A Proof of the Focusing Property of Linear Logic

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The focusing property of linear logic has been discovered by J.-M. Andreoli [And90, And92] in the beginning of the 90’s. It is one of the main properties of Linear Logic that appeared after the original paper of J.-Y. Girard [Gir87]. This property is proved in various papers [And90, Gir91, DJS97] but, as far as we know never for the usual LL sequent calculus (or with an intricate induction in Andreoli’s thesis, which is replaced here by a cut elimination property).

The proof we give is a compilation of the previous proofs and proof techniques, our goal is just to give a complete presentation of a proof of this property which appears to be more and more important in the current research in Linear Logic [Gir91, DJS97, Gir01, Lau02].

We decompose the usual focusing property into two technically different steps: weak focusing (the decompositions of two positive formulas are not interleaved, see $LL_{foc}$) and reversing (negative rules are applied as late as possible from a top-down point of view, see $LL_{Foc}$).

1 Linear Logic and the Focusing Property

Formulas of (propositional) Linear Logic are given by the usual grammar:

\[
A ::= X \mid A \otimes A \mid A \oplus A \mid 1 \mid 0 \mid !A \\
\mid X^\bot \mid A \otimes A \mid A \& A \mid \bot \mid \top \mid ?A
\]

We split this grammar into two sub-classes: positive formulas and negative formulas.

\[
P ::= X \mid A \otimes A \mid A \oplus A \mid 1 \mid 0 \mid !A \\
N ::= X^\bot \mid A \otimes A \mid A \& A \mid \bot \mid \top \mid ?A
\]

In the sequel:

- $A, B, \ldots$ denote arbitrary formulas;
- $P, Q, R$ denote positive formulas;
- $N, M, L$ denote negative formulas;

1 Around the time we were writing the first version of this note, A. Saurin also worked on focusing with a different approach. We take the opportunity of this revision to add a reference to his work [MS07].
• $X, X^\perp, ...$ are called atoms or atomic formulas.

We consider the usual rules of Linear Logic [Gir87]:

\[
\begin{align*}
& \vdash A, A \quad \text{ax} \quad \vdash A, \Delta, A \quad \text{cut} \\
& \vdash \Gamma, A, B \quad \gamma \quad \vdash \Gamma, A, \Delta, B \quad \otimes \\
& \vdash \Gamma, A \quad \vdash \Gamma, B \quad \& \quad \vdash \Gamma, A \quad \vdash \Gamma, B \quad \oplus_1 \quad \vdash \Gamma, A \quad \vdash \Gamma, B \quad \oplus_2 \\
& \vdash ?\Gamma, A \quad \vdash ?\Gamma, !A \quad ?d \quad \vdash ?\Gamma, ?A \quad \vdash ?w \quad \vdash ?\Gamma, ?A \quad ?c
\end{align*}
\]

The main connectives of positive (resp. negative) formulas are called positive connectives (resp. negative connectives), that is $X, \otimes, \oplus, 1, 0$ and $!$ (resp. $X^\perp, \gamma, \&, \perp, \top$ and $?$). The rules introducing positive (resp. negative) connectives are called positive rules (resp. negative rules).

A positive (resp. negative) formula is strictly positive (resp. strictly negative) if its main connective is not $!$ (resp. $?$). A positive (resp. negative) rule is strictly positive (resp. strictly negative) if it is not introducing a $!A$ formula (resp. $?A$ formula).

**Definition 1** (Main Positive Tree)
If $A$ is a formula, its main positive tree $T^+(A)$ is defined by:

\[
\begin{align*}
T^+(N) &= \emptyset \\
T^+(X) &= X \\
T^+(A \otimes B) &= T^+(A) \otimes T^+(B) \\
T^+(A \oplus B) &= T^+(A) \oplus T^+(B) \\
T^+(1) &= 1 \\
T^+(0) &= 0 \\
T^+(!A) &= !
\end{align*}
\]

$T^+(A)$ is a (possibly empty) tree whose nodes are $\otimes, \oplus$ (with arity at most 2) and $!, 1, 0$ or $X$ (with arity 0).

**Definition 2** (Weakly $+$-Focused Proof)
A proof $\pi$ in LL is weakly $+$-focused if it is cut-free and, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, A$, the only positive rules of $\pi'$ between two rules introducing connectives of $T^+(A)$ are rules introducing connectives of $T^+(A)$.

**Definition 3** (Strongly $+$-Focused Proof)
A proof $\pi$ in LL is strongly $+$-focused if it is cut-free and, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, A$, the only rules of $\pi'$ between two rules introducing connectives of $T^+(A)$ are rules introducing connectives of $T^+(A)$.
**Definition 4** (Reversed Proof)
A proof $\pi$ in LL is *reversed* if, for any subproof $\pi'$ of $\pi$ with conclusion $\vdash \Gamma, N$ with $N$ a strictly negative non-atomic formula, the last rule of $\pi'$ is a strictly negative rule (i.e. a $\exists$, $\&$, $\bot$ or $\top$ rule).

**Definition 5** (Focused Proof)
A proof $\pi$ in LL is *focused* if it is both strongly $+\cdot$-focused and reversed.

**Remark:** Various systems for classical logic related with focusing constraints have been introduced. In particular Girard’s LC [Gir91] corresponds to weak $+\cdot$-focusing, Danos-Joinet-Schellinx’s LK$^\eta$ [DJS97] corresponds to strong $+\cdot$-focusing and Quatrini-Tortora de Falco’s LK$^{\eta,\rho}_{pol}$ [QTdF96] corresponds to focusing since their $\rho$-constraint is a reversing constraint.

## 2 Weakly Focused Linear Logic

### 2.1 Sequent Calculus $\text{LL}_{foc}$

A sequent of $\text{LL}_{foc}$ has the shape $\vdash \Gamma ; \Pi$ where $\Gamma$ is a multi-set of formulas and $\Pi$ is either empty or contains a unique positive formula.

The rules are a linear logic version of the rules of Girard’s LC [Gir91]:

- $\vdash P^\perp; P$ \hspace{1cm} $\vdash \Gamma; P; \cdot_{foc}$
- $\vdash \Gamma; P \vdash \Delta, P^\perp; \Pi$ \hspace{1cm} $\vdash \Gamma, \Delta; \Pi$ \hspace{1cm} $\vdash \Gamma; P; \Pi \vdash \Delta, P^\perp; \Pi$

- $\vdash \Gamma, \Delta; \Pi$ \hspace{1cm} $\vdash \Gamma, \Delta, P \vdash \Gamma, \Delta, P \vdash \Gamma, \Delta, \Pi$

- $\vdash \Gamma, P \vdash \Gamma, \Delta, P \vdash \Gamma, \Delta, \Pi$

- $\vdash \Gamma, A, B; \Pi \vdash A \exists B; \Pi$ \hspace{1cm} $\vdash \Gamma ; \exists \psi ; \Gamma$ \hspace{1cm} $\vdash \Gamma; P \vdash \Delta, Q \vdash \Gamma, \Delta, P \vdash \Gamma, \Delta, \Pi$

- $\vdash \Gamma, A \exists B; \Pi \vdash \Delta, M; \Pi \vdash \Gamma, A \& B; \Pi$

- $\vdash \Gamma, A \& B; \Pi \vdash \Gamma, A \& B; \Pi$

- $\vdash \Gamma, A \& B; \Pi \vdash \Gamma, A \& B; \Pi$

- $\vdash \Gamma, A \& B; \Pi \vdash \Gamma, A \& B; \Pi$

- $\vdash 1; \Gamma; \Pi \vdash \Gamma, \bot; \Pi \vdash \Gamma, \top; \Pi$

- $\vdash \top; \Gamma, \bot; \Pi \vdash \Gamma, \bot; \Pi \vdash \Gamma, \top; \Pi$

- $\vdash \neg \exists \psi ; \Gamma$ \hspace{1cm} $\vdash \neg \exists \psi ; \Gamma$ \hspace{1cm} $\vdash \neg \exists \psi ; \Gamma$

- $\vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi$

- $\vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi$

- $\vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi$

- $\vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi$

- $\vdash \exists \psi ; \Gamma ; A; \Pi \vdash \exists \psi ; \Gamma ; A; \Pi$
2.2 Expansion of Axioms

Given a positive formula $P$, the sequent $\vdash P^\perp; P$ is provable in $\mathbb{LL}_{\text{foc}}$ by means of the $\text{ax}$ rule. Moreover, if $P$ is not atomic, it is also possible to give a proof based only on $\text{ax}$ rules applied to the sub-formulas of $P$ (and thus, by an easy induction, to the atomic sub-formulas of $P$):

\[
\frac{\vdash P^\perp; P}{\vdash P^\perp; Q^\perp; Q} \quad \frac{\vdash N^\perp; N^\perp}{\vdash N^\perp, Q^\perp; Q} \quad \frac{\vdash N^\perp, N^\perp}{\vdash N^\perp \& N^\perp, Q^\perp; Q}
\]

\[
\frac{\vdash P^\perp; \frac{\vdash M^\perp; M^\perp}{\vdash P^\perp, M^\perp; P \& M}}{\vdash \Gamma, A, B, \top; \Pi, \top} \quad \frac{\vdash N^\perp; \frac{\vdash M^\perp; M^\perp}{\vdash N^\perp, M^\perp; N \& M}}{\vdash \Gamma, A, B, \top; \Pi, \top}
\]

Similarly it is possible to restrict the application of the $\top$ rule to the case where the context $\Gamma$ does not contain any formula of the shape $A \& B$, $A \& B$ or $\perp$. This is proved by induction on the sum of the sizes of formulas in $\Gamma$, using the expansions:

\[
\frac{\vdash P^\perp; P}{\vdash P^\perp, P^\perp; ?d} \quad \frac{\vdash N^\perp; N^\perp}{\vdash N^\perp, N^\perp; ?d} \quad \frac{\vdash ?P^\perp, P^\perp}{\vdash N^\perp, \top; !N^\perp} \quad \frac{\vdash ?P^\perp, P^\perp}{\vdash N^\perp, \top; !N^\perp}
\]

\[
\frac{\vdash \Gamma, A \& B, \top; \Pi, \top}{\vdash \Gamma, A \& B, \top; \Pi, \top} \quad \frac{\vdash \Gamma, A, \top; \Pi, \top}{\vdash \Gamma, A \& B, \top; \Pi} \quad \frac{\vdash \Gamma, B, \top; \Pi, \top}{\vdash \Gamma, A \& B, \top; \Pi} \quad \frac{\vdash \Gamma, \top; \Pi, \top}{\vdash \Gamma, A \& B, \top; \Pi}
\]

Lemma 1 (Reversibility)

Let $\pi$ be a proof in $\mathbb{LL}_{\text{foc}}$ with expanded axioms and expanded $\top$ rules.

- If the conclusion of $\pi$ is $\vdash \Gamma, A \& B; \Pi$, there exists a smaller (with respect to the number of rules) proof of $\vdash \Gamma, A, B; \Pi$. 

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• If the conclusion of $\pi$ is $\vdash \Gamma, \perp; \Pi$, there exists a smaller proof of $\vdash \Gamma; \Pi$.

• If the conclusion of $\pi$ is $\vdash \Gamma, A \& B; \Pi$, there exist smaller proofs of $\vdash \Gamma, A; \Pi$ and $\vdash \Gamma, B; \Pi$.

2.3 Embedding of $\mathbf{LL}$ (Weak $\rightarrow$ Focusing)

The translation ($\star$) of $\mathbf{LL}$ into $\mathbf{LL}_{\text{foc}}$ does not modify formulas, translates the sequent $\vdash \Gamma$ as $\vdash \Gamma$; and acts on proofs by adding a lot of cuts:
If $\pi$ is a proof in $\text{LL}_\text{Foc}$, then $\pi^0$ is the $\text{LL}$ proof obtained by erasing all the “;” in the sequents.

**Proposition 1** (Cut-Free Weak $\text{+-Focusing}$)

If $\pi$ is a cut-free proof of $\vdash \Gamma ; \Pi$ in $\text{LL}_\text{Foc}$ then $\pi^0$ is a weakly $\text{+-focused}$ proof of $\vdash \Gamma, \Pi$ in $\text{LL}$.

**Corollary 1.1** (Weak $\text{+-Focusing}$)

If $\vdash \Gamma$ is provable in $\text{LL}$, $\vdash \Gamma$ is provable with a weakly $\text{+-focused}$ proof.

**Proof:** Starting from a proof $\pi$ of $\vdash \Gamma$, we translate it into the proof $\pi^*$ of $\vdash \Gamma$ in $\text{LL}_\text{Foc}$. Using the cut elimination property given in Appendix A (Corollary 5.2), this leads to a cut-free proof $\pi^{*'}$ of $\vdash \Gamma$ in $\text{LL}_\text{Foc}$. By Proposition 1, $(\pi^{*'})^c$ is a weakly $\text{+-focused}$ proof of $\vdash \Gamma$ in $\text{LL}$.

### 3 Focused Linear Logic

#### 3.1 Sequent Calculus $\text{LL}_\text{Foc}$

In the spirit of [Gir01], we define a sub-system $\text{LL}_\text{Foc}$ of $\text{LL}_\text{Foc}$ in which the strictly negative formulas in the context of positive rules must be atomic. We consider a cut-free version of the system since it is sufficient for the present development (cut elimination is performed in $\text{LL}_\text{Foc}$).

There are two kinds of sequents: $\vdash \mathcal{P}, \mathcal{N};$ and $\vdash \mathcal{P}, ?\mathcal{G}, \mathcal{X}^\perp; \mathcal{P}$ where $\mathcal{P}$ contains positive formulas only, $\mathcal{N}$ contains negative formulas only, and $\mathcal{X}^\perp$ contains negative atoms only. In order to simplify the notations we will write $\vdash \mathcal{P}, \mathcal{N}; \Pi$ for either a sequent with $\Pi$ empty or a sequent...
with $\Pi = P$ and $\mathcal{N} = ?\Gamma, \mathcal{X}^\downarrow$.

$$
\begin{align*}
\vdash X^\downarrow; X^\uparrow &\quad \text{ax} \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; P^\uparrow &\quad \text{foc} \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow, P^\uparrow &\quad \mathcal{P}, X^\downarrow, A, B; \quad \mathcal{P}, X^\downarrow, \mathcal{X} \otimes B; \quad \mathcal{P}, \mathcal{X} \otimes \mathcal{X}^\downarrow \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow, P^\uparrow; P &\quad \mathcal{P}, ?\Gamma, X^\downarrow, P^\uparrow &\quad \mathcal{P}, \mathcal{P}', ?\Gamma, X^\downarrow, X^\downarrow; P \otimes Q \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow, P^\uparrow; N &\quad \mathcal{P}, ?\Gamma, X^\downarrow, P^\uparrow &\quad \mathcal{P}, \mathcal{P}', ?\Gamma, X^\downarrow, X^\downarrow; P \otimes M \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; Q &\quad \mathcal{P}, ?\Gamma, X^\downarrow; A \otimes Q \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \otimes Q \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \otimes M \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \otimes M \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \otimes M
\end{align*}
$$

$$
\begin{align*}
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; P &\quad \mathcal{P}, ?\Gamma, X^\downarrow; P \oplus B &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \oplus B \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; Q &\quad \mathcal{P}, ?\Gamma, X^\downarrow; A \oplus Q \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; M \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; N &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; N &\quad \mathcal{P}, ?\Gamma, X^\downarrow; N \\
\vdash \mathcal{P}, ?\Gamma, X^\downarrow; A &\quad \mathcal{P}, ?\Gamma, X^\downarrow; A
\end{align*}
$$

3.2 Embedding of $\mathbb{LL}_{\text{foc}}$

Let us first define the relation $\Gamma' \leq \Gamma$ on multi-sets by: any element $A$ in $\Gamma'$ also belongs to $\Gamma$ (or equivalently, we have an inclusion of the underlying sets (or supports) of $\Gamma'$ and $\Gamma$).

**Lemma 2** (Exponential Inclusion)

*If $\vdash \mathcal{P}, \mathcal{N}, ?\Gamma'$; is provable in $\mathbb{LL}_{\text{Foc}}$ and $\Gamma' \leq \Gamma$ then $\vdash \mathcal{P}, \mathcal{N}, ?\Gamma'$; is provable in $\mathbb{LL}_{\text{Foc}}$.***

We want to embed proofs of $\mathbb{LL}_{\text{foc}}$ of sequents of the shape $\vdash \Gamma$; into $\mathbb{LL}_{\text{Foc}}$. We consider a cut-free proof $\pi$ with expanded axioms (according to Section 2.2). We proceed by induction on the size (number of rules) of $\pi$ by showing simultaneously:

- if the conclusion of $\pi$ is $\vdash \Gamma$; then it is also provable in $\mathbb{LL}_{\text{Foc}}$;
- if the conclusion of $\pi$ is $\vdash \mathcal{P}, ?\Gamma, X^\downarrow; P$ then $\vdash \mathcal{P}, ?\Gamma', X^\downarrow; P$ is provable in $\mathbb{LL}_{\text{Foc}}$ for some $\Gamma' \leq \Gamma$;

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• if the conclusion of \( \pi \) is \( \vdash \mathcal{P}, \mathcal{N}; P \) and \( \mathcal{N} \) contains at least one non-atomic non-\( ? \) formula, then both \( \vdash \mathcal{P}, \mathcal{N}, P \) and \( \vdash \mathcal{P}, \mathcal{N}, ? P \) are provable in \( \text{LL}_{\text{Foc}} \).

We look at the last rule of the proof. The key cases are the \( \text{foc} \) rules and \( ?d \) rules on positive formulas (called a positive \( ?d \) rule). We only consider the \( \text{foc} \) rule (the case of the positive \( ?d \) rule is very similar). By induction hypothesis, either we have \( \vdash \mathcal{P}, ?\Gamma, \mathcal{N}; P \) in \( \text{LL}_{\text{foc}} \) thus \( \vdash \mathcal{P}, ?\Gamma', \mathcal{N}; P \) is provable in \( \text{LL}_{\text{foc}} \) with \( \Gamma' \leq \Gamma \), and we apply a \( \text{foc} \) rule and Lemma 2, or we have \( \vdash \mathcal{P}, \mathcal{N}; P \) and \( \mathcal{N} \) contains at least one non-atomic non-\( ? \) formula and we directly have \( \vdash \mathcal{P}, \mathcal{N}, P \); in \( \text{LL}_{\text{foc}} \).

In order to turn a proof of \( \vdash \mathcal{P}, \mathcal{N}; P \) in \( \text{LL}_{\text{foc}} \) into a proof of \( \vdash \mathcal{P}, \mathcal{N}, P \); in \( \text{LL}_{\text{Foc}} \), we use Lemma 1 and the induction hypothesis which, when \( \mathcal{N} \) becomes of the shape \( ?\Gamma, \mathcal{N}; P \), requires to use a \( \text{foc} \) rule and Lemma 2.

### 3.3 Focusing in \( \text{LL}_{\text{Foc}} \)

If \( \pi \) is a proof in \( \text{LL}_{\text{Foc}} \) then \( \pi^\circ \) is the \( \text{LL} \) proof obtained by erasing all the “;” in the sequents.

**Proposition 2** (Cut-Free Focusing)

If \( \pi \) is a cut-free proof of \( \vdash \mathcal{P}, \mathcal{N}; \Pi \) in \( \text{LL}_{\text{Foc}} \) then \( \pi^\circ \) is a focused proof of \( \vdash \mathcal{P}, \mathcal{N}, \Pi \) in \( \text{LL} \).

**Corollary 2.1** (Focusing)

If \( \vdash \Gamma \) is provable in \( \text{LL} \), \( \vdash \Gamma \) is provable with a focused proof.

**Proof:** Starting from a proof \( \pi \) of \( \vdash \Gamma \), we translate it into the proof \( \pi^\star \) of \( \vdash \Gamma \); in \( \text{LL}_{\text{foc}} \) and then into a cut-free proof \( \pi' \) of \( \vdash \Gamma \); in \( \text{LL}_{\text{Foc}} \). By Proposition 2, \( \pi^\circ \) is a focused proof of \( \vdash \Gamma \) in \( \text{LL} \). \( \square \)

**Remark:** It is possible to define the embedding of \( \text{LL} \) proofs into \( \text{LL}_{\text{Foc}} \) directly (without using \( \text{LL}_{\text{foc}} \) as an intermediary step). We have chosen to decompose it into two steps in order to show that the key property is \textit{weak} \( + \)-focusing. Strong \( +-\)-focusing is then obtained through reversing which is in general easy to do.

### 4 Additional Remarks

#### 4.1 Decomposition of Exponentials

The attentive reader has certainly remarked that an hidden decomposition of the exponential connectives underlies the whole text (as suggested by Girard [Gir01]):

\[
!A = \downarrow\sharp A \quad \quad ?A = \uparrow\flat A
\]

with \( \sharp A \) negative and \( \flat A \) positive, and \( \downarrow \) and \( \uparrow \) are used as the corresponding connectives of \( \text{LL}_{\text{pol}}^{\uparrow\downarrow} \) (see Appendix A.2.1).

The associated rules in \( \text{LL}_{\text{foc}} \) would be:

\[
\begin{align*}
\vdash ?\Gamma, A &; \quad \vdash \Gamma; P & \quad \vdash \Gamma; N \; \downarrow
\\vdash ?\Gamma, \sharp A \downarrow & \quad \vdash \Gamma; \flat P & \quad \vdash \Gamma; \uparrow N \; \uparrow
\end{align*}
\]
However it is difficult to give a meaning to ♯A (resp. ♯♭A) under any other connective than ↓ (resp. ↑).

4.2 Quantifiers

Our method can perfectly be extended to quantification (of any order) by adding ∃A (resp. ∀A) in positive (resp. negative) formulas and the following rules in LL_foc:

\[
\frac{}{\Gamma, P \vdash \Gamma, \exists \alpha P} \quad \frac{}{\Gamma, N \vdash \Gamma, \exists \alpha N} \quad \frac{}{\Gamma, A ; \Pi \vdash \Gamma, \forall A ; \Pi}
\]

with α free neither in Γ nor in Π in the ∀ rule.

The corresponding rules in LL_Foc are:

\[
\frac{}{\Gamma \vdash \exists \alpha P} \quad \frac{}{\exists \alpha P \vdash \exists \alpha N} \quad \frac{}{\Gamma, \exists \alpha A ; \Pi \vdash \exists \alpha N ; \Pi}
\]

with α free neither in P nor in N in the ∀ rule.

References


A Cut Elimination in $\mathbb{LL}_{fo\!c}$

A.1 Cut Elimination Steps

Key Steps

\[
\begin{array}{l}
\vdash P^\perp; P \frac{ax}{\vdash \Gamma, P^\perp; \Pi} \quad p-cut \\
\vdash \Gamma; P \frac{ax}{\vdash P^\perp; P} \quad p-cut \\
\vdash \Gamma; P \frac{foc}{\vdash \Delta, P^\perp; \Pi} \quad n-cut \\
\vdash \Gamma; P \frac{\Sigma, P^\perp, Q^\perp; \Pi}{\vdash \Sigma, Q^\perp; \Pi} \quad p-cut \\
\vdash \Gamma; P \frac{\Sigma, P^\perp, M^\perp; \Pi}{\vdash \Sigma, M^\perp; \Pi} \quad p-cut \\
\vdash \Gamma; P \frac{\Sigma, N^\perp, M^\perp; \Pi}{\vdash \Sigma, M^\perp; \Pi} \quad p-cut \\
\vdash \Gamma; P \frac{\Delta, M; \Pi}{\vdash \Delta, \Sigma; \Pi} \quad n-cut \\
\vdash \Gamma; P \frac{\Delta, P^\perp; \Pi}{\vdash \Delta, B^\perp; \Pi} \quad \& \quad p-cut \\
\vdash \Gamma; P \frac{\Delta, N^\perp; \Pi}{\vdash \Delta, B^\perp; \Pi} \quad \& \quad p-cut \\
\end{array}
\]
\[\vdash \Gamma, ?A, ?A; P \quad \vdash \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, ?A; \Pi \quad \vdash \Gamma, ?A, ?A; P \quad \vdash \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, ?A; \Pi \quad \vdash \Gamma, \Delta, ?A; \Pi \quad ?c\]

**Right Commutative p-Steps**

\[\vdash \Gamma; P \quad \vdash \Delta, P^\perp; P' \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, A, B, P^\perp; \Pi \quad \vdash \Delta, A \& B, P^\perp; \Pi \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, A, B, P^\perp; \Pi \quad \vdash \Delta, A \& B, P^\perp; \Pi \]

\[\vdash \Gamma; P \quad \vdash \Delta, P^\perp; P'; \quad \vdash \Delta, A, B, P^\perp; \Pi \quad \vdash \Delta, A \& B, P^\perp; \Pi \quad \vdash \Delta, P^\perp; P'; \quad \vdash \Delta, A, B, P^\perp; \Pi \quad \vdash \Delta, A \& B, P^\perp; \Pi \quad \vdash \Delta, P^\perp; P'; \quad \vdash \Delta, A, B, P^\perp; \Pi \quad \vdash \Delta, A \& B, P^\perp; \Pi \]

\[\vdash \Gamma; P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]

\[\vdash \Gamma, P \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Delta, P^\perp, P'; \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \quad \vdash \Gamma, \Delta, P^\perp; \Pi \]
Commutative $n$-Steps

\[
\vdash \Gamma, P ; P' ; P'' \quad \text{foc} \quad \vdash \Delta, P^\perp ; P'' \quad n\text{-cut}
\]

\[
\vdash \Gamma, A, B, P ; P' \quad \vdash \Delta, P^\perp ; A, B \quad \text{n-cut}
\]

\[
\vdash \Gamma, P ; P' ; P'' \quad \vdash \Delta, P^\perp ; \Sigma, P' \quad \text{n-cut}
\]

\[
\vdash \Gamma, P, N ; \vdash \Delta, P^\perp ; \Sigma, P' \quad \text{n-cut}
\]

\[
\vdash \Gamma, P, P' ; \Gamma, P^\perp; P'' \quad \text{n-cut}
\]

\[
\vdash \Gamma, A, B, P ; P' \quad \vdash \Delta, P^\perp ; \Sigma, P' \quad \text{n-cut}
\]

\[
\vdash \Gamma, P, N ; \vdash \Delta, P^\perp ; \Sigma, P' \quad \text{n-cut}
\]

\[
\vdash \Gamma, P, P' ; \Gamma, P^\perp; P'' \quad \text{n-cut}
\]
A.2 Cut Elimination Property

A.2.1 $\text{LL}_{\text{pol}}^{\uparrow\downarrow}$

The system $\text{LL}_{\text{pol}}^{\uparrow\downarrow}$ is obtained from $\text{LL}$ by “restricting” formulas to the following grammar:

$$P ::= X \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid !N \mid \bot N$$

$$N ::= X \perp \mid N \not\exists N \mid N \& N \mid \bot \mid \top \mid ?P \mid \uparrow P$$

with the following rules for the $\uparrow$ and $\downarrow$ connectives:

$$\vdash N, N \downarrow \quad \vdash \Gamma, P \uparrow$$

where $N$ is a multi-set of negative formulas.

A.2.2 LLP

The system LLP is obtained from LL by restricting formulas to the following grammar:

$$P ::= X \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid !N$$

$$N ::= X \perp \mid N \not\exists N \mid N \& N \mid \bot \mid \top \mid ?P$$

with the following generalizations of the exponential rules:

$$\vdash N, !N \quad \vdash \Gamma, ?P \quad \vdash \Gamma, ?w \quad \vdash \Gamma, N, !N \quad \vdash \Gamma, N, \bot N$$

where $N$ is a multi-set of negative formulas of LLP.

**Proposition 3** (Strong Normalization)

*There is no infinite sequence of reductions in LLP if we forbid commutations of cuts with cuts.*

**Proof:** Such a sequence of reductions can only contain finitely many steps between two steps that correspond to a reduction step in proof-nets. Thus by strong normalization for proof-nets [Lau02] we can conclude. \qed

**Corollary 3.1** (Cut Elimination)

*If $\vdash \Gamma$ is provable in LLP, then $\vdash \Gamma$ is provable without the cut rule.*

A.2.3 Simulations

The translation $(\cdot)\uparrow\downarrow$ of $\text{LL}_{\text{pol}}^{\uparrow\downarrow}$ into LLP is obtained by replacing $\downarrow N$ by $!N$ and $\uparrow P$ by $?P$ and the two lifting rules by promotion and dereliction.

**Lemma 3** (Polarized Formulas)

*If $A$ is a positive (resp. negative) formula in $\text{LL}_{\text{pol}}^{\uparrow\downarrow}$ then $A\uparrow\downarrow$ is a positive (resp. negative) formula in LLP.*
Proposition 4 (One-to-One Simulation)
If \( \pi \) reduces to \( \pi' \) in \( \text{LL}^{\uparrow\downarrow}_{\text{pol}} \) by one step of reduction then \( \pi^{\uparrow} \) reduces to \( \pi'^{\uparrow} \) in \( \text{LLP} \) by one step of reduction.

Corollary 4.1 (Strong Normalization)
There is no infinite sequence of reductions in \( \text{LL}^{\uparrow\downarrow}_{\text{pol}} \) if we forbid commutations of cuts with cuts.

Corollary 4.2 (Cut Elimination)
If \( \vdash \Gamma \) is provable in \( \text{LL}^{\uparrow\downarrow}_{\text{pol}} \), then \( \vdash \Gamma \) is provable without the cut rule.

The translation \((\cdot)^{\uparrow}\) of \( \text{LL}_{\text{foc}} \) into \( \text{LL}^{\uparrow\downarrow}_{\text{pol}} \) is obtained by adding to formulas exactly the required liftings to get a polarized formula, and by translating the sequent \( \vdash \mathcal{P}, \mathcal{N} ; \Pi \) by \( \vdash \uparrow \mathcal{P}^{\uparrow}, \mathcal{N}^{\uparrow}, \Pi^{\uparrow} \).

In particular \((!P \otimes 1)^{\uparrow} = \uparrow !P^{\uparrow} \otimes 1^{\uparrow}\).

Proposition 5 (Strict Simulation)
If \( \pi \) reduces to \( \pi' \) in \( \text{LL}_{\text{foc}} \) by one step of reduction then \( \pi^{\uparrow} \) reduces to \( \pi'^{\uparrow} \) in \( \text{LL}^{\uparrow\downarrow}_{\text{pol}} \) by at least one step of reduction.

Corollary 5.1 (Strong Normalization)
There is no infinite sequence of reductions in \( \text{LL}_{\text{foc}} \) if we forbid commutations of cuts with cuts.

Corollary 5.2 (Cut Elimination)
If \( \vdash \Gamma ; \Pi \) is provable in \( \text{LL}_{\text{foc}} \), then \( \vdash \Gamma ; \Pi \) is provable without cuts.

Proof: We consider a cut without any cut above it. We look at the two different cases:

- If it is a \( n \)-cut, we look at the rule above the premise \( \vdash \Gamma, \mathcal{P} ; \Pi \). If the rule above it introduces \( \mathcal{P} \), it is either a \( \text{foc} \) rule or a \( \top \) rule and we apply the corresponding key step (and the \( n \)-cut becomes a \( p \)-cut) or commutative \( n \)-step. Otherwise this rule cannot be an \( \text{ax} \) rule, a \( 1 \) rule or a \( ! \) rule and we can apply the corresponding commutative \( n \)-step.
- If it is a \( p \)-cut, we first look at the premise \( \vdash \Gamma ; \mathcal{P} \). If \( \mathcal{P} \) is not a main formula, we can apply a left commutative \( p \)-step. If \( \mathcal{P} \) is a main formula and \( \mathcal{P}^{\perp} \) is not, we can apply the corresponding right commutative \( p \)-step (notice that the rule above \( \mathcal{P}^{\perp} \) cannot be a \( 1 \) rule). We just have to verify that we can apply the right commutative \( p \)-step in the case of a \( ! \) rule above \( \mathcal{P}^{\perp} \): since \( \mathcal{P} \) is main, the rule above it is either an \( \text{ax} \) rule or a \( ! \) rule and we can apply the reduction step. If both \( \mathcal{P} \) and \( \mathcal{P}^{\perp} \) are main, we apply the corresponding key step.

So that, either the proof is cut-free or a reduction step can be applied, and we conclude by strong normalization. \( \square \)