.

- the *ax*-rule is the identity morphism
- the *cut*-rule is the composition of morphisms

Plus equalities on proofs:

- none: no isomorphisms
- cut elimination: trivial isomorphisms [BÖHM–DEZANI]
- cut elimination and axiom expansion: *now*
- maximal: back to equivalence, *i.e.* degenerated
  (at most one morphism in $[A, B] \iff$ pre-order)

Syntax vs. Denotational Semantics

- From soundness

  if $A \vdash B$ then $[A, B]$ is not empty

  to faithfulness

  two different proofs have different interpretations

- From completeness

  if $[A, B]$ is not empty then $A \vdash B$

  to full completeness

  any element of $[A, B]$ is the interpretation of a proof

Minimal logic

Cartesian Closed Categories

$$A_1, \ldots, A_n \vdash B \quad \rightsquigarrow \quad A_1 \times \cdots \times A_n \to B$$

- Constructors $\quad \mathtt{I} \quad A \times B \quad B^A$
- Product

$$\left.\begin{array}{l} A \xrightarrow{f} C \\ B \xrightarrow{g} D \end{array}\right\} A \times B \xrightarrow{f \times g} C \times D$$

$$A \times \mathtt{I} \simeq A \qquad A \xrightarrow{\Delta} A \times A \qquad A \xrightarrow{t} \mathtt{I}$$

- Curryfication

$$[A \times B, C] \simeq [A, C^B]$$

Example: sets and functions *thus non-degenerated*

Intuitionistic logic

Cartesian Closed Categories with initial object

$$A_1, \ldots, A_n \vdash B \quad \rightsquigarrow \quad A_1 \times \cdots \times A_n \to B$$
$$A_1, \ldots, A_n \vdash \quad \rightsquigarrow \quad A_1 \times \cdots \times A_n \to 0$$

- Additional constructors 0
- Initial object

$$0 \xrightarrow{i} A$$

Example: sets and functions ($0 = \emptyset$) *thus non-degenerated*

Classical logic

Cartesian Closed Categories with involution

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m \quad \rightsquigarrow \quad A_1 \times \cdots \times A_n \times 0^{B_1} \times \cdots \times 0^{B_m} \to 0$$

- Involution

$$0^{0^A} \simeq A$$

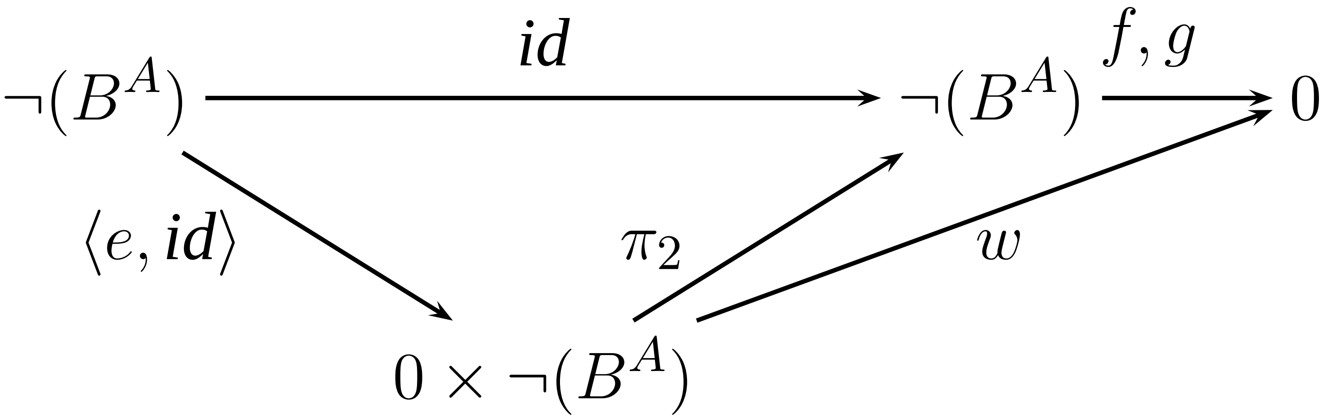
Degenerated !!! back to Boolean algebras

Proof sketch

$[A, B] \simeq [\mathtt{I} \times A, B] \simeq [\mathtt{I}, B^A] \simeq [\neg(B^A), 0]$

$\sharp[0 \times C, D] = \sharp[0, D^C] = 1$

If $e, f, g \in [\neg(B^A), 0]$**:**



thus $f = g$ **or** $[A, B] = \emptyset$

$$\Longrightarrow \quad \sharp[A, B] \leq 1 \qquad \square$$

Linear logic

$\star$-autonomous categories with products, co-monad, etc…

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m \quad \rightsquigarrow \quad A_1 \otimes \cdots \otimes A_n \to B_1 ⅋ \cdots ⅋ B_m$$

- Constructors $\quad 1 \quad \bot \quad A \otimes B \quad A \multimap B \quad !A$
- Tensor product, curryfication and involution

$$\left.\begin{array}{l} A \xrightarrow{f} C \\ B \xrightarrow{g} D \end{array}\right\} A \otimes B \xrightarrow{f \otimes g} C \otimes D$$

$$A \otimes 1 \simeq A \qquad [A \otimes B, C] \simeq [A, B \multimap C] \qquad (A \multimap \bot) \multimap \bot \simeq A$$

- Co-monad…

$$!A \xrightarrow{c} !A \otimes !A \qquad !A \xrightarrow{w} 1 \qquad !A \xrightarrow{d} A \quad \ldots$$

Example: sets and relations *thus non-degenerated*

Curry-Howard isomorphism

- The computational meaning of proof theory

Logic		Programming
formula	$\leftrightsquigarrow$	type
rule	$\leftrightsquigarrow$	instruction
proof of A	$\leftrightsquigarrow$	program of type A
cut elimination	$\leftrightsquigarrow$	computation
cut-free proof	$\leftrightsquigarrow$	result

- Example: booleans

$$\dfrac{\dfrac{}{X \vdash X}\,ax}{X, X \vdash X} \quad \leftrightsquigarrow \quad \texttt{tt} : \mathrm{BOOL} \qquad \texttt{ff} : \mathrm{BOOL}$$

Computational meaning $\Longrightarrow$ non-trivial equality of proofs

Any hope for classical logic?

- Not a Cartesian Closed Category
  - [FÜHRMANN–PYM]
  - [LAMARCHE–STRASSBURGER]

- Breaking some symmetry
  - Two dual classes of formulas and constrained rules (in an asymmetric way)
  - Negation is not an involution

Extending Minimal Logic inside LL

$$(A \to B)^\star = ?A^{\star\perp} \parr B^\star$$

- Structural rules valid for $?$-formulas:

$$\perp \multimap ?A \qquad\qquad ?A \parr ?A \multimap ?A$$

- Required for [the translation of] all formulas
- Preserved by $\parr$, valid for $\perp$: negative formulas (reversibility)

$$N \quad ::= \quad \perp \quad | \quad N \parr N \quad | \quad ?A \quad | \quad \top \quad | \quad N \,\&\, N \quad | \quad \forall \alpha N$$

- Dual notion: positive formulas (focalization [ANDREOLI])
- Two negations:
  - $(.)^\perp$ exchange positive and negative: involutive
  - $N \to \mathbf{F} \;\simeq\; ?N^\perp$: not involutive $\qquad ?!N \not\simeq N$
    (using $\mathbf{F}^\star = \perp$)

Polarized Linear Logic and LC

- Polarized formulas are expressive enough for minimal logic
- Restrict LL to polarized formulas: LL_{pol}

$$
\begin{array}{rcccccccccccc}
N & ::= & \bot & | & N \mathbin{⅋} N & | & \top & | & N \mathbin{\&} N & | & \forall\alpha N & | & ?P \\
P & ::= & 1 & | & P \otimes P & | & 0 & | & P \oplus P & | & \exists\alpha P & | & !N
\end{array}
$$

- Alternative syntax without explicit ? and !: LC

$$\vdash ?P, ?Q, ?M, ?N, P', Q', M', N'$$

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$$\vdash ?P, ?Q, ?M, ?N, M', N'; P', Q'$$

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$$\vdash ?P, ?Q, ?M, ?N, M', N'; P'$$

lemma: *at most one* positive formula in $\vdash_{\mathsf{LL}_{\text{pol}}} \Gamma$

Polarized Linear Logic and LC

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\end{array}
$$

- Alternative syntax without explicit ? and !: LC

$$\vdash \underline{P}, \underline{Q}, \underline{M}, \underline{N}, \underline{M'}, \underline{N'}; \underline{P'}$$

lemma: *at most one* positive formula in $\vdash_{\mathsf{LL}_{\text{pol}}} \Gamma$

LC sequents: $\vdash \Gamma; \Pi$ (with polarities on formulas)

Constructive Classical Logic

so many systems

$\lambda\mathcal{C}$-calculus [Felleisen]

LC [Girard]

$\lambda\mu$-calculus [Parigot, Ong–Stewart]

λc-calculus [Krivine]

LKT/LKQ [Danos–Joinet–Schellinx]

Call-by-Push-Value [Levy]

$\bar{\lambda}\mu\tilde{\mu}$-calculus [Curien–Herbelin]

dual calculus [Wadler]

...

Denotational semantics

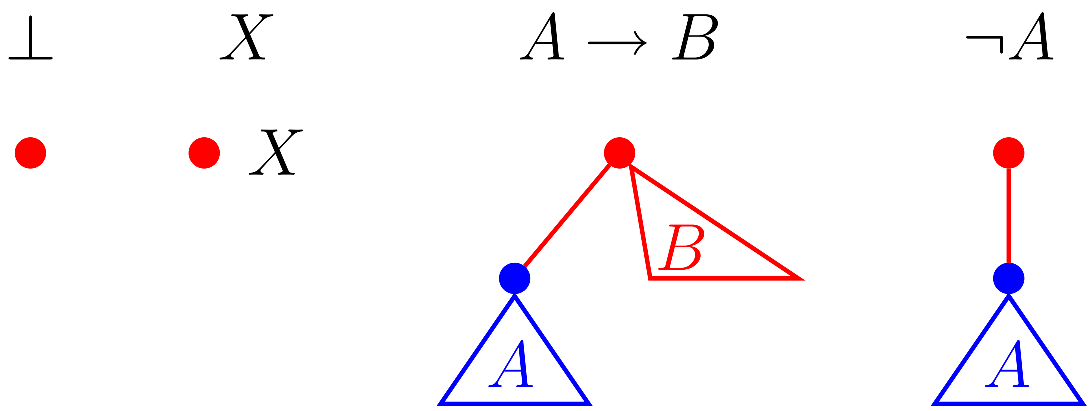
- Correlation spaces
  - Coherent spaces (*the* model of linear logic)
    $A = (|A|, \coh_A)$: a set and a reflexive relation
  - $\parr$-monoids: $N \parr N \multimap N$ and $\bot \multimap N$
    $\otimes$-comonoids: $P \multimap P \otimes P$ and $P \multimap 1$
- Control categories [SELINGER]
  - Additional constructors $\bot$ $A \parr B$
  - Pre-tensor
    $$\left.\begin{array}{l} A \xrightarrow{f} C \\ B \xrightarrow{g} D \end{array}\right\} A \parr B \xrightarrow{f \parr g} C \parr D$$
    $$A \parr \bot \simeq A \qquad\qquad B^A \simeq B \parr \bot^A$$
  - ...
- $\bot$ is not initial, $\bot^{\bot^A} \not\simeq A$ but $A \lhd \bot^{\bot^A}$

Towards completeness

- The completeness conjecture of correlation domains
  - Introduce a notion of totality
  - Failure: [QUATRINI]
- Game semantics
  - Closer to the syntax (cf. *syntax vs. semantics*)
  - Focus on interaction
  - Full completeness for constructive classical logic
  - Faithfulness for constructive classical logic
    (*e.g.* proof-nets)

Game semantics

- From logic to games
  - polarization $\Rightarrow$ clusters of connectives
  - successive moves of two players: Player and Opponent
- Moves: “sub-formulas” (tree structure)
- Game constructions

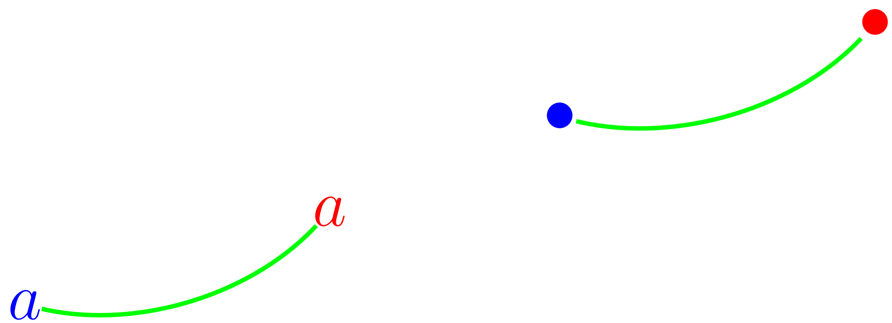


- Strategies: a proof is a winning strategy for Player
  such that. . .
- Dynamics:

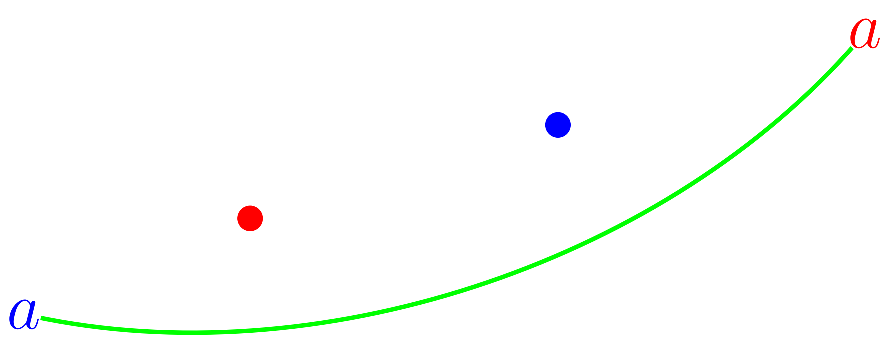
  cut elimination by composition/interaction between strategies

Let's play (1)

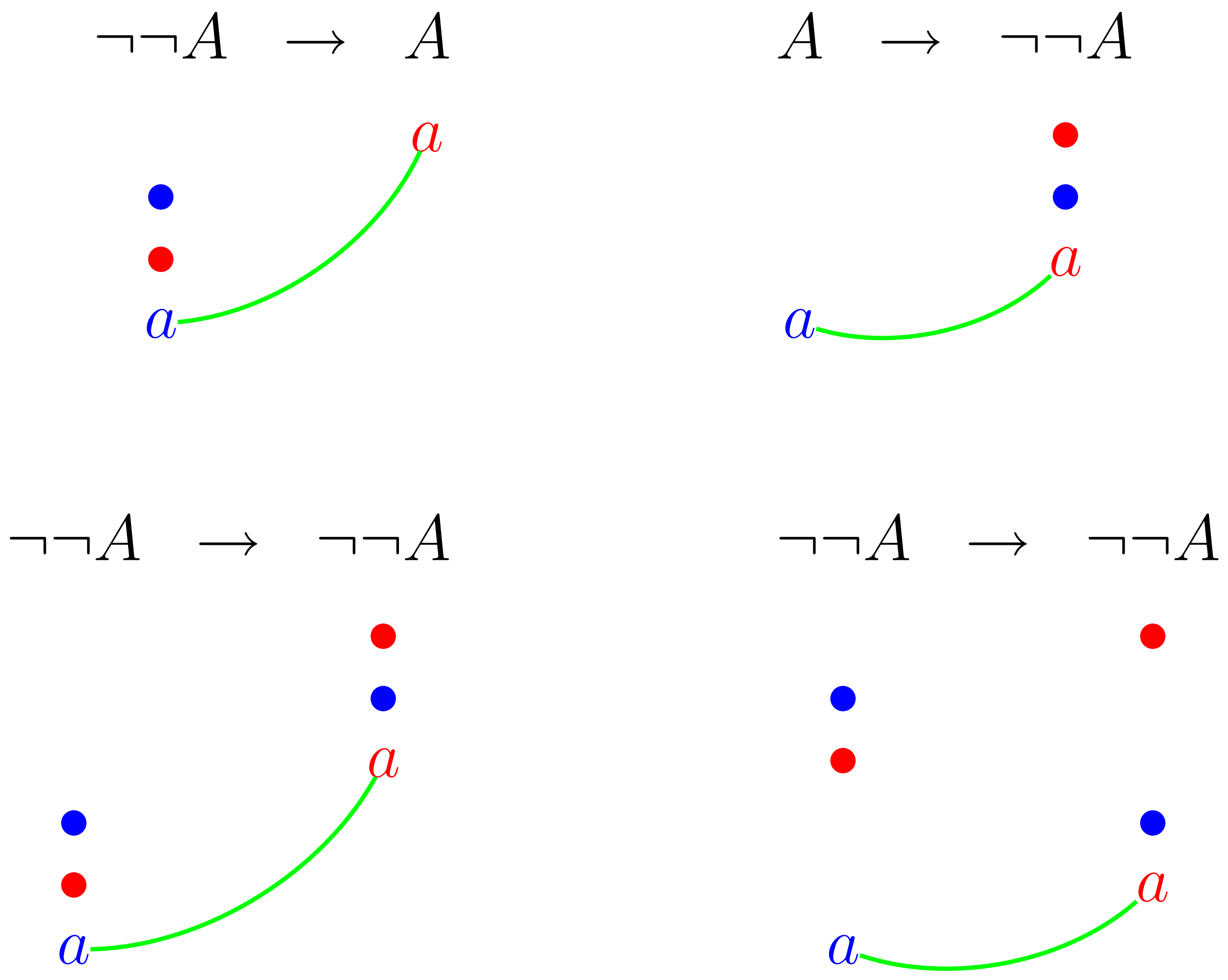
$$A \rightarrow (A \rightarrow X) \rightarrow X$$



$$((A \rightarrow \bot) \rightarrow \bot) \rightarrow A$$



Let's play (2)



$\neg\neg A \;\rightarrow\; A$
a
a
$A \;\rightarrow\; \neg\neg A$
a
a
$\neg\neg A \;\rightarrow\; \neg\neg A$
a
a
$\neg\neg A \;\rightarrow\; \neg\neg A$
a
a

Type isomorphisms

- Equational characterization
  - Cartesian Closed Categories

    [Soloviev, Bruce–Longo, Di Cosmo, . . .]

$$(A \times B) \to C \simeq A \to (B \to C)$$

$$A \to (B \times C) \simeq (A \to B) \times (A \to C)$$

$$\cdots$$

  - Control categories

$$A \parr (B \times C) \simeq (A \parr B) \times (A \parr C)$$

$$A \to B \simeq \neg A \parr B$$

$$\cdots$$

- Geometric characterization (using game semantics)

  $A \simeq B \iff$ the corresponding trees are isomorphic

Second order quantification

- Church style [Di Cosmo, De Lataillade]

$$\forall X (A \to B) \simeq A \to \forall X B \qquad X \notin A$$

$$\forall X (A \times B) \simeq \forall X A \times \forall X B$$

$$\cdots$$

- Curry style [De Lataillade]

$$\forall X A \simeq A[{}^{\forall X X}/_{X}] \qquad X \in^{+} A$$