A proof of the focalization property of Linear Logic

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The focalization property of linear logic has been discovered by Jean-Marc Andreoli [1] in the beginning of the 90's. It is one of the main properties of Linear Logic that appeared after the original paper of Jean-Yves Girard [3]. This property is proved in various papers [1, 4, 2] but, as far as we know never for the usual LL sequent calculus (or with an intricate induction in Andreoli's thesis, which is replaced here by a cut elimination property).

The proof we give is a compilation of the previous proofs and proof techniques, our goal is just to give a complete presentation of a proof of this property which appears to be more and more important in the current research in Linear Logic [4, 2, 5, 6].

We decompose the usual focalization property into two technically different steps: weak focalization (the decompositions of two positive formulas are not interleaved, see $\mathsf{LL}_{\mathrm{foc}}$) and reversing (negative rules are applied as late as possible from a top-down point of view, see $\mathsf{LL}_{\mathrm{Foc}}$).

1 Linear Logic and the focalization property

Formulas of Linear Logic are given by the usual grammar:

$$
\begin{array}{rcccccccccccc}
A & ::= & X & \mid & A \otimes A & \mid & A \oplus A & \mid & 1 & \mid & 0 & \mid & !A \\
 & \mid & X^{\perp} & \mid & A \parr A & \mid & A \,\&\, A & \mid & \perp & \mid & \top & \mid & ?A
\end{array}
$$

We split this grammar into two sub-classes: positive formulas and negative formulas.

$$
\begin{array}{rcccccccccccc}
P & ::= & X & \mid & A \otimes A & \mid & A \oplus A & \mid & 1 & \mid & 0 & \mid & !A \\
N & ::= & X^{\perp} & \mid & A \parr A & \mid & A \,\&\, A & \mid & \perp & \mid & \top & \mid & ?A
\end{array}
$$

In the sequel:

- A, B, ... denote arbitrary formulas;
- P, Q, R denote positive formulas;
- N, M, L denote negative formulas;
- X, $X^{\perp}$, ... are called *atoms* or *atomic formulas.*

We consider the usual rules of Linear Logic [3]:

$$\frac{}{\vdash A^\perp, A}\,ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2$$

$$\frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash \Gamma, \top}\,\top$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c$$

The main connectives of positive (resp. negative) formulas are called *positive connectives* (resp. *negative connectives*), that is X, $\otimes$, $\oplus$, 1, 0 and ! (resp. $X^\perp$, $\parr$, &, $\perp$, $\top$ and ?). The rules introducing positive (resp. negative) connectives are called *positive rules* (resp. *negative rules*).

A positive (resp. negative) formula is *strictly positive* (resp. *strictly negative*) if its main connective is not ! (resp. ?). A positive (resp. negative) connective is *strictly positive* (resp. *strictly negative*) if it is not ! (resp. ?). A positive (resp. negative) rule is *strictly positive* (resp. *strictly negative*) if it is not introducing a $!A$ formula (resp. $?A$ formula).

Definition 1 (Main positive tree)
If A is a formula, its *main positive tree* $\mathcal{T}^+(A)$ is defined by:

$$\begin{aligned}
\mathcal{T}^+(N) &= \emptyset \\
\mathcal{T}^+(X) &= X \\
\mathcal{T}^+(A \otimes B) &= \mathcal{T}^+(A) \otimes \mathcal{T}^+(B) \\
\mathcal{T}^+(A \oplus B) &= \mathcal{T}^+(A) \oplus \mathcal{T}^+(B) \\
\mathcal{T}^+(1) &= 1 \\
\mathcal{T}^+(0) &= 0 \\
\mathcal{T}^+(!A) &= !
\end{aligned}$$

$\mathcal{T}^+(A)$ is a (possibly empty) tree whose nodes are $\otimes$, $\oplus$ (with arity at most 2) and !, 1, 0 or X (with arity 0).

Definition 2 (Weakly +-focalized proof)
A proof π in LL is *weakly +-focalized* if it is cut-free and, for any subproof π' of π with conclusion $\vdash \Gamma, A$, the only *positive* rules of π' between two rules introducing connectives of $\mathcal{T}^+(A)$ are rules introducing connectives of $\mathcal{T}^+(A)$.

Definition 3 (Strongly +-focalized proof)
A proof π in LL is *strongly +-focalized* if it is cut-free and, for any subproof π' of π with conclusion $\vdash \Gamma, A$, the only rules of π' between two rules introducing connectives of $\mathcal{T}^+(A)$ are rules introducing connectives of $\mathcal{T}^+(A)$.

Definition 4 (Reversed proof)
A proof π in LL is *reversed* if, for any subproof π' of π with conclusion $\vdash \Gamma, N$ with N a strictly negative non-atomic formula, the last rule of π' is a strictly negative rule.

Definition 5 (Focalized proof)
A proof π in LL is *focalized* if it is both strongly +-focalized and reversed.

Remark: Various systems for classical logic related with focalization constraints have been introduced. In particular Girard's LC [4] corresponds to weak +-focalization, Danos-Joinet-Schellinx's LK^η [2] corresponds to strong +-focalization and Quatrini-Tortora de Falco's $\mathsf{LK}^{\eta,\rho}_{\mathrm{pol}}$ [7] corresponds to focalization since their ρ-constraint is a reversing constraint.

2 Weakly Focalized Linear Logic

2.1 Sequent calculus $\mathsf{LL}_{\mathrm{foc}}$

A sequent of $\mathsf{LL}_{\mathrm{foc}}$ has the shape $\vdash \Gamma \,;\, \Pi$ where Γ is a multi-set of formulas and Π is either empty or contains a unique positive formula.

The rules are a linear logic version of the rules of Girard's LC [4]:

$$\frac{}{\vdash P^\perp \,;\, P}\,ax \qquad \frac{\vdash \Gamma \,;\, P}{\vdash \Gamma, P \,;}\,foc$$

$$\frac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^\perp \,;\, \Pi}{\vdash \Gamma, \Delta \,;\, \Pi}\,p\text{-}cut \qquad \frac{\vdash \Gamma, P \,;\, \Pi \qquad \vdash \Delta, P^\perp \,;}{\vdash \Gamma, \Delta \,;\, \Pi}\,n\text{-}cut$$

$$\frac{\vdash \Gamma, A, B \,;\, \Pi}{\vdash \Gamma, A \parr B \,;\, \Pi}\,\parr \qquad \frac{\vdash \Gamma \,;\, P \qquad \vdash \Delta \,;\, Q}{\vdash \Gamma, \Delta \,;\, P \otimes Q}\,\otimes$$

$$\frac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, M \,;}{\vdash \Gamma, \Delta \,;\, P \otimes M}\,\otimes \qquad \frac{\vdash \Gamma, N \,; \qquad \vdash \Delta \,;\, Q}{\vdash \Gamma, \Delta \,;\, N \otimes Q}\,\otimes \qquad \frac{\vdash \Gamma, N \,; \qquad \vdash \Delta, M \,;}{\vdash \Gamma, \Delta \,;\, N \otimes M}\,\otimes$$

$$\frac{\vdash \Gamma, A \,;\, \Pi \qquad \vdash \Gamma, B \,;\, \Pi}{\vdash \Gamma, A \,\&\, B \,;\, \Pi}\,\&$$

$$\frac{\vdash \Gamma \,;\, P}{\vdash \Gamma \,;\, P \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, N \,;}{\vdash \Gamma \,;\, N \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma \,;\, Q}{\vdash \Gamma \,;\, A \oplus Q}\,\oplus_2 \qquad \frac{\vdash \Gamma, M \,;}{\vdash \Gamma \,;\, A \oplus M}\,\oplus_2$$

$$\frac{}{\vdash \,;\, 1}\,1 \qquad \frac{\vdash \Gamma \,;\, \Pi}{\vdash \Gamma, \perp \,;\, \Pi}\,\perp \qquad \frac{}{\vdash \Gamma, \top \,;\, \Pi}\,\top$$

$$\frac{\vdash ?\Gamma, A \,;}{\vdash ?\Gamma \,;\, !A}\,! \qquad \frac{\vdash \Gamma \,;\, P}{\vdash \Gamma, ?P \,;}\,?d \qquad \frac{\vdash \Gamma, N \,;}{\vdash \Gamma, ?N \,;}\,?d$$

$$\frac{\vdash \Gamma \,;\, \Pi}{\vdash \Gamma, ?A \,;\, \Pi}\,?w \qquad \frac{\vdash \Gamma, ?A, ?A \,;\, \Pi}{\vdash \Gamma, ?A \,;\, \Pi}\,?c$$

2.2 Expansion of axioms

Given a positive formula P, the sequent $\vdash P^\perp ; P$ is provable in LL_{foc} by means of the *ax* rule. Moreover, if P is not atomic, it is also possible to give a proof based only on *ax* rules applied to the sub-formulas of P (and thus, by an easy induction, to the atomic sub-formulas of P):

$$\dfrac{\dfrac{\vdash P^\perp ; P \qquad \vdash Q^\perp ; Q}{\vdash P^\perp, Q^\perp ; P \otimes Q}\,\otimes}{\vdash P^\perp \parr Q^\perp ; P \otimes Q}\,\parr \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\,foc \qquad \vdash Q^\perp ; Q}{\vdash N^\perp, Q^\perp ; N \otimes Q}\,\otimes}{\vdash N^\perp \parr Q^\perp ; N \otimes Q}\,\parr$$

$$\dfrac{\dfrac{\vdash P^\perp ; P \qquad \dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\,foc}{\vdash P^\perp, M^\perp ; P \otimes M}\,\otimes}{\vdash P^\perp \parr M^\perp ; P \otimes M}\,\parr \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\,foc \qquad \dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\,foc}{\vdash N^\perp, M^\perp ; N \otimes M}\,\otimes}{\vdash N^\perp \parr M^\perp ; N \otimes M}\,\parr$$

$$\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp ; P \oplus Q}\,\oplus_1 \qquad \dfrac{\vdash Q^\perp ; Q}{\vdash Q^\perp ; P \oplus Q}\,\oplus_2}{\vdash P^\perp \,\&\, Q^\perp ; P \oplus Q}\,\& \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\,foc}{\vdash N^\perp ; N \oplus Q}\,\oplus_1 \qquad \dfrac{\vdash Q^\perp ; Q}{\vdash Q^\perp ; N \oplus Q}\,\oplus_2}{\vdash N^\perp \,\&\, Q^\perp ; N \oplus Q}\,\&$$

$$\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp ; P \oplus M}\,\oplus_1 \qquad \dfrac{\dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\,foc}{\vdash M^\perp ; P \oplus M}\,\oplus_2}{\vdash P^\perp \,\&\, M^\perp ; P \oplus M}\,\& \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\,foc}{\vdash N^\perp ; N \oplus M}\,\oplus_1 \qquad \dfrac{\dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\,foc}{\vdash M^\perp ; N \oplus M}\,\oplus_2}{\vdash N^\perp \,\&\, M^\perp ; N \oplus M}\,\&$$

$$\dfrac{\dfrac{}{\vdash ; 1}\,1}{\vdash \bot ; 1}\,\bot \qquad \dfrac{}{\vdash \top ; 0}\,\top$$

$$\dfrac{\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp, P ;}\,foc}{\vdash ?P^\perp, P ;}\,?d}{\vdash ?P^\perp ; !P}\,! \qquad \dfrac{\dfrac{\vdash N ; N^\perp}{\vdash ?N^\perp, N ;}\,?d}{\vdash ?N^\perp ; !N}\,!$$

2.3 Embedding of LL (weak +-focalization)

The translation $(.)^\bullet$ of LL into LL_{foc} does not modify formulas, translates the sequent $\vdash \Gamma$ as $\vdash \Gamma ;$ and acts on proofs by adding a lot of cut rules:

$$\dfrac{}{\vdash P^\perp, P}\,ax \quad \rightsquigarrow \quad \dfrac{}{\vdash P^\perp ; P}\,ax$$

$$\dfrac{\vdash \Gamma, P \qquad \vdash \Delta, P^\perp}{\vdash \Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, P ; \qquad \vdash \Delta, P^\perp ;}{\vdash \Gamma, \Delta ;}\,n\text{-}cut$$

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, A, B ;}{\vdash \Gamma, A \parr B ;}\,\parr$$

$$\dfrac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\otimes \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, P\,; \qquad \dfrac{\vdash \Delta, Q\,; \qquad \dfrac{\dfrac{\dfrac{}{\vdash P^{\perp}\,; P}\mathit{ax} \qquad \dfrac{}{\vdash Q^{\perp}\,; Q}\mathit{ax}}{\vdash P^{\perp}, Q^{\perp}\,; P \otimes Q}\otimes}{\vdash P^{\perp}, Q^{\perp}, P \otimes Q\,;}\mathit{foc}}{\vdash \Delta, P^{\perp}, P \otimes Q\,;}\mathit{n\text{-}cut}}{\vdash \Gamma, \Delta, P \otimes Q\,;}\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, N \otimes Q}\otimes \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, N\,; \qquad \dfrac{\vdash \Delta, Q\,; \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N\,; N^{\perp}}\mathit{ax}}{\vdash N, N^{\perp}\,;}\mathit{foc} \qquad \dfrac{}{\vdash Q^{\perp}\,; Q}\mathit{ax}}{\vdash N^{\perp}, Q^{\perp}\,; N \otimes Q}\otimes}{\vdash N^{\perp}, Q^{\perp}, N \otimes Q\,;}\mathit{foc}}{\vdash \Delta, N^{\perp}, N \otimes Q\,;}\mathit{n\text{-}cut}}{\vdash \Gamma, \Delta, N \otimes Q\,;}\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N \qquad \vdash \Delta, M}{\vdash \Gamma, \Delta, N \otimes M}\otimes \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, N\,; \qquad \dfrac{\vdash \Delta, M\,; \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N\,; N^{\perp}}\mathit{ax}}{\vdash N, N^{\perp}\,;}\mathit{foc} \qquad \dfrac{\dfrac{}{\vdash M\,; M^{\perp}}\mathit{ax}}{\vdash M, M^{\perp}\,;}\mathit{foc}}{\vdash N^{\perp}, M^{\perp}\,; N \otimes M}\otimes}{\vdash N^{\perp}, M^{\perp}, N \otimes M\,;}\mathit{foc}}{\vdash \Delta, N^{\perp}, N \otimes M\,;}\mathit{n\text{-}cut}}{\vdash \Gamma, \Delta, N \otimes M\,;}\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, A\,; \qquad \vdash \Gamma, B\,;}{\vdash \Gamma, A \,\&\, B\,;}\,\&$$

$$\dfrac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus B}\oplus_1 \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, P\,; \qquad \dfrac{\dfrac{\dfrac{}{\vdash P^{\perp}\,; P}\mathit{ax}}{\vdash P^{\perp}\,; P \oplus B}\oplus_1}{\vdash P^{\perp}, P \oplus B\,;}\mathit{foc}}{\vdash \Gamma, P \oplus B\,;}\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N}{\vdash \Gamma, N \oplus B}\oplus_1 \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, N\,; \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash N\,; N^{\perp}}\mathit{ax}}{\vdash N, N^{\perp}\,;}\mathit{foc}}{\vdash N^{\perp}\,; N \oplus B}\oplus_1}{\vdash N^{\perp}, N \oplus B\,;}\mathit{foc}}{\vdash \Gamma, N \oplus B\,;}\mathit{n\text{-}cut}$$

$$\dfrac{}{\vdash 1}\,1 \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash\,; 1}\,1}{\vdash 1\,;}\mathit{foc}$$

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma\,;}{\vdash \Gamma, \bot\,;}\,\bot$$

$$\dfrac{}{\vdash \Gamma, \top}\,\top \quad \rightsquigarrow \quad \dfrac{}{\vdash \Gamma, \top\,;}\,\top$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\vdash ?\Gamma, A\,;}{\vdash ?\Gamma\,;\,!A}\,!}{\vdash ?\Gamma, !A\,;}\,\mathit{foc}$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,?d \qquad \rightsquigarrow \qquad \cfrac{\vdash \Gamma, P\,; \qquad \cfrac{\cfrac{}{\vdash P^\perp\,;\,P}\,ax}{\vdash P^\perp, ?P\,;}\,?d}{\vdash \Gamma, ?P\,;}\,n\text{-}cut$$

$$\frac{\vdash \Gamma, N}{\vdash \Gamma, ?N}\,?d \qquad \rightsquigarrow \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma, ?N\,;}\,?d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w \qquad \rightsquigarrow \qquad \frac{\vdash \Gamma\,;}{\vdash \Gamma, ?A\,;}\,?w$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c \qquad \rightsquigarrow \qquad \frac{\vdash \Gamma, ?A, ?A\,;}{\vdash \Gamma, ?A\,;}\,?c$$

2.4 Focalization in LL_{foc}

If π is a proof in LL_{foc} then π° is the LL proof obtained by erasing all the "$;$" in the sequents.

Proposition 1 (Cut-free weak $+$-focalization)
If π is a cut-free proof of $\vdash \Gamma\,;\,\Pi$ in LL_{foc} then π° is a weakly $+$-focalized proof of $\vdash \Gamma, \Pi$ in LL.

Corollary 1.1 (Weak $+$-focalization)
If $\vdash \Gamma$ is provable in LL, $\vdash \Gamma$ is provable with a weakly $+$-focalized proof.

PROOF: Starting from a proof π of $\vdash \Gamma$, we translate it into the proof $\pi^\bullet$ of $\vdash \Gamma\,;$ in LL_{foc}. Using the cut elimination property given in appendix A (corollary 5.2), this leads to a cut-free proof $\pi^{\bullet\prime}$ of $\vdash \Gamma\,;$ in LL_{foc}. By proposition 1, $(\pi^{\bullet\prime})^\circ$ is a weakly $+$-focalized proof of $\vdash \Gamma$ in LL. □

3 Focalized Linear Logic

3.1 Sequent calculus LL_{Foc}

In the spirit of [5], we define a sub-system LL_{Foc} of LL_{foc} in which the strictly negative formulas in the context of positive rules must be atomic.

There are two kinds of sequents: $\vdash \mathcal{P}, \mathcal{N}\,;$ and $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\,P$ where $\mathcal{X}^\perp$ contains only negative atoms. In order to simplify the notations we will write $\vdash \mathcal{P}, \mathcal{N}\,;\,\Pi$ for either a sequent with Π empty or a sequent with $\Pi = P$ and $\mathcal{N} = ?\Gamma, \mathcal{X}^\perp$.

$$\frac{}{\vdash X^\perp\,;\,X}\,ax \qquad\qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\,P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, P\,;}\,\mathit{foc}$$

$$\frac{\vdash \mathcal{P}, \mathcal{N}, A, B\,;}{\vdash \mathcal{P}, \mathcal{N}, A \parr B\,;}\,\parr$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}\,;\,P \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp\,;\,Q}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\,P \otimes Q}\,\otimes \qquad\qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}\,;\,P \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp, M\,;}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\,P \otimes M}\,\otimes$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}, N\,; \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^{\perp}\,; Q}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^{\perp}, \mathcal{X}'^{\perp}\,; N \otimes Q}\otimes \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}, N\,; \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^{\perp}, M\,;}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^{\perp}, \mathcal{X}'^{\perp}\,; N \otimes M}\otimes$$

$$\frac{\vdash \mathcal{P}, \mathcal{N}, A\,; \qquad \vdash \mathcal{P}, \mathcal{N}, B\,;}{\vdash \mathcal{P}, \mathcal{N}, A \,\&\, B\,;}\,\&$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; P \oplus B}\oplus_1 \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, N\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; N \oplus B}\oplus_1$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; Q}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; A \oplus Q}\oplus_2 \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, M\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; A \oplus M}\oplus_2$$

$$\frac{}{\vdash\,; 1}\,1 \qquad \frac{\vdash \mathcal{P}, \mathcal{N}\,;}{\vdash \mathcal{P}, \mathcal{N}, \perp\,;}\perp \qquad \frac{}{\vdash \mathcal{P}, \mathcal{N}, \top\,;}\top$$

$$\frac{\vdash ?\Gamma, A\,;}{\vdash ?\Gamma\,; !A}\,! \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?P\,;}\,?d \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, N\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?N\,;}\,?d$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A\,;}\,?w \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A, ?A\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A\,;}\,?c$$

3.2 Embedding of LL_{foc}

We embed proofs of LL_{foc} of sequents of the shape $\vdash \Gamma\,;$ into LL_{Foc}. We proceed in two steps by first showing that LL_{Foc} sequents (that is of the shape $\vdash \mathcal{P}, \mathcal{N}\,;$ or $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; P$) provable in LL_{foc} are provable in LL_{Foc} enriched with the two rules:

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,; P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A\,; P}\,?w \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A, ?A\,; P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}, ?A\,; P}\,?c$$

The size of a sequent $\vdash \Gamma\,;$ is the sum of the sizes of the formulas of Γ plus one. The size of a sequent $\vdash \Gamma\,; P$ is the sum of the sizes of the formulas of Γ plus the size of P.

We consider a cut-free proof with expanded axioms (according to section 2.2). We proceed by induction on the size of the final sequent. We look at the last rule of the proof. The non trivial cases are the *foc* rules and $?d$ rules on positive formulas (called a *positive* $?d$ rule), the other rules are immediately valid in LL_{Foc} (in our restricted case where their conclusion is an LL_{Foc} sequent).

We only consider the *foc* rule (the case of the positive $?d$ rule is very similar). If the context is of the shape $\mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}$, the rule is valid in LL_{Foc}. Otherwise this context contains a formula with main connective $⅋$, &, $\perp$ or $\top$:

- If we have a context of the shape $\Gamma, A ⅋ B$, using the required proof of section 2.2 without the dashed rule, we build:

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma, A ⅋ B\,; P} \qquad \vdash A, B\,; A^{\perp} \otimes B^{\perp}}{\vdash \Gamma, A, B\,; P}\,p\text{-}cut}{\vdash \Gamma, A, B, P\,;}\,foc$$

By induction hypothesis, after cut elimination and expansion of axioms in LL_{foc}, this proof gives a proof π' of LL_{Foc}. We conclude with:

$$\dfrac{\vdash \Gamma, A, B, P\,;}{\vdash \Gamma, A \parr B, P\,;}\ \parr$$

- If we have a context of the shape $\Gamma, A \mathbin{\&} B$, in the same way we build:

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma, A \mathbin{\&} B\,;\,P} \qquad \vdash A\,;\,A^\perp \oplus B^\perp}{\vdash \Gamma, A\,;\,P}\ p\text{-}cut}{\vdash \Gamma, A, P\,;}\ foc$$

and

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma, A \mathbin{\&} B\,;\,P} \qquad \vdash B\,;\,A^\perp \oplus B^\perp}{\vdash \Gamma, B\,;\,P}\ p\text{-}cut}{\vdash \Gamma, B, P\,;}\ foc$$

After cut elimination and expansion of axioms, we apply the induction hypothesis and we conclude with:

$$\dfrac{\vdash \Gamma, A, P\,; \qquad \vdash \Gamma, B, P\,;}{\vdash \Gamma, A \mathbin{\&} B, P\,;}\ \&$$

- If we have a context of the shape $\Gamma, \perp$, we build:

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma, \perp\,;\,P} \qquad \vdash\,;\,1}{\vdash \Gamma\,;\,P}\ p\text{-}cut}{\vdash \Gamma, P\,;}\ foc$$

After cut elimination and expansion of axioms, we apply the induction hypothesis and we conclude with:

$$\dfrac{\vdash \Gamma, P\,;}{\vdash \Gamma, \perp, P\,;}\ \perp$$

- If we have a context of the shape $\Gamma, \top$, we can immediately use the LL_{Foc} proof:

$$\dfrac{}{\vdash \Gamma, \top, P\,;}\ \top$$

We now have to eliminate the additional $?w$ and $?c$ rules. We show by induction on the size of the proof that if $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp} ; P$ (resp. $\vdash \mathcal{P}, \mathcal{N} ;$) is provable in $\mathsf{LL}_{\mathrm{Foc}}$ with the two additional rules then $\vdash \mathcal{P}, ?\Gamma', \mathcal{X}^{\perp} ; P$ (resp. $\vdash \mathcal{P}, \mathcal{N} ;$) is provable in $\mathsf{LL}_{\mathrm{Foc}}$ with $?\Gamma' \leq ?\Gamma$ (where $?\Gamma' \leq ?\Gamma$ means that for each formula $?A$ in $?\Gamma'$ there is at least an occurrence of $?A$ in $?\Gamma$, or equivalently that we have an inclusion of the underlying sets).

If the last rule is not one of the two additional rules and neither a *foc* rule nor a positive $?d$ rule, we translate the rule by itself. Otherwise:

- If the last rule is a *foc* rule or a positive $?d$ rule, we apply the corresponding rule and we apply the required $?w$ and $?c$ rules to move from $?\Gamma'$ to $?\Gamma$.
- In the case of the added $?w$ rule, by induction hypothesis, $\vdash \mathcal{P}, ?\Gamma', \mathcal{X}^{\perp} ; P$ is provable in $\mathsf{LL}_{\mathrm{Foc}}$ with $?\Gamma' \leq ?\Gamma$, thus we have the result since $?\Gamma' \leq ?\Gamma, ?A$.
- In the case of the added $?c$ rule, by induction hypothesis, $\vdash \mathcal{P}, ?\Gamma', \mathcal{X}^{\perp} ; P$ is provable in $\mathsf{LL}_{\mathrm{Foc}}$ with $?\Gamma' \leq ?\Gamma, ?A, ?A$, thus we have the result since $?\Gamma' \leq ?\Gamma, ?A$.

This shows that if $\vdash \Gamma ;$ is provable in $\mathsf{LL}_{\mathrm{foc}}$, then it is also provable in $\mathsf{LL}_{\mathrm{Foc}}$.

3.3 Focalization in $\mathsf{LL}_{\mathrm{Foc}}$

If π is a proof in $\mathsf{LL}_{\mathrm{Foc}}$ then π° is the LL proof obtained by erasing all the "$;$" in the sequents.

Proposition 2 (Cut-free focalization)
If π is a cut-free proof of $\vdash \mathcal{P}, \mathcal{N} ; \Pi$ in $\mathsf{LL}_{\mathrm{Foc}}$ then π° is a focalized proof of $\vdash \mathcal{P}, \mathcal{N}, \Pi$ in LL.

Corollary 2.1 (Focalization)
If $\vdash \Gamma$ is provable in LL, $\vdash \Gamma$ is provable with a focalized proof.

PROOF: Starting from a proof π of $\vdash \Gamma$, we translate it into the proof $\pi^{\bullet}$ of $\vdash \Gamma ;$ in $\mathsf{LL}_{\mathrm{foc}}$ and then into a cut-free proof π' of $\vdash \Gamma ;$ in $\mathsf{LL}_{\mathrm{Foc}}$. By proposition 2, π'° is a focalized proof of $\vdash \Gamma$ in LL. ☐

Remark: It is possible to define the embedding of LL proofs into $\mathsf{LL}_{\mathrm{Foc}}$ directly (without using $\mathsf{LL}_{\mathrm{foc}}$ as an intermediary step). We have chosen to decompose it into two steps in order to show that the key property is *weak +-focalization*. Strong +-focalization is then obtained through reversing which is in general easy to do.

4 Additional remarks

4.1 Decomposition of exponentials

The attentive reader has certainly remarked that an hidden decomposition of the exponential connectives underlies the whole text (as suggested by Girard [5]):

$$!A = \downarrow\sharp A \qquad\qquad ?A = \uparrow\flat A$$

with $\sharp A$ negative and $\flat A$ positive, and $\downarrow$ and $\uparrow$ are used as the same connectives of $\mathsf{LL}_{\mathrm{pol}}^{\uparrow\downarrow}$ (see appendix A.2.1).

The corresponding rules in LL_{foc} would be:

$$\frac{\vdash ?\Gamma, A\,;\,(\Pi)}{\vdash ?\Gamma, \sharp A\,;\,(\Pi)}\,\sharp \qquad \frac{\vdash \Gamma\,;\,P}{\vdash \Gamma\,;\,\flat P}\,\flat \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma\,;\,\flat N}\,\flat$$

$$\frac{\vdash \Gamma, A\,;}{\vdash \Gamma\,;\,\downarrow A}\,\downarrow \qquad \frac{\vdash \Gamma\,;\,P}{\vdash \Gamma, \uparrow P\,;}\,\uparrow \qquad \frac{\vdash \Gamma, N\,;\,\Pi}{\vdash \Gamma, \uparrow N\,;\,\Pi}\,\uparrow$$

However it is difficult to give a meaning to $\sharp A$ (resp. $\flat A$) under any other connective than $\downarrow$ (resp. $\uparrow$).

4.2 Quantifiers

Our method can perfectly be extended to quantification (of any order) by adding $\exists\alpha A$ (resp. $\forall\alpha A$) in positive (resp. negative) formulas and the following rules in LL_{foc}:

$$\frac{\vdash \Gamma\,;\,P}{\vdash \Gamma\,;\,\exists\alpha P}\,\exists \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma\,;\,\exists\alpha N}\,\exists \qquad \frac{\vdash \Gamma, A\,;\,\Pi}{\vdash \Gamma, \forall\alpha A\,;\,\Pi}\,\forall$$

with α free neither in Γ nor in Π in the $\forall$ rule.

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A Cut elimination in LL_{foc}

A.1 Cut elimination steps

Key steps

$$\dfrac{\dfrac{}{\vdash P^\perp ; P}\,ax \qquad \vdash \Gamma, P^\perp ; \Pi}{\vdash \Gamma, P^\perp ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \vdash \Gamma, P^\perp ; \Pi$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{}{\vdash P^\perp ; P}\,ax}{\vdash \Gamma ; P}\,p\text{-}cut \quad \rightsquigarrow \quad \vdash \Gamma ; P$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, P ;}\,foc \qquad \vdash \Delta, P^\perp ;}{\vdash \Gamma, \Delta, P^\perp ;}\,n\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ;}{\vdash \Gamma, \Delta, P^\perp ;}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta ; P \otimes Q}\,\otimes \qquad \dfrac{\vdash \Xi, P^\perp, Q^\perp ; \Pi}{\vdash \Xi, P^\perp \parr Q^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta ; Q \qquad \vdash \Xi, P^\perp, Q^\perp ; \Pi}{\vdash \Delta, \Xi, P^\perp ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, M ;}{\vdash \Gamma, \Delta ; P \otimes M}\,\otimes \qquad \dfrac{\vdash \Xi, P^\perp, M^\perp ; \Pi}{\vdash \Xi, P^\perp \parr M^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, M ; \qquad \vdash \Xi, P^\perp, M^\perp ; \Pi}{\vdash \Delta, \Xi, P^\perp ; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N ; \qquad \vdash \Delta, M ;}{\vdash \Gamma, \Delta ; N \otimes M}\,\otimes \qquad \dfrac{\vdash \Xi, N^\perp, M^\perp ; \Pi}{\vdash \Xi, N^\perp \parr M^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma, N ; \qquad \dfrac{\vdash \Delta, M ; \qquad \vdash \Xi, N^\perp, M^\perp ; \Pi}{\vdash \Delta, \Xi, N^\perp ; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, \Xi ; \Pi}\,n\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma ; P \oplus B}\,\oplus_1 \qquad \dfrac{\vdash \Delta, P^\perp ; \Pi \qquad \vdash \Delta, B^\perp ; \Pi}{\vdash \Delta, P^\perp \mathbin{\&} B^\perp ; \Pi}\,\&}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N ;}{\vdash \Gamma ; N \oplus B}\,\oplus_1 \qquad \dfrac{\vdash \Delta, N^\perp ; \Pi \qquad \vdash \Delta, B^\perp ; \Pi}{\vdash \Delta, N^\perp \mathbin{\&} B^\perp ; \Pi}\,\&}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, N ; \qquad \vdash \Delta, N^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,n\text{-}cut$$

$$\dfrac{\dfrac{}{\vdash ; 1}\,1 \qquad \dfrac{\vdash \Gamma ; \Pi}{\vdash \Gamma, \bot ; \Pi}\,\bot}{\vdash \Gamma ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \vdash \Gamma ; \Pi$$

$$\dfrac{\dfrac{\vdash ?\Gamma, N ;}{\vdash ?\Gamma ; !N}\,! \qquad \dfrac{\vdash \Delta ; N^{\perp}}{\vdash \Delta, ?N^{\perp} ;}\,?d}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\vdash ?\Gamma, N ; \qquad \vdash \Delta ; N^{\perp}}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, P ;}{\vdash ?\Gamma ; !P}\,! \qquad \dfrac{\vdash \Delta, P^{\perp} ;}{\vdash \Delta, ?P^{\perp} ;}\,?d}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\vdash ?\Gamma, P ; \qquad \vdash \Delta, P^{\perp} ;}{\vdash ?\Gamma, \Delta ;}\,n\text{-}cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\vdash \Delta ; \Pi}{\vdash \Delta, ?A^{\perp} ; \Pi}\,?w}{\vdash ?\Gamma, \Delta ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\vdash \Delta ; \Pi}{\vdash ?\Gamma, \Delta ; \Pi}\,?w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\vdash \Delta, ?A^{\perp}, ?A^{\perp} ; \Pi}{\vdash \Delta, ?A^{\perp} ; \Pi}\,?c}{\vdash ?\Gamma, \Delta ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \vdash \Delta, ?A^{\perp}, ?A^{\perp} ; \Pi}{\vdash ?\Gamma, \Delta, ?A^{\perp} ; \Pi}\,p\text{-}cut}{\vdash ?\Gamma, ?\Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash ?\Gamma, \Delta ; \Pi}\,?c$$

Left commutative p-steps

$$\dfrac{\dfrac{\vdash \Gamma, A, B ; P}{\vdash \Gamma, A \parr B ; P}\,\parr \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A, B ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, A, B ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,\parr$$

$$\dfrac{\dfrac{\vdash \Gamma, A ; P \qquad \vdash \Gamma, B ; P}{\vdash \Gamma, A \,\&\, B ; P}\,\& \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, A \,\&\, B ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, A ; \Pi}\,p\text{-}cut \qquad \dfrac{\vdash \Gamma, B ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, B ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \,\&\, B ; \Pi}\,\&$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, \bot ; P}\,\bot \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, \bot ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, \bot ; \Pi}\,\bot$$

$$\dfrac{\dfrac{}{\vdash \Gamma, \top ; P}\,\top \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, \top ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{}{\vdash \Gamma, \Delta, \top ; \Pi}\,\top$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, ?A ; P}\,?w \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta, ?A ; \Pi}\,p\text{-}cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?w$$

$$\dfrac{\dfrac{\vdash \Gamma, ?A, ?A\,;\, P}{\vdash \Gamma, ?A\,;\, P}\,?c \qquad \vdash \Delta, P^{\perp}\,;\, \Pi}{\vdash \Gamma, \Delta, ?A\,;\, \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, ?A, ?A\,;\, P \qquad \vdash \Delta, P^{\perp}\,;\, \Pi}{\vdash \Gamma, \Delta, ?A, ?A\,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A\,;\, \Pi}\,?c$$

Right commutative p-steps

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}\,;\, P'}{\vdash \Delta, P^{\perp}, P'\,;}\,foc}{\vdash \Gamma, \Delta, P'\,;}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}\,;\, P'}{\vdash \Gamma, \Delta\,;\, P'}\,p\text{-}cut}{\vdash \Gamma, \Delta, P'\,;}\,foc$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, A, B, P^{\perp}\,;\, \Pi}{\vdash \Delta, A \parr B, P^{\perp}\,;\, \Pi}\,\parr}{\vdash \Gamma, \Delta, A \parr B\,;\, \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, A, B, P^{\perp}\,;\, \Pi}{\vdash \Gamma, \Delta, A, B\,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \parr B\,;\, \Pi}\,\parr$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}\,;\, P' \qquad \vdash \Xi\,;\, Q}{\vdash \Delta, \Xi, P^{\perp}\,;\, P' \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Xi\,;\, P' \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}\,;\, P'}{\vdash \Gamma, \Delta\,;\, P'}\,p\text{-}cut \qquad \vdash \Xi\,;\, Q}{\vdash \Gamma, \Delta, \Xi\,;\, P' \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N\,; \qquad \vdash \Xi\,;\, Q}{\vdash \Delta, \Xi, P^{\perp}\,;\, N \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}, N\,;}{\vdash \Gamma, \Delta, N\,;}\,p\text{-}cut \qquad \vdash \Xi\,;\, Q}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, N\,; \qquad \vdash \Xi, P^{\perp}\,;\, Q}{\vdash \Delta, \Xi, P^{\perp}\,;\, N \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\vdash \Delta, N\,; \qquad \dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Xi, P^{\perp}\,;\, Q}{\vdash \Gamma, \Xi\,;\, Q}\,p\text{-}cut}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N\,; \qquad \vdash \Xi, M\,;}{\vdash \Delta, \Xi, P^{\perp}\,;\, N \otimes M}\,\otimes}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes M}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}, N\,;}{\vdash \Gamma, \Delta, N\,;}\,p\text{-}cut \qquad \vdash \Xi, M\,;}{\vdash \Gamma, \Delta, \Xi\,;\, N \otimes M}\,\otimes$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, A\,;\, \Pi \qquad \vdash \Delta, P^{\perp}, B\,;\, \Pi}{\vdash \Delta, P^{\perp}, A \,\&\, B\,;\, \Pi}\,\&}{\vdash \Gamma, \Delta, A \,\&\, B\,;\, \Pi}\,p\text{-}cut$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}, A\,;\, \Pi}{\vdash \Gamma, \Delta, A\,;\, \Pi}\,p\text{-}cut \qquad \dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}, B\,;\, \Pi}{\vdash \Gamma, \Delta, B\,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \,\&\, B\,;\, \Pi}\,\&$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}\,;\, P'}{\vdash \Delta, P^{\perp}\,;\, P' \oplus B}\,\oplus_1}{\vdash \Gamma, \Delta\,;\, P' \oplus B}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}\,;\, P'}{\vdash \Gamma, \Delta\,;\, P'}\,p\text{-}cut}{\vdash \Gamma, \Delta\,;\, P' \oplus B}\,\oplus_1$$

$$\dfrac{\vdash \Gamma\,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N\,;}{\vdash \Delta, P^{\perp}\,;\, N \oplus B}\,\oplus_1}{\vdash \Gamma, \Delta\,;\, N \oplus B}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma\,;\, P \qquad \vdash \Delta, P^{\perp}, N\,;}{\vdash \Gamma, \Delta, N\,;}\,p\text{-}cut}{\vdash \Gamma, \Delta\,;\, N \oplus B}\,\oplus_1$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; \Pi}{\vdash \Delta, P^{\perp}, \bot ; \Pi}\,\bot}{\vdash \Gamma, \Delta, \bot ; \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, \bot ; \Pi}\,\bot$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{}{\vdash \Delta, \top, P^{\perp} ; \Pi}\,\top}{\vdash \Gamma, \Delta, \top ; \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{}{\vdash \Gamma, \Delta, \top ; \Pi}\,\top$$

$$\dfrac{\vdash ?\Gamma ; !A \qquad \dfrac{\vdash ?\Delta, ?A^{\perp}, B ;}{\vdash ?\Delta, ?A^{\perp} ; !B}\,!}{\vdash ?\Gamma, ?\Delta ; !B}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash ?\Gamma ; !A \qquad \vdash ?\Delta, ?A^{\perp}, B ;}{\vdash ?\Gamma, ?\Delta, B ;}\,p\text{-}cut}{\vdash ?\Gamma, ?\Delta ; !B}\,!$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; P'}{\vdash \Delta, P^{\perp}, ?P' ;}\,?d}{\vdash \Gamma, \Delta, ?P' ;}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; P'}{\vdash \Gamma, \Delta ; P'}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?P' ;}\,?d$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp}, N ;}{\vdash \Delta, P^{\perp}, ?N ;}\,?d}{\vdash \Gamma, \Delta, ?N ;}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp}, N ;}{\vdash \Gamma, \Delta, N ;}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?N ;}\,?d$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; \Pi}{\vdash \Delta, P^{\perp}, ?A ; \Pi}\,?w}{\vdash \Gamma, \Delta, ?A ; \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?w$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp}, ?A, ?A ; \Pi}{\vdash \Delta, P^{\perp}, ?A ; \Pi}\,?c}{\vdash \Gamma, \Delta, ?A ; \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp}, ?A, ?A ; \Pi}{\vdash \Gamma, \Delta, ?A, ?A ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?c$$

Commutative n-steps

$$\dfrac{\dfrac{\vdash \Gamma, P ; P'}{\vdash \Gamma, P, P' ;}\,foc \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, P' ;}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta ; P'}\,n\text{-}cut}{\vdash \Gamma, \Delta, P' ;}\,foc$$

$$\dfrac{\dfrac{\vdash \Gamma, A, B, P ; \Pi}{\vdash \Gamma, A \parr B, P ; \Pi}\,\parr \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A, B, P ; \Pi \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, A, B ; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,\parr$$

$$\dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, P ; P' \otimes Q}\,\otimes \qquad \vdash \Xi, P^{\perp} ;}{\vdash \Gamma, \Delta, \Xi ; P' \otimes Q}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Xi, P^{\perp} ;}{\vdash \Gamma, \Xi ; P'}\,n\text{-}cut \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, \Xi ; P' \otimes Q}\,\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N ; \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, P ; N \otimes Q}\,\otimes \qquad \vdash \Xi, P^{\perp} ;}{\vdash \Gamma, \Delta, \Xi ; N \otimes Q}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, P, N ; \qquad \vdash \Xi, P^{\perp} ;}{\vdash \Gamma, \Xi, N ;}\,n\text{-}cut \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, \Xi ; N \otimes Q}\,\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, N \,; \qquad \vdash \Delta, P \,; Q}{\vdash \Gamma, \Delta, P \,; N \otimes Q}\otimes \qquad \vdash \Xi, P^{\perp} \,;}{\vdash \Gamma, \Delta, \Xi \,; N \otimes Q}\,n\text{-}cut \quad\leadsto\quad \dfrac{\vdash \Gamma, N \,; \qquad \dfrac{\vdash \Delta, P \,; Q \qquad \vdash \Xi, P^{\perp} \,;}{\vdash \Delta, \Xi \,; Q}\,n\text{-}cut}{\vdash \Gamma, \Delta, \Xi \,; N \otimes Q}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N \,; \qquad \vdash \Delta, M \,;}{\vdash \Gamma, \Delta, P \,; N \otimes M}\otimes \qquad \vdash \Xi, P^{\perp} \,;}{\vdash \Gamma, \Delta, \Xi \,; N \otimes M}\,n\text{-}cut$$

$$\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P, N \,; \qquad \vdash \Xi, P^{\perp} \,;}{\vdash \Gamma, \Xi, N \,;}\,n\text{-}cut \qquad \vdash \Delta, M \,;}{\vdash \Gamma, \Delta, \Xi \,; N \otimes M}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, A \,; \Pi \qquad \vdash \Gamma, P, B \,; \Pi}{\vdash \Gamma, P, A \,\&\, B \,; \Pi}\,\& \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, A \,\&\, B \,; \Pi}\,n\text{-}cut$$

$$\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P, A \,; \Pi \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, A \,; \Pi}\,n\text{-}cut \qquad \dfrac{\vdash \Gamma, P, B \,; \Pi \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, B \,; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, A \,\&\, B \,; \Pi}\,\&$$

$$\dfrac{\dfrac{\vdash \Gamma, P \,; P'}{\vdash \Gamma, P \,; P' \oplus B}\oplus_1 \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; P' \oplus B}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P \,; P' \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; P'}\,n\text{-}cut}{\vdash \Gamma, \Delta \,; P' \oplus B}\oplus_1$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N \,;}{\vdash \Gamma, P \,; N \oplus B}\oplus_1 \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; N \oplus B}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P, N \,; \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, N \,;}\,n\text{-}cut}{\vdash \Gamma, \Delta \,; N \oplus B}\oplus_1$$

$$\dfrac{\dfrac{\vdash \Gamma, P \,; \Pi}{\vdash \Gamma, P, \bot \,; \Pi}\bot \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, \bot \,; \Pi}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P \,; \Pi \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, \bot \,; \Pi}\bot$$

$$\dfrac{\dfrac{}{\vdash \Gamma, P, \top \,; \Pi}\top \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, \top \,; \Pi}\,n\text{-}cut \quad\leadsto\quad \dfrac{}{\vdash \Gamma, \Delta, \top \,; \Pi}\top$$

$$\dfrac{\dfrac{\vdash \Gamma, P \,; P'}{\vdash \Gamma, P, ?P' \,;}\,?d \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, ?P' \,;}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P \,; P' \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; P'}\,n\text{-}cut}{\vdash \Gamma, \Delta, ?P' \,;}\,?d$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N \,;}{\vdash \Gamma, P, ?N \,;}\,?d \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, ?N \,;}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P, N \,; \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, N \,;}\,n\text{-}cut}{\vdash \Gamma, \Delta, ?N \,;}\,?d$$

$$\dfrac{\dfrac{\vdash \Gamma, P \,; \Pi}{\vdash \Gamma, P, ?A \,; \Pi}\,?w \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, ?A \,; \Pi}\,n\text{-}cut \quad\leadsto\quad \dfrac{\dfrac{\vdash \Gamma, P \,; \Pi \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta \,; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, ?A \,; \Pi}\,?w$$

$$\dfrac{\dfrac{\vdash \Gamma, P, ?A, ?A\,;\,\Pi}{\vdash \Gamma, P, ?A\,;\,\Pi}\,?c \qquad \vdash \Delta, P^{\perp}\,;}{\vdash \Gamma, \Delta, ?A\,;\,\Pi}\,n\text{-}cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, P, ?A, ?A\,;\,\Pi \qquad \vdash \Delta, P^{\perp}\,;}{\vdash \Gamma, \Delta, ?A, ?A\,;\,\Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, ?A\,;\,\Pi}\,?c$$

A.2 Cut elimination property

A.2.1 $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$

The system $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ is obtained from LL by "restricting" formulas to the following grammar:

$$\begin{array}{rcccccccccccccc} P & ::= & X & \mid & P \otimes P & \mid & P \oplus P & \mid & 1 & \mid & 0 & \mid & !N & \mid & \downarrow N \\ N & ::= & X^{\perp} & \mid & N \parr N & \mid & N \,\&\, N & \mid & \perp & \mid & \top & \mid & ?P & \mid & \uparrow P \end{array}$$

with the following rules for the $\uparrow$ and $\downarrow$ connectives:

$$\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\downarrow \qquad \dfrac{\vdash \Gamma, P}{\vdash \Gamma, \uparrow P}\uparrow$$

where $\mathcal{N}$ is a multi-set of negative formulas.

A.2.2 LLP

The system LLP is obtained from LL by restricting formulas to the following grammar:

$$\begin{array}{rcccccccccccc} P & ::= & X & \mid & P \otimes P & \mid & P \oplus P & \mid & 1 & \mid & 0 & \mid & !N \\ N & ::= & X^{\perp} & \mid & N \parr N & \mid & N \,\&\, N & \mid & \perp & \mid & \top & \mid & ?P \end{array}$$

with the following generalizations of the exponential rules:

$$\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \dfrac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,?d \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, N}\,?w \qquad \dfrac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\,?c$$

where $\mathcal{N}$ is a multi-set of negative formulas.

Proposition 3 (Strong normalization)
There is no infinite sequence of reductions in LLP *if we forbid commutations of cuts.*

PROOF: Such a sequence of reductions can only contain finitely many steps between two steps that correspond to a reduction step in proof-nets. Thus by strong normalization for proof-nets [6] we can conclude. □

Corollary 3.1 (Cut elimination)
If $\vdash \Gamma$ *is provable in* LLP, *then* $\vdash \Gamma$ *is provable without the cut rule.*

A.2.3 Simulations

The translation $(.)^!$ of $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ into LLP is obtained by replacing $\downarrow N$ by $!N$ and $\uparrow P$ by $?P$ and the two lifting rules by promotion and dereliction.

Lemma 1 (Polarized formulas)
If A is a positive (resp. negative) formula in $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ *then $A^!$ is a positive (resp. negative) formula in* LLP.

Proposition 4 (One-to-one simulation)
If π reduces to π' in $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ by one step of reduction then $\pi^!$ reduces to $\pi'^!$ in LLP by one step of reduction.

Corollary 4.1 (Strong normalization)
There is no infinite sequence of reductions in $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ if we forbid commutations of cuts.

Corollary 4.2 (Cut elimination)
If $\vdash \Gamma$ is provable in $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$, then $\vdash \Gamma$ is provable without the cut rule.

The translation $(.)^{\uparrow}$ of LL_{foc} into $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ is obtained by adding to formulas exactly the required liftings to get a polarized formula, and by translating the sequent $\vdash \mathcal{P}, \mathcal{N} \,;\, \Pi$ by $\vdash \uparrow\mathcal{P}^{\uparrow}, \mathcal{N}^{\uparrow}, \Pi^{\uparrow}$.

In particular $(!P \parr 1)^{\uparrow} = \uparrow!\uparrow P^{\uparrow} \parr \uparrow 1$.

Proposition 5 (Strict simulation)
If π reduces to π' in LL_{foc} by one step of reduction then $\pi^{\uparrow}$ reduces to $\pi'^{\uparrow}$ in $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ by at least one step of reduction.

Corollary 5.1 (Strong normalization)
There is no infinite sequence of reductions in LL_{foc} if we forbid commutations of cuts.

Corollary 5.2 (Cut elimination)
If $\vdash \Gamma \,;\, \Pi$ is provable in LL_{foc}, then $\vdash \Gamma \,;\, \Pi$ is provable without the cut rules.

PROOF: We consider a cut rule without any cut above it. We look at the two different cases:

- If it is a *n-cut*, we look at the rule above the premise $\vdash \Gamma, P \,;\, \Pi$. If the rule above it introduces P, it is either a *foc* rule or a $\top$ rule and we apply the corresponding key step (and the *n-cut* becomes a *p-cut*) or commutative n-step. Otherwise this rule cannot be an *ax* rule, a 1 rule or a ! rule and we can apply the corresponding commutative n-step.
- If it is a *p-cut*, we first look at the premise $\vdash \Gamma \,;\, P$. If P is not a main formula, we can apply a left commutative p-step. If P is a main formula and $P^{\perp}$ is not, we can apply the corresponding right commutative p-step (notice that the rule above $P^{\perp}$ cannot be a 1 rule). We just have to verify that we can apply the right commutative p-step in the case of a ! rule above $P^{\perp}$: since P is main, the rule above it is either an *ax* rule or a ! rule and we can apply the reduction step. If both P and $P^{\perp}$ are main, we apply the corresponding key step.

So that, either the proof is cut-free or a reduction step can be applied, and we conclude by strong normalization. □