# A Proof of the Focusing Property of Linear Logic

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The focusing property of linear logic has been discovered by J.-M. Andreoli [And90, And92] in the beginning of the 90's. It is one of the main properties of Linear Logic that appeared after the original paper of J.-Y. Girard [Gir87]. This property is proved in various papers [And90, Gir91, DJS97][1] but, as far as we know never for the usual $\mathsf{LL}$ sequent calculus (or with an intricate induction in Andreoli's thesis, which is replaced here by a cut elimination property).

The proof we give is a compilation of the previous proofs and proof techniques, our goal is just to give a complete presentation of a proof of this property which appears to be more and more important in the current research in Linear Logic [Gir91, DJS97, Gir01, Lau02].

We decompose the usual focusing property into two technically different steps: weak focusing (the decompositions of two positive formulas are not interleaved, see $\mathsf{LL}_{\mathrm{foc}}$) and reversing (negative rules are applied as late as possible from a top-down point of view, see $\mathsf{LL}_{\mathrm{Foc}}$).

## 1 Linear Logic and the Focusing Property

Formulas of (propositional) Linear Logic are given by the usual grammar:

$$\begin{array}{rcccccccccccccc} A & ::= & X & \mid & A \otimes A & \mid & A \oplus A & \mid & 1 & \mid & 0 & \mid & !A \\ & \mid & X^{\perp} & \mid & A \parr A & \mid & A \,\&\, A & \mid & \perp & \mid & \top & \mid & ?A \end{array}$$

We split this grammar into two sub-classes: positive formulas and negative formulas.

$$\begin{array}{rcccccccccccccc} P & ::= & X & \mid & A \otimes A & \mid & A \oplus A & \mid & 1 & \mid & 0 & \mid & !A \\ N & ::= & X^{\perp} & \mid & A \parr A & \mid & A \,\&\, A & \mid & \perp & \mid & \top & \mid & ?A \end{array}$$

In the sequel:

- $A$, $B$, ... denote arbitrary formulas;
- $P$, $Q$, $R$ denote positive formulas;
- $N$, $M$, $L$ denote negative formulas;

[1] Around the time we were writing the first version of this note, A. Saurin also worked on focusing with a different approach. We take the opportunity of this revision to add a reference to his work [MS07].

- $X$, $X^{\perp}$, ... are called *atoms* or *atomic formulas*.

We consider the usual rules of Linear Logic [Gir87]:

$$\frac{}{\vdash A^{\perp}, A}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2$$

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c$$

The main connectives of positive (resp. negative) formulas are called *positive connectives* (resp. *negative connectives*), that is $X$, $\otimes$, $\oplus$, 1, 0 and ! (resp. $X^{\perp}$, $\parr$, &, $\bot$, $\top$ and ?). The rules introducing positive (resp. negative) connectives are called *positive rules* (resp. *negative rules*).

A positive (resp. negative) formula is *strictly positive* (resp. *strictly negative*) if its main connective is not ! (resp. ?). A positive (resp. negative) rule is *strictly positive* (resp. *strictly negative*) if it is not introducing a $!A$ formula (resp. $?A$ formula).

**Definition 1** (Main Positive Tree)
If $A$ is a formula, its *main positive tree* $\mathcal{T}^+(A)$ is defined by:

$$\begin{aligned}
\mathcal{T}^+(N) &= \emptyset \\
\mathcal{T}^+(X) &= X \\
\mathcal{T}^+(A \otimes B) &= \mathcal{T}^+(A) \otimes \mathcal{T}^+(B) \\
\mathcal{T}^+(A \oplus B) &= \mathcal{T}^+(A) \oplus \mathcal{T}^+(B) \\
\mathcal{T}^+(1) &= 1 \\
\mathcal{T}^+(0) &= 0 \\
\mathcal{T}^+(!A) &= !
\end{aligned}$$

$\mathcal{T}^+(A)$ is a (possibly empty) tree whose nodes are $\otimes$, $\oplus$ (with arity at most 2) and !, 1, 0 or $X$ (with arity 0).

**Definition 2** (Weakly +-Focused Proof)
A proof $\pi$ in LL is *weakly +-focused* if it is cut-free and, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, A$, the only *positive* rules of $\pi'$ between two rules introducing connectives of $\mathcal{T}^+(A)$ are rules introducing connectives of $\mathcal{T}^+(A)$.

**Definition 3** (Strongly +-Focused Proof)
A proof $\pi$ in LL is *strongly +-focused* if it is cut-free and, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, A$, the only rules of $\pi'$ between two rules introducing connectives of $\mathcal{T}^+(A)$ are rules introducing connectives of $\mathcal{T}^+(A)$.

**Definition 4** (Reversed Proof)
A proof $\pi$ in $\mathsf{LL}$ is *reversed* if, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, N$ with $N$ a strictly negative non-atomic formula, the last rule of $\pi'$ is a strictly negative rule (*i.e.* a $\parr$, $\&$, $\bot$ or $\top$ rule).

**Definition 5** (Focused Proof)
A proof $\pi$ in $\mathsf{LL}$ is *focused* if it is both strongly $+$-focused and reversed.

**Remark:** Various systems for classical logic related with focusing constraints have been introduced. In particular Girard's $\mathsf{LC}$ [Gir91] corresponds to weak $+$-focusing, Danos-Joinet-Schellinx's $\mathsf{LK}^{\eta}$ [DJS97] corresponds to strong $+$-focusing and Quatrini-Tortora de Falco's $\mathsf{LK}^{\eta;\rho}_{\text{pol}}$ [QTdF96] corresponds to focusing since their $\rho$-constraint is a reversing constraint.

# 2 Weakly Focused Linear Logic

## 2.1 Sequent Calculus $\mathsf{LL}_{\text{foc}}$

A sequent of $\mathsf{LL}_{\text{foc}}$ has the shape $\vdash \Gamma\,;\Pi$ where $\Gamma$ is a multi-set of formulas and $\Pi$ is either empty or contains a unique positive formula.

The rules are a linear logic version of the rules of Girard's $\mathsf{LC}$ [Gir91]:

$$\frac{}{\vdash P^{\perp}\,;P}\,ax \qquad \frac{\vdash \Gamma\,;P}{\vdash \Gamma, P\,;}\,foc$$

$$\frac{\vdash \Gamma\,;P \qquad \vdash \Delta, P^{\perp}\,;\Pi}{\vdash \Gamma, \Delta\,;\Pi}\,p\text{-}cut \qquad \frac{\vdash \Gamma, P\,;\Pi \qquad \vdash \Delta, P^{\perp}\,;}{\vdash \Gamma, \Delta\,;\Pi}\,n\text{-}cut$$

$$\frac{\vdash \Gamma, A, B\,;\Pi}{\vdash \Gamma, A \parr B\,;\Pi}\,\parr \qquad \frac{\vdash \Gamma\,;P \qquad \vdash \Delta\,;Q}{\vdash \Gamma, \Delta\,;P \otimes Q}\,\otimes$$

$$\frac{\vdash \Gamma\,;P \qquad \vdash \Delta, M\,;}{\vdash \Gamma, \Delta\,;P \otimes M}\,\otimes \qquad \frac{\vdash \Gamma, N\,; \qquad \vdash \Delta\,;Q}{\vdash \Gamma, \Delta\,;N \otimes Q}\,\otimes \qquad \frac{\vdash \Gamma, N\,; \qquad \vdash \Delta, M\,;}{\vdash \Gamma, \Delta\,;N \otimes M}\,\otimes$$

$$\frac{\vdash \Gamma, A\,;\Pi \qquad \vdash \Gamma, B\,;\Pi}{\vdash \Gamma, A \,\&\, B\,;\Pi}\,\&$$

$$\frac{\vdash \Gamma\,;P}{\vdash \Gamma\,;P \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma\,;N \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma\,;Q}{\vdash \Gamma\,;A \oplus Q}\,\oplus_2 \qquad \frac{\vdash \Gamma, M\,;}{\vdash \Gamma\,;A \oplus M}\,\oplus_2$$

$$\frac{}{\vdash\,;1}\,1 \qquad \frac{\vdash \Gamma\,;\Pi}{\vdash \Gamma, \bot\,;\Pi}\,\bot \qquad \frac{}{\vdash \Gamma, \top\,;\Pi}\,\top$$

$$\frac{\vdash ?\Gamma, A\,;}{\vdash ?\Gamma\,;!A}\,! \qquad \frac{\vdash \Gamma\,;P}{\vdash \Gamma, ?P\,;}\,?d \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma, ?N\,;}\,?d$$

$$\frac{\vdash \Gamma\,;\Pi}{\vdash \Gamma, ?A\,;\Pi}\,?w \qquad \frac{\vdash \Gamma, ?A, ?A\,;\Pi}{\vdash \Gamma, ?A\,;\Pi}\,?c$$

### 2.2 Expansion of Axioms

Given a positive formula $P$, the sequent $\vdash P^\perp ; P$ is provable in $\mathsf{LL}_{\text{foc}}$ by means of the *ax* rule. Moreover, if $P$ is not atomic, it is also possible to give a proof based only on *ax* rules applied to the sub-formulas of $P$ (and thus, by an easy induction, to the atomic sub-formulas of $P$):

$$\dfrac{\dfrac{\vdash P^\perp ; P \qquad \vdash Q^\perp ; Q}{\vdash P^\perp, Q^\perp ; P \otimes Q}\otimes}{\vdash P^\perp \parr Q^\perp ; P \otimes Q}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\mathit{foc} \qquad \vdash Q^\perp ; Q}{\vdash N^\perp, Q^\perp ; N \otimes Q}\otimes}{\vdash N^\perp \parr Q^\perp ; N \otimes Q}\parr$$

$$\dfrac{\dfrac{\vdash P^\perp ; P \qquad \dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\mathit{foc}}{\vdash P^\perp, M^\perp ; P \otimes M}\otimes}{\vdash P^\perp \parr M^\perp ; P \otimes M}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\mathit{foc} \qquad \dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\mathit{foc}}{\vdash N^\perp, M^\perp ; N \otimes M}\otimes}{\vdash N^\perp \parr M^\perp ; N \otimes M}\parr$$

$$\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp ; P \oplus Q}\oplus_1 \qquad \dfrac{\vdash Q^\perp ; Q}{\vdash Q^\perp ; P \oplus Q}\oplus_2}{\vdash P^\perp \,\&\, Q^\perp ; P \oplus Q}\& \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\mathit{foc}}{\vdash N^\perp ; N \oplus Q}\oplus_1 \qquad \dfrac{\vdash Q^\perp ; Q}{\vdash Q^\perp ; N \oplus Q}\oplus_2}{\vdash N^\perp \,\&\, Q^\perp ; N \oplus Q}\&$$

$$\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp ; P \oplus M}\oplus_1 \qquad \dfrac{\dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\mathit{foc}}{\vdash M^\perp ; P \oplus M}\oplus_2}{\vdash P^\perp \,\&\, M^\perp ; P \oplus M}\& \qquad \dfrac{\dfrac{\dfrac{\vdash N ; N^\perp}{\vdash N, N^\perp ;}\mathit{foc}}{\vdash N^\perp ; N \oplus M}\oplus_1 \qquad \dfrac{\dfrac{\vdash M ; M^\perp}{\vdash M, M^\perp ;}\mathit{foc}}{\vdash M^\perp ; N \oplus M}\oplus_2}{\vdash N^\perp \,\&\, M^\perp ; N \oplus M}\&$$

$$\dfrac{\dfrac{}{\vdash ; 1}1}{\vdash \bot ; 1}\bot \qquad \dfrac{}{\vdash \top ; 0}\top$$

$$\dfrac{\dfrac{\dfrac{\vdash P^\perp ; P}{\vdash P^\perp, P ;}\mathit{foc}}{\vdash ?P^\perp, P ;}?d}{\vdash ?P^\perp ; !P}! \qquad \dfrac{\dfrac{\vdash N ; N^\perp}{\vdash ?N^\perp, N ;}?d}{\vdash ?N^\perp ; !N}!$$

Similarly it is possible to restrict the application of the $\top$ rule to the case where the context $\Gamma$ does not contain any formula of the shape $A \parr B$, $A \,\&\, B$ or $\bot$. This is proved by induction on the sum of the sizes of formulas in $\Gamma$, using the expansions:

$$\dfrac{\dfrac{}{\vdash \Gamma, A, B, \top ; \Pi}\top}{\vdash \Gamma, A \parr B, \top ; \Pi}\parr \qquad \dfrac{\dfrac{}{\vdash \Gamma, A, \top ; \Pi}\top \qquad \dfrac{}{\vdash \Gamma, B, \top ; \Pi}\top}{\vdash \Gamma, A \,\&\, B, \top ; \Pi}\& \qquad \dfrac{\dfrac{}{\vdash \Gamma, \top ; \Pi}\top}{\vdash \Gamma, \bot, \top ; \Pi}\bot$$

**Lemma 1** (Reversibility)
*Let $\pi$ be a proof in $\mathsf{LL}_{\text{foc}}$ with expanded axioms and expanded $\top$ rules.*

- *If the conclusion of $\pi$ is $\vdash \Gamma, A \parr B ; \Pi$, there exists a smaller (with respect to the number of rules) proof of $\vdash \Gamma, A, B ; \Pi$.*

- *If the conclusion of* $\pi$ *is* $\vdash \Gamma, \bot \,;\, \Pi$*, there exists a smaller proof of* $\vdash \Gamma \,;\, \Pi$*.*
- *If the conclusion of* $\pi$ *is* $\vdash \Gamma, A \& B \,;\, \Pi$*, there exist smaller proofs of* $\vdash \Gamma, A \,;\, \Pi$ *and* $\vdash \Gamma, B \,;\, \Pi$*.*

### 2.3 Embedding of LL (Weak +-Focusing)

The translation $(.)^\bullet$ of LL into $\mathsf{LL}_{\text{foc}}$ does not modify formulas, translates the sequent $\vdash \Gamma$ as $\vdash \Gamma \,;$ and acts on proofs by adding a lot of cuts:

$$\dfrac{}{\vdash P^\perp, P}\,\mathit{ax} \quad\rightsquigarrow\quad \dfrac{}{\vdash P^\perp \,;\, P}\,\mathit{ax}$$

$$\dfrac{\vdash \Gamma, P \qquad \vdash \Delta, P^\perp}{\vdash \Gamma, \Delta}\,\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, P \,; \qquad \vdash \Delta, P^\perp \,;}{\vdash \Gamma, \Delta \,;}\,\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, A, B \,;}{\vdash \Gamma, A \parr B \,;}\,\parr$$

$$\dfrac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\,\otimes \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, P \,; \qquad \dfrac{\vdash \Delta, Q \,; \qquad \dfrac{\dfrac{\dfrac{}{\vdash P^\perp \,;\, P}\,\mathit{ax} \qquad \dfrac{}{\vdash Q^\perp \,;\, Q}\,\mathit{ax}}{\vdash P^\perp, Q^\perp \,;\, P \otimes Q}\,\otimes}{\vdash P^\perp, Q^\perp, P \otimes Q \,;}\,\mathit{foc}}{\vdash \Delta, P^\perp, P \otimes Q \,;}\,\mathit{n\text{-}cut}}{\vdash \Gamma, \Delta, P \otimes Q \,;}\,\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, N \otimes Q}\,\otimes \quad\rightsquigarrow\quad \dfrac{\vdash \Delta, Q \,; \qquad \dfrac{\dfrac{\vdash \Gamma, N \,; \qquad \dfrac{}{\vdash Q^\perp \,;\, Q}\,\mathit{ax}}{\vdash \Gamma, Q^\perp \,;\, N \otimes Q}\,\otimes}{\vdash \Gamma, Q^\perp, N \otimes Q \,;}\,\mathit{foc}}{\vdash \Gamma, \Delta, N \otimes Q \,;}\,\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N \qquad \vdash \Delta, M}{\vdash \Gamma, \Delta, N \otimes M}\,\otimes \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, N \,; \qquad \vdash \Delta, M \,;}{\vdash \Gamma, \Delta \,;\, N \otimes M}\,\otimes}{\vdash \Gamma, \Delta, N \otimes M \,;}\,\mathit{foc}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, A \,; \qquad \vdash \Gamma, B \,;}{\vdash \Gamma, A \,\&\, B \,;}\,\&$$

$$\dfrac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus B}\,\oplus_1 \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, P \,; \qquad \dfrac{\dfrac{\dfrac{}{\vdash P^\perp \,;\, P}\,\mathit{ax}}{\vdash P^\perp \,;\, P \oplus B}\,\oplus_1}{\vdash P^\perp, P \oplus B \,;}\,\mathit{foc}}{\vdash \Gamma, P \oplus B \,;}\,\mathit{n\text{-}cut}$$

$$\dfrac{\vdash \Gamma, N}{\vdash \Gamma, N \oplus B}\,\oplus_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, N \,;}{\vdash \Gamma \,;\, N \oplus B}\,\oplus_1}{\vdash \Gamma, N \oplus B \,;}\,\mathit{foc}$$

$$\dfrac{}{\vdash 1}\,1 \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\vdash \,;\, 1}\,1}{\vdash 1 \,;}\,\mathit{foc}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \quad\rightsquigarrow\quad \frac{\vdash \Gamma\,;}{\vdash \Gamma, \bot\,;}\,\bot$$

$$\frac{}{\vdash \Gamma, \top}\,\top \quad\rightsquigarrow\quad \frac{}{\vdash \Gamma, \top\,;}\,\top$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash ?\Gamma, A\,;}{\vdash ?\Gamma\,;\, !A}\,!}{\vdash ?\Gamma, !A\,;}\,\mathit{foc}$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,?d \quad\rightsquigarrow\quad \dfrac{\vdash \Gamma, P\,; \qquad \dfrac{\dfrac{}{\vdash P^{\perp}\,;\, P}\,\mathit{ax}}{\vdash P^{\perp}, ?P\,;}\,?d}{\vdash \Gamma, ?P\,;}\,n\text{-}\mathit{cut}$$

$$\frac{\vdash \Gamma, N}{\vdash \Gamma, ?N}\,?d \quad\rightsquigarrow\quad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma, ?N\,;}\,?d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w \quad\rightsquigarrow\quad \frac{\vdash \Gamma\,;}{\vdash \Gamma, ?A\,;}\,?w$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c \quad\rightsquigarrow\quad \frac{\vdash \Gamma, ?A, ?A\,;}{\vdash \Gamma, ?A\,;}\,?c$$

### 2.4 Focusing in $\mathsf{LL}_{\mathrm{foc}}$

If $\pi$ is a proof in $\mathsf{LL}_{\mathrm{foc}}$ then $\pi^\circ$ is the $\mathsf{LL}$ proof obtained by erasing all the "$;$" in the sequents.

**Proposition 1** (Cut-Free Weak $+$-Focusing)
*If $\pi$ is a cut-free proof of $\vdash \Gamma\,;\,\Pi$ in $\mathsf{LL}_{\mathrm{foc}}$ then $\pi^\circ$ is a weakly $+$-focused proof of $\vdash \Gamma, \Pi$ in $\mathsf{LL}$.*

**Corollary 1.1** (Weak $+$-Focusing)
*If $\vdash \Gamma$ is provable in $\mathsf{LL}$, $\vdash \Gamma$ is provable with a weakly $+$-focused proof.*

Proof: Starting from a proof $\pi$ of $\vdash \Gamma$, we translate it into the proof $\pi^\bullet$ of $\vdash \Gamma\,;$ in $\mathsf{LL}_{\mathrm{foc}}$. Using the cut elimination property given in Appendix A (Corollary 5.2), this leads to a cut-free proof $\pi^{\bullet\prime}$ of $\vdash \Gamma\,;$ in $\mathsf{LL}_{\mathrm{foc}}$. By Proposition 1, $(\pi^{\bullet\prime})^\circ$ is a weakly $+$-focused proof of $\vdash \Gamma$ in $\mathsf{LL}$. □

## 3 Focused Linear Logic

### 3.1 Sequent Calculus $\mathsf{LL}_{\mathrm{Foc}}$

In the spirit of [Gir01], we define a sub-system $\mathsf{LL}_{\mathrm{Foc}}$ of $\mathsf{LL}_{\mathrm{foc}}$ in which the strictly negative formulas in the context of positive rules must be atomic. We consider a cut-free version of the system since it is sufficient for the present development (cut elimination is performed in $\mathsf{LL}_{\mathrm{foc}}$).

There are two kinds of sequents: $\vdash \mathcal{P}, \mathcal{N}\,;$ and $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^{\perp}\,;\, P$ where $\mathcal{P}$ contains positive formulas only, $\mathcal{N}$ contains negative formulas only, and $\mathcal{X}^{\perp}$ contains negative atoms only. In order to simplify the notations we will write $\vdash \mathcal{P}, \mathcal{N}\,;\,\Pi$ for either a sequent with $\Pi$ empty or a sequent with $\Pi = P$ and $\mathcal{N} = ?\Gamma, \mathcal{X}^\perp$.

$$\frac{}{\vdash X^\perp\,;\, X}\,ax \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, P\,;}\,foc$$

$$\frac{\vdash \mathcal{P}, \mathcal{N}, A, B\,;}{\vdash \mathcal{P}, \mathcal{N}, A \parr B\,;}\,\parr$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp\,;\, Q}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\, P \otimes Q}\,\otimes \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp, M\,;}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\, P \otimes M}\,\otimes$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, N\,; \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp\,;\, Q}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\, N \otimes Q}\,\otimes \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, N\,; \qquad \vdash \mathcal{P}', ?\Gamma', \mathcal{X}'^\perp, M\,;}{\vdash \mathcal{P}, \mathcal{P}', ?\Gamma, ?\Gamma', \mathcal{X}^\perp, \mathcal{X}'^\perp\,;\, N \otimes M}\,\otimes$$

$$\frac{\vdash \mathcal{P}, \mathcal{N}, A\,; \qquad \vdash \mathcal{P}, \mathcal{N}, B\,;}{\vdash \mathcal{P}, \mathcal{N}, A \,\&\, B\,;}\,\&$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P \oplus B}\,\oplus_1 \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, N\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, N \oplus B}\,\oplus_1$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, Q}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, A \oplus Q}\,\oplus_2 \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, M\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, A \oplus M}\,\oplus_2$$

$$\frac{}{\vdash\,;\, 1}\,1 \qquad \frac{\vdash \mathcal{P}, \mathcal{N}\,;}{\vdash \mathcal{P}, \mathcal{N}, \bot\,;}\,\bot \qquad \frac{}{\vdash \mathcal{P}, \mathcal{N}, \top\,;}\,\top$$

$$\frac{\vdash ?\Gamma, A\,;}{\vdash ?\Gamma\,;\, !A}\,! \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, ?P\,;}\,?d \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, N\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, ?N\,;}\,?d$$

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, ?A\,;}\,?w \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, ?A, ?A\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, ?A\,;}\,?c$$

## 3.2 Embedding of $\mathsf{LL}_{\text{foc}}$

Let us first define the relation $\Gamma' \leq \Gamma$ on multi-sets by: any element $A$ in $\Gamma'$ also belongs to $\Gamma$ (or equivalently, we have an inclusion of the underlying sets (or supports) of $\Gamma'$ and $\Gamma$).

**Lemma 2** (Exponential Inclusion)
*If* $\vdash \mathcal{P}, \mathcal{N}, ?\Gamma'\,;$ *is provable in* $\mathsf{LL}_{\text{Foc}}$ *and* $\Gamma' \leq \Gamma$ *then* $\vdash \mathcal{P}, \mathcal{N}, ?\Gamma\,;$ *is provable in* $\mathsf{LL}_{\text{Foc}}$.

We want to embed proofs of $\mathsf{LL}_{\text{foc}}$ of sequents of the shape $\vdash \Gamma\,;$ into $\mathsf{LL}_{\text{Foc}}$. We consider a cut-free proof $\pi$ with expanded axioms (according to Section 2.2). We proceed by induction on the size (number of rules) of $\pi$ by showing simultaneously:

- if the conclusion of $\pi$ is $\vdash \Gamma\,;$ then it is also provable in $\mathsf{LL}_{\text{Foc}}$;
- if the conclusion of $\pi$ is $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\, P$ then $\vdash \mathcal{P}, ?\Gamma', \mathcal{X}^\perp\,;\, P$ is provable in $\mathsf{LL}_{\text{Foc}}$ for some $\Gamma' \leq \Gamma$;

- if the conclusion of $\pi$ is $\vdash \mathcal{P}, \mathcal{N} ; P$ and $\mathcal{N}$ contains at least one non-atomic non-? formula, then both $\vdash \mathcal{P}, \mathcal{N}, P ;$ and $\vdash \mathcal{P}, \mathcal{N}, ?P ;$ are provable in $\mathsf{LL}_{\text{Foc}}$.

We look at the last rule of the proof. The key cases are the *foc* rules and $?d$ rules on positive formulas (called a *positive* $?d$ rule). We only consider the *foc* rule (the case of the positive $?d$ rule is very similar). By induction hypothesis, either we have $\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp ; P$ in $\mathsf{LL}_{\text{foc}}$ thus $\vdash \mathcal{P}, ?\Gamma', \mathcal{X}^\perp ; P$ is provable in $\mathsf{LL}_{\text{Foc}}$ with $\Gamma' \leq \Gamma$, and we apply a *foc* rule and Lemma 2, or we have $\vdash \mathcal{P}, \mathcal{N} ; P$ and $\mathcal{N}$ contains at least one non-atomic non-? formula and we directly have $\vdash \mathcal{P}, \mathcal{N}, P ;$ in $\mathsf{LL}_{\text{Foc}}$.

In order to turn a proof of $\vdash \mathcal{P}, \mathcal{N} ; P$ in $\mathsf{LL}_{\text{foc}}$ into a proof of $\vdash \mathcal{P}, \mathcal{N}, P ;$ in $\mathsf{LL}_{\text{Foc}}$, we use Lemma 1 and the induction hypothesis which, when $\mathcal{N}$ becomes of the shape $?\Gamma, \mathcal{X}^\perp$, requires to use a *foc* rule and Lemma 2.

### 3.3 Focusing in $\mathsf{LL}_{\text{Foc}}$

If $\pi$ is a proof in $\mathsf{LL}_{\text{Foc}}$ then $\pi^\circ$ is the $\mathsf{LL}$ proof obtained by erasing all the "$;$" in the sequents.

**Proposition 2** (Cut-Free Focusing)
*If $\pi$ is a cut-free proof of $\vdash \mathcal{P}, \mathcal{N} ; \Pi$ in $\mathsf{LL}_{\text{Foc}}$ then $\pi^\circ$ is a focused proof of $\vdash \mathcal{P}, \mathcal{N}, \Pi$ in $\mathsf{LL}$.*

**Corollary 2.1** (Focusing)
*If $\vdash \Gamma$ is provable in $\mathsf{LL}$, $\vdash \Gamma$ is provable with a focused proof.*

PROOF: Starting from a proof $\pi$ of $\vdash \Gamma$, we translate it into the proof $\pi^\bullet$ of $\vdash \Gamma ;$ in $\mathsf{LL}_{\text{foc}}$ and then into a cut-free proof $\pi'$ of $\vdash \Gamma ;$ in $\mathsf{LL}_{\text{Foc}}$. By Proposition 2, $\pi'^\circ$ is a focused proof of $\vdash \Gamma$ in $\mathsf{LL}$. □

**Remark:** It is possible to define the embedding of $\mathsf{LL}$ proofs into $\mathsf{LL}_{\text{Foc}}$ directly (without using $\mathsf{LL}_{\text{foc}}$ as an intermediary step). We have chosen to decompose it into two steps in order to show that the key property is *weak +-focusing*. Strong +-focusing is then obtained through reversing which is in general easy to do.

## 4 Additional Remarks

### 4.1 Decomposition of Exponentials

The attentive reader has certainly remarked that an hidden decomposition of the exponential connectives underlies the whole text (as suggested by Girard [Gir01]):

$$!A = \downarrow\sharp A \qquad\qquad ?A = \uparrow\flat A$$

with $\sharp A$ negative and $\flat A$ positive, and $\downarrow$ and $\uparrow$ are used as the corresponding connectives of $\mathsf{LL}_{\text{pol}}^{\uparrow\downarrow}$ (see Appendix A.2.1).

The associated rules in $\mathsf{LL}_{\text{foc}}$ would be:

$$\frac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma, \sharp A ;}\ \sharp \qquad \frac{\vdash \Gamma ; P}{\vdash \Gamma ; \flat P}\ \flat \qquad \frac{\vdash \Gamma, N ;}{\vdash \Gamma ; \flat N}\ \flat$$

$$\frac{\vdash \Gamma, A ;}{\vdash \Gamma ; \downarrow A}\ \downarrow \qquad \frac{\vdash \Gamma ; P}{\vdash \Gamma, \uparrow P ;}\ \uparrow \qquad \frac{\vdash \Gamma, N ;}{\vdash \Gamma, \uparrow N ;}\ \uparrow$$

However it is difficult to give a meaning to $\sharp A$ (resp. $\flat A$) under any other connective than $\downarrow$ (resp. $\uparrow$).

### 4.2 Quantifiers

Our method can perfectly be extended to quantification (of any order) by adding $\exists\alpha A$ (resp. $\forall\alpha A$) in positive (resp. negative) formulas and the following rules in $\mathsf{LL}_{\text{foc}}$:

$$\frac{\vdash \Gamma\,;\,P}{\vdash \Gamma\,;\,\exists\alpha P}\,\exists \qquad \frac{\vdash \Gamma, N\,;}{\vdash \Gamma\,;\,\exists\alpha N}\,\exists \qquad \frac{\vdash \Gamma, A\,;\,\Pi}{\vdash \Gamma, \forall\alpha A\,;\,\Pi}\,\forall$$

with $\alpha$ free neither in $\Gamma$ nor in $\Pi$ in the $\forall$ rule.

The corresponding rules in $\mathsf{LL}_{\text{Foc}}$ are:

$$\frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\,P}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\,\exists\alpha P}\,\exists \qquad \frac{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp, N\,;}{\vdash \mathcal{P}, ?\Gamma, \mathcal{X}^\perp\,;\,\exists\alpha N}\,\exists \qquad \frac{\vdash \mathcal{P}, \mathcal{N}, A\,;}{\vdash \mathcal{P}, \mathcal{N}, \forall\alpha A\,;}\,\forall$$

with $\alpha$ free neither in $\mathcal{P}$ nor in $\mathcal{N}$ in the $\forall$ rule.

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# A Cut Elimination in $\mathsf{LL}_{\text{foc}}$

## A.1 Cut Elimination Steps

### Key Steps

$$\dfrac{\dfrac{}{\vdash P^\perp ; P}\,ax \qquad \vdash \Gamma, P^\perp ; \Pi}{\vdash \Gamma, P^\perp ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \vdash \Gamma, P^\perp ; \Pi$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{}{\vdash P^\perp ; P}\,ax}{\vdash \Gamma ; P}\,p\text{-}cut \quad \rightsquigarrow \quad \vdash \Gamma ; P$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, P ;}\,foc \qquad \vdash \Delta, P^\perp ;}{\vdash \Gamma, \Delta, P^\perp ;}\,n\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ;}{\vdash \Gamma, \Delta, P^\perp ;}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta ; P \otimes Q}\,\otimes \qquad \dfrac{\vdash \Sigma, P^\perp, Q^\perp ; \Pi}{\vdash \Sigma, P^\perp \parr Q^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta ; Q \qquad \vdash \Sigma, P^\perp, Q^\perp ; \Pi}{\vdash \Delta, \Sigma, P^\perp ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, M ;}{\vdash \Gamma, \Delta ; P \otimes M}\,\otimes \qquad \dfrac{\vdash \Sigma, P^\perp, M^\perp ; \Pi}{\vdash \Sigma, P^\perp \parr M^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, M ; \qquad \vdash \Sigma, P^\perp, M^\perp ; \Pi}{\vdash \Delta, \Sigma, P^\perp ; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N ; \qquad \vdash \Delta, M ;}{\vdash \Gamma, \Delta ; N \otimes M}\,\otimes \qquad \dfrac{\vdash \Sigma, N^\perp, M^\perp ; \Pi}{\vdash \Sigma, N^\perp \parr M^\perp ; \Pi}\,\parr}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma, N ; \qquad \dfrac{\vdash \Delta, M ; \qquad \vdash \Sigma, N^\perp, M^\perp ; \Pi}{\vdash \Delta, \Sigma, N^\perp ; \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, \Sigma ; \Pi}\,n\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma ; P \oplus B}\,\oplus_1 \qquad \dfrac{\vdash \Delta, P^\perp ; \Pi \qquad \vdash \Delta, B^\perp ; \Pi}{\vdash \Delta, P^\perp \,\&\, B^\perp ; \Pi}\,\&}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash \Gamma, N ;}{\vdash \Gamma ; N \oplus B}\,\oplus_1 \qquad \dfrac{\vdash \Delta, N^\perp ; \Pi \qquad \vdash \Delta, B^\perp ; \Pi}{\vdash \Delta, N^\perp \,\&\, B^\perp ; \Pi}\,\&}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, N ; \qquad \vdash \Delta, N^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,n\text{-}cut$$

$$\dfrac{\dfrac{}{\vdash ; 1}\,1 \qquad \dfrac{\vdash \Gamma ; \Pi}{\vdash \Gamma, \bot ; \Pi}\,\bot}{\vdash \Gamma ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \vdash \Gamma ; \Pi$$

$$\dfrac{\dfrac{\vdash ?\Gamma, N ;}{\vdash ?\Gamma ; !N}\,! \qquad \dfrac{\vdash \Delta ; N^\perp}{\vdash \Delta, ?N^\perp ;}\,?d}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\vdash ?\Gamma, N ; \qquad \vdash \Delta ; N^\perp}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, P ;}{\vdash ?\Gamma ; !P}\,! \qquad \dfrac{\vdash \Delta, P^\perp ;}{\vdash \Delta, ?P^\perp ;}\,?d}{\vdash ?\Gamma, \Delta ;}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\vdash ?\Gamma, P ; \qquad \vdash \Delta, P^\perp ;}{\vdash ?\Gamma, \Delta ;}\,n\text{-}cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\vdash \Delta ; \Pi}{\vdash \Delta, ?A^\perp ; \Pi}\,?w}{\vdash ?\Gamma, \Delta ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\vdash \Delta ; \Pi}{\vdash ?\Gamma, \Delta ; \Pi}\,?w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\vdash \Delta, ?A^\perp, ?A^\perp ; \Pi}{\vdash \Delta, ?A^\perp ; \Pi}\,?c}{\vdash ?\Gamma, \Delta ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \dfrac{\dfrac{\vdash ?\Gamma, A ;}{\vdash ?\Gamma ; !A}\,! \qquad \vdash \Delta, ?A^\perp, ?A^\perp ; \Pi}{\vdash ?\Gamma, \Delta, ?A^\perp ; \Pi}\,p\text{-}cut}{\vdash ?\Gamma, ?\Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash ?\Gamma, \Delta ; \Pi}\,?c$$

**Left Commutative $p$-Steps**

$$\dfrac{\dfrac{\vdash \Gamma, A, B ; P}{\vdash \Gamma, A \parr B ; P}\,\parr \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\vdash \Gamma, A, B ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, A, B ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,\parr$$

$$\dfrac{\dfrac{\vdash \Gamma, A ; P \qquad \vdash \Gamma, B ; P}{\vdash \Gamma, A \,\&\, B ; P}\,\& \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, A \,\&\, B ; \Pi}\,p\text{-}cut$$

$$\rightsquigarrow\qquad \dfrac{\dfrac{\vdash \Gamma, A ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, A ; \Pi}\,p\text{-}cut \qquad \dfrac{\vdash \Gamma, B ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, B ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \,\&\, B ; \Pi}\,\&$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, \bot ; P}\,\bot \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, \bot ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, \bot ; \Pi}\,\bot$$

$$\dfrac{\dfrac{}{\vdash \Gamma, \top ; P}\,\top \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, \top ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{}{\vdash \Gamma, \Delta, \top ; \Pi}\,\top$$

$$\dfrac{\dfrac{\vdash \Gamma ; P}{\vdash \Gamma, ?A ; P}\,?w \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta, ?A ; \Pi}\,p\text{-}cut \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^\perp ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?w$$

$$\dfrac{\dfrac{\vdash \Gamma, ?A, ?A \,;\, P}{\vdash \Gamma, ?A \,;\, P}\,?c \qquad \vdash \Delta, P^{\perp} \,;\, \Pi}{\vdash \Gamma, \Delta, ?A \,;\, \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, ?A, ?A \,;\, P \qquad \vdash \Delta, P^{\perp} \,;\, \Pi}{\vdash \Gamma, \Delta, ?A, ?A \,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, ?A \,;\, \Pi}\,?c$$

**Right Commutative $p$-Steps**

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp} \,;\, P'}{\vdash \Delta, P^{\perp}, P' \,;}\,foc}{\vdash \Gamma, \Delta, P' \,;}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp} \,;\, P'}{\vdash \Gamma, \Delta \,;\, P'}\,p\text{-}cut}{\vdash \Gamma, \Delta, P' \,;}\,foc$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, A, B, P^{\perp} \,;\, \Pi}{\vdash \Delta, A \parr B, P^{\perp} \,;\, \Pi}\,\parr}{\vdash \Gamma, \Delta, A \parr B \,;\, \Pi}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, A, B, P^{\perp} \,;\, \Pi}{\vdash \Gamma, \Delta, A, B \,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \parr B \,;\, \Pi}\,\parr$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp} \,;\, P' \qquad \vdash \Sigma \,;\, Q}{\vdash \Delta, \Sigma, P^{\perp} \,;\, P' \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Sigma \,;\, P' \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp} \,;\, P'}{\vdash \Gamma, \Delta \,;\, P'}\,p\text{-}cut \qquad \vdash \Sigma \,;\, Q}{\vdash \Gamma, \Delta, \Sigma \,;\, P' \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N \,; \qquad \vdash \Sigma \,;\, Q}{\vdash \Delta, \Sigma, P^{\perp} \,;\, N \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp}, N \,;}{\vdash \Gamma, \Delta, N \,;}\,p\text{-}cut \qquad \vdash \Sigma \,;\, Q}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, N \,; \qquad \vdash \Sigma, P^{\perp} \,;\, Q}{\vdash \Delta, \Sigma, P^{\perp} \,;\, N \otimes Q}\,\otimes}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes Q}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\vdash \Delta, N \,; \qquad \dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Sigma, P^{\perp} \,;\, Q}{\vdash \Gamma, \Sigma \,;\, Q}\,p\text{-}cut}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes Q}\,\otimes$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N \,; \qquad \vdash \Sigma, M \,;}{\vdash \Delta, \Sigma, P^{\perp} \,;\, N \otimes M}\,\otimes}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes M}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp}, N \,;}{\vdash \Gamma, \Delta, N \,;}\,p\text{-}cut \qquad \vdash \Sigma, M \,;}{\vdash \Gamma, \Delta, \Sigma \,;\, N \otimes M}\,\otimes$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, A \,;\, \Pi \qquad \vdash \Delta, P^{\perp}, B \,;\, \Pi}{\vdash \Delta, P^{\perp}, A \,\&\, B \,;\, \Pi}\,\&}{\vdash \Gamma, \Delta, A \,\&\, B \,;\, \Pi}\,p\text{-}cut$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp}, A \,;\, \Pi}{\vdash \Gamma, \Delta, A \,;\, \Pi}\,p\text{-}cut \qquad \dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp}, B \,;\, \Pi}{\vdash \Gamma, \Delta, B \,;\, \Pi}\,p\text{-}cut}{\vdash \Gamma, \Delta, A \,\&\, B \,;\, \Pi}\,\&$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp} \,;\, P'}{\vdash \Delta, P^{\perp} \,;\, P' \oplus B}\,\oplus_1}{\vdash \Gamma, \Delta \,;\, P' \oplus B}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp} \,;\, P'}{\vdash \Gamma, \Delta \,;\, P'}\,p\text{-}cut}{\vdash \Gamma, \Delta \,;\, P' \oplus B}\,\oplus_1$$

$$\dfrac{\vdash \Gamma \,;\, P \qquad \dfrac{\vdash \Delta, P^{\perp}, N \,;}{\vdash \Delta, P^{\perp} \,;\, N \oplus B}\,\oplus_1}{\vdash \Gamma, \Delta \,;\, N \oplus B}\,p\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma \,;\, P \qquad \vdash \Delta, P^{\perp}, N \,;}{\vdash \Gamma, \Delta, N \,;}\,p\text{-}cut}{\vdash \Gamma, \Delta \,;\, N \oplus B}\,\oplus_1$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; \Pi}{\vdash \Delta, P^{\perp}, \bot ; \Pi}\,\bot}{\vdash \Gamma, \Delta, \bot ; \Pi}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,\textit{p-cut}}{\vdash \Gamma, \Delta, \bot ; \Pi}\,\bot$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{}{\vdash \Delta, \top, P^{\perp} ; \Pi}\,\top}{\vdash \Gamma, \Delta, \top ; \Pi}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{}{\vdash \Gamma, \Delta, \top ; \Pi}\,\top$$

$$\dfrac{\vdash ?\Gamma ; !A \qquad \dfrac{\vdash ?\Delta, ?A^{\perp}, B ;}{\vdash ?\Delta, ?A^{\perp} ; !B}\,!}{\vdash ?\Gamma, ?\Delta ; !B}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash ?\Gamma ; !A \qquad \vdash ?\Delta, ?A^{\perp}, B ;}{\vdash ?\Gamma, ?\Delta, B ;}\,\textit{p-cut}}{\vdash ?\Gamma, ?\Delta ; !B}\,!$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; P'}{\vdash \Delta, P^{\perp}, ?P' ;}\,?d}{\vdash \Gamma, \Delta, ?P' ;}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; P'}{\vdash \Gamma, \Delta ; P'}\,\textit{p-cut}}{\vdash \Gamma, \Delta, ?P' ;}\,?d$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp}, N ;}{\vdash \Delta, P^{\perp}, ?N ;}\,?d}{\vdash \Gamma, \Delta, ?N ;}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp}, N ;}{\vdash \Gamma, \Delta, N ;}\,\textit{p-cut}}{\vdash \Gamma, \Delta, ?N ;}\,?d$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp} ; \Pi}{\vdash \Delta, P^{\perp}, ?A ; \Pi}\,?w}{\vdash \Gamma, \Delta, ?A ; \Pi}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; \Pi}{\vdash \Gamma, \Delta ; \Pi}\,\textit{p-cut}}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?w$$

$$\dfrac{\vdash \Gamma ; P \qquad \dfrac{\vdash \Delta, P^{\perp}, ?A, ?A ; \Pi}{\vdash \Delta, P^{\perp}, ?A ; \Pi}\,?c}{\vdash \Gamma, \Delta, ?A ; \Pi}\,\textit{p-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp}, ?A, ?A ; \Pi}{\vdash \Gamma, \Delta, ?A, ?A ; \Pi}\,\textit{p-cut}}{\vdash \Gamma, \Delta, ?A ; \Pi}\,?c$$

**Commutative $n$-Steps**

$$\dfrac{\dfrac{\vdash \Gamma, P ; P'}{\vdash \Gamma, P, P' ;}\,\textit{foc} \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, P' ;}\,\textit{n-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta ; P'}\,\textit{n-cut}}{\vdash \Gamma, \Delta, P' ;}\,\textit{foc}$$

$$\dfrac{\dfrac{\vdash \Gamma, A, B, P ; \Pi}{\vdash \Gamma, A \parr B, P ; \Pi}\,\parr \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,\textit{n-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A, B, P ; \Pi \qquad \vdash \Delta, P^{\perp} ;}{\vdash \Gamma, \Delta, A, B ; \Pi}\,\textit{n-cut}}{\vdash \Gamma, \Delta, A \parr B ; \Pi}\,\parr$$

$$\dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, P ; P' \otimes Q}\,\otimes \qquad \vdash \Sigma, P^{\perp} ;}{\vdash \Gamma, \Delta, \Sigma ; P' \otimes Q}\,\textit{n-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, P ; P' \qquad \vdash \Sigma, P^{\perp} ;}{\vdash \Gamma, \Sigma ; P'}\,\textit{n-cut} \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, \Sigma ; P' \otimes Q}\,\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N ; \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, P ; N \otimes Q}\,\otimes \qquad \vdash \Sigma, P^{\perp} ;}{\vdash \Gamma, \Delta, \Sigma ; N \otimes Q}\,\textit{n-cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, P, N ; \qquad \vdash \Sigma, P^{\perp} ;}{\vdash \Gamma, \Sigma, N ;}\,\textit{n-cut} \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta, \Sigma ; N \otimes Q}\,\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, N\,; \qquad \vdash \Delta, P\,;\,Q}{\vdash \Gamma,\Delta,P\,;\,N\otimes Q}\otimes \qquad \vdash \Sigma, P^\perp\,;}{\vdash \Gamma,\Delta,\Sigma\,;\,N\otimes Q}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\vdash\Gamma,N\,; \qquad \dfrac{\vdash\Delta,P\,;\,Q \qquad \vdash\Sigma,P^\perp\,;}{\vdash\Delta,\Sigma\,;\,Q}\,n\text{-}cut}{\vdash\Gamma,\Delta,\Sigma\,;\,N\otimes Q}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N\,; \qquad \vdash \Delta, M\,;}{\vdash \Gamma,\Delta,P\,;\,N\otimes M}\otimes \qquad \vdash \Sigma, P^\perp\,;}{\vdash \Gamma,\Delta,\Sigma\,;\,N\otimes M}\,n\text{-}cut$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P,N\,; \qquad \vdash\Sigma,P^\perp\,;}{\vdash\Gamma,\Sigma,N\,;}\,n\text{-}cut \qquad \vdash\Delta,M\,;}{\vdash\Gamma,\Delta,\Sigma\,;\,N\otimes M}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma, P, A\,;\,\Pi \qquad \vdash \Gamma, P, B\,;\,\Pi}{\vdash \Gamma,P,A\mathbin{\&}B\,;\,\Pi}\mathbin{\&} \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,A\mathbin{\&}B\,;\,\Pi}\,n\text{-}cut$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P,A\,;\,\Pi \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta,A\,;\,\Pi}\,n\text{-}cut \qquad \dfrac{\vdash\Gamma,P,B\,;\,\Pi \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta,B\,;\,\Pi}\,n\text{-}cut}{\vdash\Gamma,\Delta,A\mathbin{\&}B\,;\,\Pi}\mathbin{\&}$$

$$\dfrac{\dfrac{\vdash \Gamma, P\,;\,P'}{\vdash \Gamma,P\,;\,P'\oplus B}\oplus_1 \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta\,;\,P'\oplus B}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P\,;\,P' \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta\,;\,P'}\,n\text{-}cut}{\vdash\Gamma,\Delta\,;\,P'\oplus B}\oplus_1$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N\,;}{\vdash \Gamma,P\,;\,N\oplus B}\oplus_1 \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta\,;\,N\oplus B}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P,N\,; \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta,N\,;}\,n\text{-}cut}{\vdash\Gamma,\Delta\,;\,N\oplus B}\oplus_1$$

$$\dfrac{\dfrac{\vdash \Gamma, P\,;\,\Pi}{\vdash \Gamma,P,\bot\,;\,\Pi}\bot \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,\bot\,;\,\Pi}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P\,;\,\Pi \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta\,;\,\Pi}\,n\text{-}cut}{\vdash\Gamma,\Delta,\bot\,;\,\Pi}\bot$$

$$\dfrac{\dfrac{}{\vdash \Gamma,P,\top\,;\,\Pi}\top \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,\top\,;\,\Pi}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{}{\vdash\Gamma,\Delta,\top\,;\,\Pi}\top$$

$$\dfrac{\dfrac{\vdash \Gamma, P\,;\,P'}{\vdash \Gamma,P,?P'\,;}\,?d \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,?P'\,;}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P\,;\,P' \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta\,;\,P'}\,n\text{-}cut}{\vdash\Gamma,\Delta,?P'\,;}\,?d$$

$$\dfrac{\dfrac{\vdash \Gamma, P, N\,;}{\vdash \Gamma,P,?N\,;}\,?d \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,?N\,;}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P,N\,; \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta,N\,;}\,n\text{-}cut}{\vdash\Gamma,\Delta,?N\,;}\,?d$$

$$\dfrac{\dfrac{\vdash \Gamma, P\,;\,\Pi}{\vdash \Gamma,P,?A\,;\,\Pi}\,?w \qquad \vdash \Delta, P^\perp\,;}{\vdash \Gamma,\Delta,?A\,;\,\Pi}\,n\text{-}cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash\Gamma,P\,;\,\Pi \qquad \vdash\Delta,P^\perp\,;}{\vdash\Gamma,\Delta\,;\,\Pi}\,n\text{-}cut}{\vdash\Gamma,\Delta,?A\,;\,\Pi}\,?w$$

$$\dfrac{\dfrac{\vdash \Gamma, P, ?A, ?A \,;\, \Pi}{\vdash \Gamma, P, ?A \,;\, \Pi}\,?c \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, ?A \,;\, \Pi}\,n\text{-}cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, P, ?A, ?A \,;\, \Pi \qquad \vdash \Delta, P^{\perp} \,;}{\vdash \Gamma, \Delta, ?A, ?A \,;\, \Pi}\,n\text{-}cut}{\vdash \Gamma, \Delta, ?A \,;\, \Pi}\,?c$$

## A.2 Cut Elimination Property

### A.2.1 $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$

The system $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ is obtained from $\mathsf{LL}$ by "restricting" formulas to the following grammar:

$$\begin{array}{rcl} P & ::= & X \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid\ !N \mid\ \downarrow N \\ N & ::= & X^{\perp} \mid N \parr N \mid N \,\&\, N \mid \perp \mid \top \mid\ ?P \mid\ \uparrow P \end{array}$$

with the following rules for the $\uparrow$ and $\downarrow$ connectives:

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\downarrow \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, \uparrow P}\uparrow$$

where $\mathcal{N}$ is a multi-set of negative formulas.

### A.2.2 $\mathsf{LLP}$

The system $\mathsf{LLP}$ is obtained from $\mathsf{LL}$ by restricting formulas to the following grammar:

$$\begin{array}{rcl} P & ::= & X \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid\ !N \\ N & ::= & X^{\perp} \mid N \parr N \mid N \,\&\, N \mid \perp \mid \top \mid\ ?P \end{array}$$

with the following generalizations of the exponential rules:

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N}\,?w \qquad \frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\,?c$$

where $\mathcal{N}$ is a multi-set of negative formulas of $\mathsf{LLP}$.

**Proposition 3** (Strong Normalization)
*There is no infinite sequence of reductions in* $\mathsf{LLP}$ *if we forbid commutations of cuts with cuts.*

Proof: Such a sequence of reductions can only contain finitely many steps between two steps that correspond to a reduction step in proof-nets. Thus by strong normalization for proof-nets [Lau02] we can conclude. □

**Corollary 3.1** (Cut Elimination)
*If* $\vdash \Gamma$ *is provable in* $\mathsf{LLP}$*, then* $\vdash \Gamma$ *is provable without the cut rule.*

### A.2.3 Simulations

The translation $(.)^!$ of $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ into $\mathsf{LLP}$ is obtained by replacing $\downarrow N$ by $!N$ and $\uparrow P$ by $?P$ and the two lifting rules by promotion and dereliction.

**Lemma 3** (Polarized Formulas)
*If $A$ is a positive (resp. negative) formula in* $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ *then $A^!$ is a positive (resp. negative) formula in* $\mathsf{LLP}$.

**Proposition 4** (One-to-One Simulation)
*If $\pi$ reduces to $\pi'$ in $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ by one step of reduction then $\pi^!$ reduces to $\pi'^!$ in* LLP *by one step of reduction.*

**Corollary 4.1** (Strong Normalization)
*There is no infinite sequence of reductions in $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ if we forbid commutations of cuts with cuts.*

**Corollary 4.2** (Cut Elimination)
*If $\vdash \Gamma$ is provable in $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$, then $\vdash \Gamma$ is provable without the cut rule.*

The translation $(.)^{\uparrow}$ of $\mathsf{LL}_{\mathrm{foc}}$ into $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ is obtained by adding to formulas exactly the required liftings to get a polarized formula, and by translating the sequent $\vdash \mathcal{P}, \mathcal{N} \,;\, \Pi$ by $\vdash \uparrow\mathcal{P}^{\uparrow}, \mathcal{N}^{\uparrow}, \Pi^{\uparrow}$.

In particular $(!P \parr 1)^{\uparrow} = \uparrow!\uparrow P^{\uparrow} \parr \uparrow 1$.

**Proposition 5** (Strict Simulation)
*If $\pi$ reduces to $\pi'$ in $\mathsf{LL}_{\mathrm{foc}}$ by one step of reduction then $\pi^{\uparrow}$ reduces to $\pi'^{\uparrow}$ in $\mathsf{LL}^{\uparrow\downarrow}_{\mathrm{pol}}$ by at least one step of reduction.*

**Corollary 5.1** (Strong Normalization)
*There is no infinite sequence of reductions in $\mathsf{LL}_{\mathrm{foc}}$ if we forbid commutations of cuts with cuts.*

**Corollary 5.2** (Cut Elimination)
*If $\vdash \Gamma \,;\, \Pi$ is provable in $\mathsf{LL}_{\mathrm{foc}}$, then $\vdash \Gamma \,;\, \Pi$ is provable without cuts.*

Proof: We consider a cut without any cut above it. We look at the two different cases:

- If it is a *n-cut*, we look at the rule above the premise $\vdash \Gamma, P \,;\, \Pi$. If the rule above it introduces $P$, it is either a *foc* rule or a $\top$ rule and we apply the corresponding key step (and the *n-cut* becomes a *p-cut*) or commutative n-step. Otherwise this rule cannot be an *ax* rule, a 1 rule or a ! rule and we can apply the corresponding commutative n-step.
- If it is a *p-cut*, we first look at the premise $\vdash \Gamma \,;\, P$. If $P$ is not a main formula, we can apply a left commutative p-step. If $P$ is a main formula and $P^{\perp}$ is not, we can apply the corresponding right commutative p-step (notice that the rule above $P^{\perp}$ cannot be a 1 rule). We just have to verify that we can apply the right commutative p-step in the case of a ! rule above $P^{\perp}$: since $P$ is main, the rule above it is either an *ax* rule or a ! rule and we can apply the reduction step. If both $P$ and $P^{\perp}$ are main, we apply the corresponding key step.

So that, either the proof is cut-free or a reduction step can be applied, and we conclude by strong normalization. □