Polarized Proof-Nets: Proof-Nets for LC
(Extended Abstract)

Olivier Laurent

Institut de Mathématiques de Luminy
CNRS-Marseille, France
olaurent@iml.univ-mrs.fr

Abstract. We define a notion of polarization in linear logic (**LL**) coming from the polarities of Jean-Yves Girard's classical sequent calculus **LC** [4]. This allows us to define a translation between the two systems. Then we study the application of this polarization constraint to proof-nets for full linear logic described in [7]. This yields an important simplification of the correctness criterion for polarized proof-nets. In this way we obtain a system of proof-nets for **LC**.

The study of cut-elimination takes an important place in proof-theory. Much work is spent to deal with commutation of rules for cut-elimination in sequent calculi. The introduction of proof-nets (see [7] for instance) solves commutation problems and allows us to define a clear notion of reduction and complexity.

In [4], Jean-Yves Girard defines the sequent calculus **LC** using polarities. **LC** is a refinement of **LK** with a deterministic cut-elimination. J.-Y. Girard leaves open the following problem about the syntax:

*"Find a better syntax (which would be to **LC** what typed λ-calculus is to **LJ**) for normalization [...]. A kind of proof-nets could be the solution, and the fact that proof-nets are not available for full linear logic could be compensated by the fact that only certain linear configurations are used."*

In this paper we address this problem but the situation is now slightly different since proof-nets for full linear logic are given in [7]. In these proof-nets, the boxes for additives are replaced by weights on the nodes giving less sequentialization information. To use these proof-nets, we will first define a translation from **LC** to the fragment **LLP** of **LL** defined by restricting to polarized formulas. The "particular linear configurations" of **LC** correspond to the polarization of **LLP**.

We then turn to the study of proof-structures for **LLP** and show that the restriction to polarized formulas induces a natural orientation, the *orientation of polarization*, which is respected by the paths of **LL**'s correctness condition (Orientation Lemma). This yields a striking simplification of the correctness condition which allows us to get rid of the notion of switches. In particular it turns out to be cubic in the size of polarized proof-nets whereas the **LL** condition is immediately seen to be exponential.

1 Classical Logic: LC

Gentzen's classical sequent calculus **LK** has well known problems, such as the lack of a denotational semantics and the non determinism of cut-elimination. J.-Y. Girard proposed in [4] the calculus **LC** as a refinement of **LK** to solve these defects. The key point is the introduction of polarities for formulas. Let us just remind the syntax.

1.1 Formulas and Polarity

The formulas of **LC** are built from the atomic formulas and the constants V and F by using the connectives $\wedge$, $\vee$, $\neg$, $\exists$ and $\forall$. For each formula, we define its *polarity*: atomic formulas, V and F are positive; as for the compound formulas we use the following table:

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$\exists x A$	$\forall x A$
$+$	$+$	$+$	$+$	$-$	$+$	$-$
$-$	$+$	$+$	$-$	$+$	$+$	$-$
$+$	$-$	$+$	$-$			
$-$	$-$	$-$	$-$			

In the sequel, P and Q will stand for positive formulas and N and M for negative ones.

1.2 Rules of the Sequent Calculus LC

To limit the number of rules, we will use one-sided sequents. The formulas will be defined modulo the De Morgan's laws. The sequents for **LC** are written $\vdash \Gamma; \Pi$ where Γ (the *body*) is a multi-set of formulas and Π (the *stoup*) is either empty or a unique positive formula.

Then the sequent calculus is defined by the following rules:

$$\frac{}{\vdash \neg P; P} \qquad \frac{\vdash \Gamma; P \quad \vdash \neg P, \Delta; \Pi}{\vdash \Gamma, \Delta; \Pi} \qquad \frac{\vdash \Gamma, N; \quad \vdash \neg N, \Delta; \Pi}{\vdash \Gamma, \Delta; \Pi}$$

$$\frac{\vdash \Gamma; P}{\vdash \Gamma, P;} \qquad \frac{\vdash \Gamma; \Pi}{\vdash \Gamma, A; \Pi} \qquad \frac{\vdash \Gamma, A, A; \Pi}{\vdash \Gamma, A; \Pi}$$

$$\frac{}{\vdash ; V} \qquad \frac{}{\vdash \Gamma, \neg F; \Pi}$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \wedge Q} \qquad \frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma, \Delta; P \wedge N} \qquad \frac{\vdash \Gamma, M; \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; M \wedge Q}$$

$$\frac{\vdash \Gamma, M; \Pi \quad \vdash \Gamma, N; \Pi}{\vdash \Gamma, M \wedge N; \Pi}$$

$$\frac{\vdash \Gamma, A, B; \Pi}{\vdash \Gamma, A \vee B; \Pi} A \vee B \textit{ negative} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma; P \vee Q} \qquad \frac{\vdash \Gamma; Q}{\vdash \Gamma; P \vee Q}$$

$$\frac{\vdash \Gamma, A; \Pi}{\vdash \Gamma, \forall x A; \Pi} x \notin \Gamma, \Pi \qquad \frac{\vdash \Gamma, N[{}^t/_x];}{\vdash \Gamma; \exists x N} \qquad \frac{\vdash \Gamma; P[{}^t/_x]}{\vdash \Gamma; \exists x P}$$

2 Linear Logic with Polarities

We can give a translation from **LC** to **LL** using the definition of the denotational semantics described in [4]. More precisely, we will define a polarized fragment of **LL** and we will show in which way it corresponds to **LC**. We start with the definition of two polarized fragments of **LL**.

The first notion of polarization for **LL** splits the connectives into reversible and non reversible ones.

Definition 1 (Polarized formula). *We define in the same time the* positive *(denoted by P, Q) and* negative *(denoted by M, N) formulas, starting from a set of atoms (denoted by A, B):*

$$\begin{array}{l} P ::= \;!A \;\mid P \otimes P \mid P \oplus P \mid \exists x P \mid 1 \mid 0 \mid !N \\ N ::= ?A^{\perp} \mid N \parr N \mid N \,\&\, N \mid \forall x N \mid \perp \mid \top \mid ?P \end{array}$$

A polarized formula *is either a positive one or a negative one.*

The second notion of polarization is more precise and corresponds to **LC**'s polarities. It will be used for studying translations between **LC** and **LL**.

Definition 2 (Strictly polarized formula). *We define in the same time the* strictly positive *(denoted by $\mathcal{P}$, $\mathcal{Q}$) and* strictly negative *(denoted by $\mathcal{M}$, $\mathcal{N}$) formulas, starting from a set of atoms (denoted by A, B):*

$$\begin{array}{l} \mathcal{P} ::= \;!A \;\mid \mathcal{P} \otimes \mathcal{P} \mid \mathcal{P} \otimes !\mathcal{N} \mid !\mathcal{N} \otimes \mathcal{P} \mid \mathcal{P} \oplus \mathcal{P} \mid \exists x \mathcal{P} \mid \exists x !\mathcal{N} \mid 1 \mid 0 \\ \mathcal{N} ::= ?A^{\perp} \mid \mathcal{N} \parr \mathcal{N} \mid \mathcal{N} \parr ?\mathcal{P} \mid ?\mathcal{P} \parr \mathcal{N} \mid \mathcal{N} \,\&\, \mathcal{N} \mid \forall x \mathcal{N} \mid \forall x ?\mathcal{P} \mid \perp \mid \top \end{array}$$

A strictly polarized formula *is $\mathcal{P}$, $\mathcal{N}$, $?\mathcal{P}$ or $!\mathcal{N}$.*

Definition 3 (LLP and $\mathbf{LLP_C}$). *The fragment **LLP** (resp. $\mathbf{LLP_C}$) of **LL** is obtained by restricting to polarized (resp. strictly polarized) formulas and by adding the constraint that the $\top$-rule must introduce at most one positive formula.*

$\mathbf{LLP_C}$ is a fragment of **LLP** (strictly polarized formulas are polarized) so all the results we will prove on **LLP** (about proof-nets,...) will be also true for $\mathbf{LLP_C}$.

The constraint on the $\top$-rule is needed in particular for the next proposition.

Proposition 1. *If $\vdash \Gamma$ is provable in **LLP** then Γ has at most one positive formula.*

3 Translations between LC and $\mathbf{LLP_C}$

We now prove the similarity of the two systems by defining two translations between **LC** and **LL**. More precisely these translations show that **LC** and $\mathbf{LLP_C}$ are almost isomorphic.

Definition 4. ***LC***^{rev} *is the fragment of* ***LC*** *which refuses:*

- *structural rules on negative non atomic formulas;*
- *negative non atomic formulas in the context of the negative premise of:*
  - *negative cut-rule,*
  - *$\otimes$-rule between a negative and a positive formula,*
  - *$\exists$-rule on a negative formula.*

Every proof of **LC** can be transformed into a proof of **LC**$^{\text{rev}}$ by commuting some reversible rules with structural ones so we have no loss of provability in **LC**$^{\text{rev}}$. A study of these commutations of reversible rules has been done in a similar case by M. Quatrini and L. Tortora de Falco in [9] for translation of $\mathbf{LK}^{\eta,\rho}_{\text{pol}}$ into **LL**.

3.1 $\mathbf{LC^{rev}} \to \mathbf{LLP_C}$

Definition 5. *The translation $G \mapsto G^\bullet$ from* ***LC***^{rev} *into* ***LLP***$_C$ *is defined on formulas by:*

$$\begin{array}{rcl@{\qquad}rcl}
A^\bullet &=& !A & (\neg P)^\bullet &=& P^{\bullet\perp} \\
V^\bullet &=& 1 & F^\bullet &=& 0 \\
(P \wedge Q)^\bullet &=& P^\bullet \otimes Q^\bullet & (N \wedge M)^\bullet &=& N^\bullet \,\&\, M^\bullet \\
(P \wedge N)^\bullet &=& P^\bullet \otimes !N^\bullet & (N \wedge P)^\bullet &=& !N^\bullet \otimes P^\bullet \\
(\exists x P)^\bullet &=& \exists x P^\bullet & (\exists x N)^\bullet &=& \exists x !N^\bullet
\end{array}$$

Given a sequent of ***LC****, we can split the body into two parts: positive formulas and negative formulas, $\vdash \Gamma; \Pi = \vdash \Gamma^-, \Gamma^+; \Pi$. Then we can define the translation on sequents: $(\vdash \Gamma^-, \Gamma^+; \Pi)^\bullet = \vdash \Gamma^{-\bullet}, ?(\Gamma^+)^\bullet, \Pi^\bullet$.*

The translation of proofs is defined rule by rule by introducing promotion rules on the negative premise before negative cut, before $\wedge$ between a positive and a negative formula and before $\exists$ for a negative formula. For example here is the case of the negative cut:

$$\frac{\vdash \Gamma^-, \Gamma^+, N; \quad \vdash \neg N, \Delta^-, \Delta^+; \Pi}{\vdash \Gamma^-, \Delta^-, \Gamma^+, \Delta^+; \Pi}$$

$$\downarrow$$

$$\frac{\dfrac{\vdash \Gamma^{-\bullet}, ?(\Gamma^+)^\bullet, N^\bullet}{\vdash \Gamma^{-\bullet}, ?(\Gamma^+)^\bullet, !N^\bullet} \quad \vdash ?(\neg N)^\bullet, \Delta^{-\bullet}, ?(\Delta^+)^\bullet, \Pi^\bullet}{\vdash \Gamma^{-\bullet}, \Delta^{-\bullet}, ?(\Gamma^+)^\bullet, ?(\Delta^+)^\bullet, \Pi^\bullet}$$

Remark 1. An empty stoup corresponds to a $?\Gamma$ context in **LL**, i.e. to a correct context for promotion.

LC accepts structural rules on non atomic negative formulas which are not translated by $?G$ formulas in **LL**. A solution is to add the constraints of **LC**$^{\text{rev}}$ to **LC** as we have done, but another one is to introduce cuts for the translation of these rules. This has been done with linear isomorphisms in Danos-Joinet-Schellinx [1].

3.2 $\mathbf{LLP_C} \to \mathbf{LC^{rev}}$

Definition 6. *The translation $G \mapsto G^*$ from $\boldsymbol{LLP_C}$ into $\boldsymbol{LC^{rev}}$ is defined on strictly polarized formulas by:*

$$\begin{array}{rcl@{\qquad\qquad}rcl} (!A)^* &=& A & (!\mathcal{N})^* &=& \mathcal{N}^* \\ 1^* &=& V & 0^* &=& F \\ (\mathcal{P} \otimes \mathcal{Q})^* &=& \mathcal{P}^* \wedge \mathcal{Q}^* & (\mathcal{P} \oplus \mathcal{Q})^* &=& \mathcal{P}^* \vee \mathcal{Q}^* \\ (\mathcal{P} \otimes !\mathcal{N})^* &=& \mathcal{P}^* \wedge \mathcal{N}^* & (!\mathcal{N} \otimes \mathcal{P})^* &=& \mathcal{N}^* \wedge \mathcal{P}^* \\ (\exists x \mathcal{P})^* &=& \exists x \mathcal{P}^* & (\exists x !\mathcal{N})^* &=& \exists x \mathcal{N}^* \end{array}$$

$$(\mathcal{P}^\perp)^* = \neg \mathcal{P}^*$$

By Proposition 1, a sequent $\vdash \Gamma$ of $\boldsymbol{LLP_C}$ can be written $\vdash \Gamma', \Pi$ where Π is the unique strictly positive formula of Γ (if it exists). Then the translation is given on sequents by: $(\vdash \Gamma', \Pi)^* = \vdash \Gamma'^*; \Pi^*$.

There is no problem for the translation of proofs, we just have to precise the translation of the promotion rule:

$$\left(\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash ?\Gamma, \mathcal{N}\end{array}}{\vdash ?\Gamma, !\mathcal{N}} \right)^* = \begin{array}{c}\pi^* \\ \vdots \\ \vdash \Gamma^*, \mathcal{N}^*;\end{array}$$

Remark 2. This particular translation corresponds to the fact that a promotion is always followed by another rule: a *cut*-rule, a $\otimes$-rule or a $\exists$-rule. So promotion rules can be erased by the translation.

The translations $(.)^\bullet$ and $(.)^*$ are almost inverse of each other, more precisely:

- If G is a formula of $\mathbf{LC^{rev}}$, $G^{\bullet *} = G$.
- If $\mathcal{P}$ is a strictly positive formula of $\mathbf{LLP_C}$, $\mathcal{P}^{*\bullet} = \mathcal{P}$ and $(?\mathcal{P})^{*\bullet} = \mathcal{P}$.
- If $\mathcal{N}$ is a strictly negative formula of $\mathbf{LLP_C}$, $\mathcal{N}^{*\bullet} = \mathcal{N}$ and $(!\mathcal{N})^{*\bullet} = \mathcal{N}$.
- For the sequents: $(\vdash \Gamma; \Pi)^{\bullet *} = \vdash \Gamma; \Pi$ and $(\vdash \Gamma)^{*\bullet} = \vdash \Gamma$.
- If π is a proof in $\mathbf{LC^{rev}}$, $\pi^{\bullet *} = \pi$.

However the converse is wrong for proofs: $\pi^{*\bullet} \neq \pi$ because $\mathbf{LLP_C}$ is more flexible about the position of promotions. In the following example, the first **LL** proof puts weakening in between the promotion and its associated $\exists$-rule whereas the third one, being translated from **LC**, has glued the promotion with the $\exists$-rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash !A, ?A^\perp}}{\vdash ?!A, ?A^\perp}}{\vdash ?!A, !?A^\perp}}{\vdash ?B^\perp, ?!A, !?A^\perp}}{\vdash ?B^\perp, ?!A, \exists x !?A^\perp} \quad \xrightarrow{*} \quad \dfrac{\dfrac{\dfrac{\overline{\vdash \neg A; A}}{\vdash \neg A, A;}}{\vdash \neg B, \neg A, A;}}{\vdash \neg B, A; \exists x \neg A} \quad \xrightarrow{\bullet} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash !A, ?A^\perp}}{\vdash ?!A, ?A^\perp}}{\vdash ?B^\perp, ?!A, ?A^\perp}}{\vdash ?B^\perp, ?!A, !?A^\perp}}{\vdash ?B^\perp, ?!A, \exists x !?A^\perp}$$

4 Proof-Nets

Proof-nets have been introduced in [3] for the multiplicative case and then extended in [5] and [7] to full linear logic.

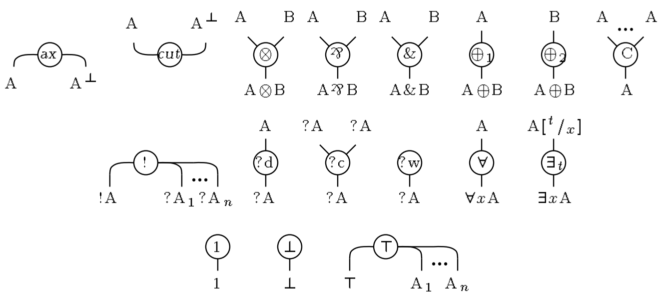
4.1 Proof-Structure

The following definitions come from [7] with just some modifications.

Definition 7 (Weight). *Given a set of* elementary weights*, i.e. boolean variables, (denoted by p, q,...), a* weight *is a product (conjunction) of elementary weights p and of negations of elementary weights $\bar{p}$.*

As a convention, we use 1 for the empty product and 0 for a product where p and $\bar{p}$ appear. We also replace p.p by p. With this convention we say that the weight w depends on *p when p or $\bar{p}$ appears in w.*

A *proof-structure* is an oriented graph with pending edges, for which each edge is associated with an **LL** formula, constructed on the following set of nodes respecting the following typing constraints. The orientation is from top to bottom.



To avoid confusion with the other orientation that we will introduce later, this orientation will be called the *geographic orientation* and we will refer to it by the terms: *top*, *bottom*, *above*, *bellow*, *to go up*, *to go down*, *premise* of a node (edge just above the node), *conclusion* of a node (edge just bellow the node),...

A *unary node* is a node with only one premise and a *binary node* is a node with two premises. The C-nodes must have at least two premises.

In such a graph:

- we associate an elementary weight to each &-node called its *eigen weight*;
- the variable used in the quantification of a ∀-node is called its *eigen variable*;
- we associate a non empty set of nodes (different from *cut*) to each ⊥-node and ?w-node. These are called the *jumps* of the node.

Eigen weights and eigen variables are supposed to be different.

We associate a weight to each node with the constraint that if two nodes have a common edge, they must have the same weight except if the edge is a premise of a &-node or of a C-node (*additive contraction*). In these particular cases the weight changes:

– if w is the weight of a &-node and p is its eigen weight then w does not depend on p and its premise nodes must have weights $w.p$ and $w.\bar{p}$;
– if w is the weight of a C-node and $w_1, \ldots, w_n$ are the weights of its premise nodes then we must have $w = w_1 + \ldots + w_n$ and $w_i w_j = 0, \forall i \neq j$.

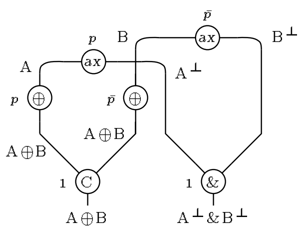
Then we can define the following notions:

– A node L with weight w is said *to depend on* p if w depends on p or if L is a C-node and one of the weights just above it depends on p.
– A node L is said *to depend on* an eigen variable x if x is free in the formula associated to the conclusion of L or if L is a $\exists_t$-node and x is free in t.

A proof-structure must also satisfy the following properties:

– a *conclusion node* (i.e. a node with pending edge) has weight 1;
– eigen variables are not free in the formulas associated to pending edges;
– if w is the weight of a &-node with eigen weight p and w' is a weight depending on p and appearing in the proof-structure then $w' \leq w$;
– if w is the weight of a $\forall$-node with eigen variable x and w' is the weight of a node depending on x then $w' \leq w$;
– if w is the weight of a $\bot$-node or of a ?w-node and w' is the weight of one of its jumps then $w \leq w'$.

With this definition we have a notion of proof-structures for full linear logic. Now to make it clear, let us look at the example of a proof-structure for $A \oplus B \multimap A \oplus B$:



4.2 Sequentialization and Correctness

An important point in the study of proof-nets is the problem of correctness criterions that is the problem to know whether a proof-structure is a proof. More technically, can you inductively deconstruct a proof-structure?

There exist different correctness criterions for multiplicative proof-structures like [3] or [2] which lead to the criterion of [7] for the full case. We present here this general criterion.

Definition 8 (Sequentialization of a proof-structure). *The relation* "L sequentializes $\mathcal{R}$ into $\mathcal{E}$" *is defined for each possible L. $\mathcal{R}$ is a proof-structure, $\mathcal{E}$ is a set of proof-structures and L is a conclusion node of $\mathcal{R}$ or a cut.*

– ax, !, 1, $\top$: *if L is the only node of $\mathcal{R}$ then L sequentializes $\mathcal{R}$ into $\emptyset$;*

- cut, $\otimes$: *if it is possible to split the graph obtained by erasing* L *into two proof-structures* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *then* L *sequentializes* $\mathcal{R}$ *into* $\{\mathcal{R}_1, \mathcal{R}_2\}$*;*
- $⅋$, $\oplus_1$, $\oplus_2$, $?d$, $?c$, $?w$, $\forall$, $\exists$, $\bot$: *if when we erase* L *in* $\mathcal{R}$*, we obtain a proof-structure* $\mathcal{R}_0$ *then* L *sequentializes* $\mathcal{R}$ *into* $\{\mathcal{R}_0\}$*;*
- &: *let* p *be the eigen weight of* L*. The graph* $\mathcal{R}_0$ *(resp.* $\mathcal{R}_1$*) is obtained by giving to* p *the value 0 (resp. 1) and just keeping nodes with non zero weights and identifying the unary* C*-nodes to the node just above. If* $\mathcal{R}_0$ *(resp.* $\mathcal{R}_1$*) is a proof-structure then* L *sequentializes* $\mathcal{R}$ *into* $\{\mathcal{R}_0, \mathcal{R}_1\}$*;*
- C: *a* C*-node never sequentializes a proof-structure.*

Definition 9 (Sequentializable proof-structure). *A proof-structure* $\mathcal{R}$ *is said to be* sequentializable *if one of its nodes sequentializes* $\mathcal{R}$ *into a set of sequentializable proof-structures or into the empty set.*

Definition 10 (Valuation). *A* valuation φ *for a proof-structure* $\mathcal{R}$ *is a function from the set of the eigen weights of* $\mathcal{R}$ *into* $\{0, 1\}$*. Such a valuation can easily be seen as a function defined on the set of all the weights of* $\mathcal{R}$*.*

Definition 11 (Slice). *Given a valuation* φ *of a proof-structure* $\mathcal{R}$*, the* slice $\varphi(\mathcal{R})$ *is the proof-structure obtained from* $\mathcal{R}$ *by keeping only the nodes with weights* w *such that* $\varphi(w) = 1$ *and the edges bellow a kept node and by identifying the unary* C*-nodes with the upper node. A slice is not really a proof-structure according to definition of the Sect. 4.1 because unary &-nodes appear.*

Definition 12 (Switch). *Given a valuation* φ *of a proof-structure* $\mathcal{R}$*, a* switch $\mathcal{S}$ *of* $\mathcal{R}$ *is defined as a non oriented graph constructed with the nodes and the edges of* $\varphi(\mathcal{R})$ *with the modifications:*

- *for each* $⅋$*- or* $?c$*-node, we keep only one premise;*
- *for each &-node* L*, we erase the premise appearing in* $\varphi(\mathcal{R})$ *and we add an edge, called* dependency edge*, from a node depending on* L *to* L *(this may change nothing);*
- *for each* $\forall$*-node* L*, we erase the premise and we add an edge, called* dependency edge*, from a node depending on its eigen variable to* L *(this may change nothing);*
- *for each* $?w$*- or* $\bot$*-node* L*, we add an edge, called* jump edge*, from a jump of* L *to* L*.*

Definition 13 (Proof-net). *A proof-structure is a* proof-net *if all its switches are acyclic and connected.*

Theorem 1 (Sequentialization – *J.-Y. Girard* in [7]**).** *A proof-structure is sequentializable iff it is a proof-net.*

5 Polarized Proof-Nets

Now we restrict proof-nets to the polarized case. This strong constraint will allow us to define a new and simpler correctness criterion.

Definition 14 (Polarized proof-structure). *A* polarized proof-structure *is a proof-structure made only of polarized formulas and with the constraint that at most one of the formulas associated to the conclusions of a* $\top$*-node can be positive.*

In other words, a polarized proof-structure is a proof-structure typed by **LLP**. As $\mathbf{LLP}_{\mathrm{C}}$ is a fragment of **LLP**, all the following results will give a notion of proof-nets for **LC** through the translations in Sect. 3.

Definition 15 (Edges). *We give here some new terminology on edges in a polarized proof-structure:*

- *a* positive *(resp.* negative*) edge is an edge with a positive (resp. negative) formula;*
- *a* principal edge *in a switch is an edge already appearing in the proof-structure; a* switching edge *is either a dependency edge or a jump edge. For switching edges, we extend the polarization and the geographic orientation by considering them negative and oriented towards the corresponding* &*-,* $\forall$*-,* $?w$*- or* $\bot$*-node.*

In the sequel, we will distinguish between two C-nodes: the C^+*-node* with positive premises and conclusion and the C^-*-node* with negative ones.

Definition 16 (Positive and negative nodes). *A* positive node *is a node with positive edges, that is* $\otimes$, $\oplus$, C^+, $\exists$ *and* 1, *and a* negative node *is a node with negative edges, that is* $⅋$, &, C^-, $?c$, $?w$, $\forall$ *and* $\bot$.

5.1 Towards Specific Criterions

The key point for the simplification of the correctness criterion in the case of polarized proof-nets is the existence of a specific orientation in these proof-nets as shown in Lemma 2. The use of this orientation allows us to forget the notion of switch and then also the notion of slice.

The idea of orientation linked to polarization in proof-nets has already been used. For example François Lamarche proposed in [8] a criterion for proof-nets for intuitionistic linear logic with Danos-Regnier polarities.

We define a new orientation on proof-structures, the *orientation of polarization* (or *p-orientation*): positive edges are oriented upwardly and negative edges downwardly. We will talk about this orientation using the terms: *to arrive to*, *to come from*, *incident edge*, *emergent edge*,...

Lemma 1. *In a switch of a polarized proof-structure, a node has at most one incident edge. Positive and negative nodes have exactly one incident edge.*

Proof. We study each node:

- the only nodes with incident switching edges are &, $\forall$, ?w and $\bot$ and by the definition of a switch these nodes have exactly one incident edge in a switch (either a premise or a switching edge);

– ⅋- and ?c-nodes have just one premise in a switch so just one incident edge;
– positive nodes, *ax*, *cut* and ! have only principal edges in a switch and the only incident one is their positive conclusion (negative premise for *cut*);
– ?d-nodes have only emergent edges;
– ⊤-nodes with a positive conclusion are like ! and those with only negative conclusions have no incident edges;
– there are no C-nodes in a switch. □

Lemma 2 (Orientation lemma). *A non bouncing path in a switch of a polarized proof-structure starting accordingly to the p-orientation always respects this orientation.*

Proof. We prove the result by induction on the length of the path, the case of length 0 being given by the starting hypothesis. Now when the path arrives to a new node, this is only possible through the incident edge so when the path continues it must be by another edge, thus an emergent one (by Lemma 1) since it does not bounce. □

Lemma 3. *A non oriented cycle in a switch of a polarized proof-structure is p-oriented.*

Definition 17 (Correction graph). *The* correction graph *of a proof-structure $\mathcal{R}$ is the oriented graph obtained by putting on $\mathcal{R}$ the p-orientation and by adding some new edges:*

– *from each node depending on an eigen weight to the corresponding &-node;*
– *from each node depending on an eigen variable to the corresponding ∀-node;*
– *from the jumps to the nodes they are associated to.*

Lemma 4. *If there is a (non oriented) cycle in a switch of a proof-structure then there is a p-oriented cycle in its correction graph.*

Definition 18 (Initial and final nodes). *In a correction graph, a node is* initial *(resp.* final*) if all the edges starting from (resp. arriving to) it are pending edges.*

Remark 3. A final node is a conclusion node so its weight is always 1. A ?d-node is always initial.

5.2 Weak Criterion

We give here our first criterion for polarized proof-nets, which is simpler than the general one but equivalent. To obtain this result we still need to use the notion of slices.

Definition 19 (Slice of a correction graph). *A* slice *of a correction graph $\mathcal{G}$ is the sub-graph of $\mathcal{G}$ made only of the nodes and the edges of a slice of the proof-structure (in other terms it is the correction graph of the slice).*

Theorem 2 (Correctness criterion). *A polarized proof-structure has all its switches acyclic and connected iff all the slices of its correction graph are acyclic (with orientation), contain exactly one initial node and all the nodes of the slice are p-accessible from the initial one (in this case we say that the correction graph is* weakly correct*).*

Proof. By Theorem 1, a proof-structure with all its switches acyclic and connected is sequentializable and by an easy induction, a sequentializable polarized proof-structure has a weakly correct correction graph. Conversely if the correction graph is weakly correct, switches cannot contain any cycle by Lemma 4.

To finish, we can prove by induction on the sum Σ of the lengths of all the paths from the initial node i of the slice to a fixed node s that in all the switches of this slice there is a path between i and s.

- If $\Sigma = 0$ then $s = i$.
- If $\Sigma = n + 1$, s is not an initial node in the slice. We choose a switch $\mathcal{S}$, there exist a node s' and an edge a from s' to s such that a appears in $\mathcal{S}$ (by definition of a switch we always keep such an edge). Then by induction hypothesis on s', there is a path in $\mathcal{S}$ between i and s' which can be extended with a into a path between i and s. □

We can apply to our polarized proof-structures all the results of the general case given in [7] about sequentialization, cut-elimination,...

5.3 Strong Criterion

Following the same direction we obtain a second and most important criterion which allows us to forget also slices.

Definition 20 (Strong correctness criterion). *The correction graph of a polarized proof-structure is* strongly correct *if it is acyclic and if for all pair of distinct initial nodes with weights w_i and w_j: $w_i.w_j = 0$.*

Theorem 3 (Strong criterion and weak criterion). *A strongly correct correction graph is weakly correct.*

Proof. No problem for acyclicity because a slice of a correction graph has less edges than the correction graph itself. Then by acyclicity of the slices we have at least one initial node in each slice. But also at most one because taking a slice does not create any initial node (a negative node is never initial and the other ones cannot lose the node under their conclusion) so the condition on initial nodes of the correction graph is sufficient.

For accessibility of nodes, we prove by induction on the sum Σ of the lengths of all the paths from an initial node to a fixed node s that s is p-accessible by the initial node in each slice where it appears:

- if $\Sigma = 0$ then s is initial;

– if $\Sigma = n+1$ then in a slice where s appears either it is initial and there is no problem or there is another node s' with an edge from s' to s. By induction hypothesis s' is accessible from the initial node in every slice where it appears. Thus in the slice we are looking at, s' is accessible and also s by adding the edge to a path arriving to s'. □

The converse is wrong, some proof-structures are weakly correct but rejected by the strong criterion because some cycles may come from the interactions between different slices. However we keep enough proof-structures to have proof-nets for all proofs of sequent calculus and the strong criterion is preserved by cut-elimination. We will see this in the Sects. 5.5 and 5.6.

5.4 Sequentialization

We will now give a proof of sequentializability of strongly correct proof-nets different from the one consisting in using the proof for the general criterion by Theorems 3, 2 and then 1.

Definition 21 (Positive tree). *A* positive tree *of a correction graph is a non empty connected set of positive nodes and positive edges maximal for inclusion.*

A positive tree $\mathcal{A}$ is terminal *when for each positive edge a of the correction graph if there is a path from $\mathcal{A}$ to a then a is in $\mathcal{A}$.*

Theorem 4 (Sequentialization). *A polarized proof-net is sequentializable.*

Proof. The first point is to sequentialize by all negative final nodes. We prove that if a ⅋-, &-, ?c-, ?w-, ⊥- or ∀-node is final then it sequentializes the proof-net. We remark that ⊗-, $\oplus_i$, ∃-, C$^+$-, C$^-$-, ?d- and *cut*-nodes are never final. So we have to sequentialize a proof-net with only ax, !, ⊤ and 1-nodes as final ones.

Lemma 5. *If the only final nodes of a polarized proof-net are* ax, !, ⊤ *and* 1 *then from each non final node there exists a path to a terminal positive tree.*

Definition 22 (Cut positive tree). *A positive tree is said to be* cut *if it has a cut-node hereditary above it.*

Proof (Theorem 4 – continued). Given a proof-net with only ax, !, ⊤ and 1-nodes as final ones, by Lemma 5 it contains a terminal positive tree. If there is no nodes under this tree, it can be sequentialized. Otherwise this is a cut positive tree and we show by terminality of the tree that the *cut*-node under it sequentializes the proof-net. □

Proposition 2. *The criterion given by Theorem 4 has a cubic complexity in the size of the proof-net (i.e. the number of its nodes).*

5.5 Translation from Sequent Calculus

To show that the strong criterion keep enough proof-structures we have to define a translation from **LLP** to polarized proof-structures and to prove the correctness of the proof-structures built in this way.

When we talked about sequentialization we used proof-structures with !-nodes just seen as generalized axioms but to talk about the translation of proofs and about cut-elimination, we need to refine our definition of proof-structure.

Definition 23 (Proof-structure and proof-net with boxes). *We define a* proof-structure with boxes *by induction, it is:*

- *either a proof-structure with no !-nodes,*
- *or a proof-structure together with a proof-structure with boxes of conclusions* $A, ?B_1, \ldots, ?B_n$ *associated to each !-node of conclusions* $!A, ?B_1, \ldots, ?B_n$*.*

We can define in the same way proof-nets with boxes *from proof-nets.*

In the sequel we will use the term proof-structure (resp. proof-net) instead of proof-structure (resp. proof-net) with boxes.

Definition 24 (Translation of proofs). *We define the translation from* ***LLP*** *to polarized proof-structures by induction on the size of the proof:*

- &*: by induction we obtain two polarized proof-structures* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *from the two proofs of the premises of the* &*-rule. We choose a new elementary weight* p *and multiply all the weights of* $\mathcal{R}_1$ *by* p *and all the weights of* $\mathcal{R}_2$ *by* $\bar{p}$*. Then we add a* &*-node (with eigen weight* p*) between the two pending edges corresponding to the formulas used by the* & *and a* C*-node for each pair of formulas of the context coming from* $\mathcal{R}_1$ *and* $\mathcal{R}_2$*;*
- !*: the new proof-structure is just a single* !*-node introducing the conclusions* $!A, ?B_1, \ldots, ?B_n$ *of the rule and the proof-structure associated to it is the one obtained at the previous step with conclusions* $A, ?B_1, \ldots, ?B_n$*;*
- $?w$*: we just add a* $?w$*-node to the proof-structure* $\mathcal{R}$ *of the previous step with a set of jumps constituted of all the conclusion nodes of* $\mathcal{R}$*;*
- $\perp$*: same as* $?w$*;*

no problem for the other rules.

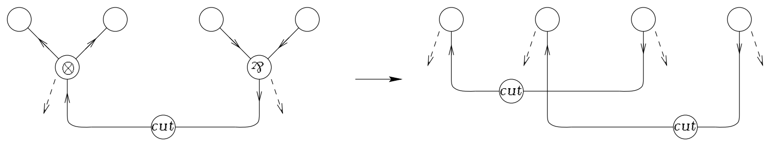
Theorem 5. *The previous translation is in fact from* ***LLP*** *to polarized proof-nets.*

5.6 Cut Elimination

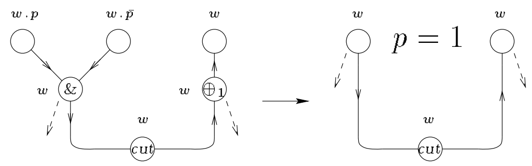
Definition 25 (Reduction step). *The different cut-elimination steps are the following ones:*

- Axiom cut: *we erase the* ax*- and* cut*-nodes and replace them by an edge, the jumps coming from the* ax*-node are moved to the other node above the cut.*

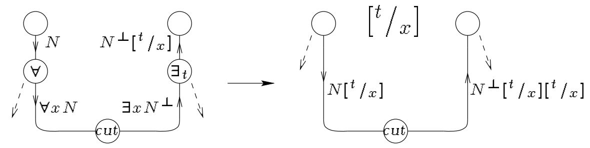
– Multiplicative cut: *we erase the* ⅋ *and the* $\otimes$*, the cut is duplicated between the two pairs of premises. All the jumps are duplicated and moved up.*



– Additive cut: *if the* $\oplus$*-node is a* $\oplus_1$*-node (resp.* $\oplus_2$*-node) we erase in the proof-structure all the nodes with null weights when* $p = 1$ *(resp.* $p = 0$*) and the cut moves up as the jumps.*



– Dereliction cut: *the box is opened and the cut moves up as the jumps.*
– Contraction cut: *the* !*-node is duplicated and also the cut to be put between each premise of the* ?c *and a box. New* ?c*-nodes are put between the pairs of conclusions of the* !*. Jumps from the* ! *and from the* ?c *are duplicated.*
– Weakening cut: *we just erase the box and put new* ?w*-nodes above its conclusions. The jumps of these new nodes are the jumps of the cut one.*
– Commutative exponential cut: *the box with the cut* !*-node comes into the other one and the other* !*-node is extended with the conclusions of the first one. All the jumps coming from the two* !*-nodes are put on the second one.*
– Quantifier cut: *we erase the two nodes* $\forall$ *and* $\exists_t$*, the cut goes up as the jumps. In all the proof-structure we make the substitution of* x *by* t*.*



– Multiplicative constant cut: *we erase the three nodes:* 1, $\perp$ *and cut. The jumps starting from them are duplicated and moved to the jumps of* $\perp$*.*

The cases of a cut with a $\top$- or a C-node are still to be studied. A solution for the additive contraction is proposed in [7] but is not uniform with the other reduction steps. However with the restriction on the steps defined above, we have the same result as in [7]:

Theorem 6. *A proof-net without* $\top$*-node and without* &*-connectives in the formulas associated to its pending edges, which cannot be reduce by any step described above, is in normal form (i.e. without cut-node).*

This has been already proved by J.-Y. Girard for the multiplicative-additive case but we give here a really different proof using the p-orientation.

Proof. If the proof-net contains no &-nodes, all the weights are 1 and there are no problems. Otherwise let L be a terminal &-node, that is with no paths to another &-node. By the hypothesis, there must be a *cut*-node (hereditary) under L. Then this *cut*-node can be reduced by terminality of L. □

Theorem 7 (Cut-elimination). *Strong correctness is preserved by the cut-elimination procedure.*

Proof. The steps are well defined in a proof-net (x is not free in $N^{\perp}[^t/_x]$ for the quantifier step by acyclicity). Then each step preserves the strong criterion. □

Conclusion

The polarization constraint, coming from **LC**, gives a system of proof-nets with a correctness criterion which is really simpler than the one in the general case [7]. Through the translation between **LC** and **LLP**, this gives proof-nets for the sequent calculus **LC**, solving our starting problem.

The last section of this paper is devoted to cut-elimination where the problem of commutative additive contraction appears. A full solution has still to be found.

Much work is now possible such as an extension of our approach to second order quantifiers, the study of a geometry of interaction or of a game semantics for such proof-nets, the continuation of this work towards the intuitionistic polarities as defined in [6],...

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