Formalizing the meta-theory of focusing[1]

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Context

- Project: formalizing deductive & computational systems
- Encode: formulas, sequents, proof systems
- Prove: meta-theorems (think cut-admissibility)
- Tool: Abella (`http://abella-prover.org`)

A sequent calculus for linear logic

$$\frac{}{\vdash a, \bar{a}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \frac{}{\vdash 1} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \mathbin{\&} B} \qquad \frac{}{\vdash \Gamma, \top} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

A sequent calculus for linear logic

Classical, propositional, one-sided, multiplicative-additive

$$\frac{}{\vdash a, \bar{a}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \frac{}{\vdash 1} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \qquad \frac{}{\vdash \Gamma, \top} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

A sequent calculus for linear logic

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$$\frac{}{\vdash a, \bar{a}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \frac{}{\vdash 1} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \qquad \frac{}{\vdash \Gamma, \top} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta} \text{ cut}$$

Commutative cases

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdash \Gamma_1, B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \vdash \Gamma_2, C, A\end{array}}{\vdash \Gamma_1, \Gamma_2, B \otimes C, A} \qquad \begin{array}{c}\mathcal{E}\\ \vdash \Delta, \overline{A}\end{array}}{\vdash \Gamma_1, \Gamma_2, B \otimes C, D} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdash \Gamma_1, B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \vdash \Gamma_2, C, A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \vdash \Delta, \overline{A}\end{array}}{\vdash \Gamma_2, C, \Delta}}{\vdash \Gamma_1, \Gamma_2, B \otimes C, \Delta}$$

Principal cases

$$
\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash \Gamma_1, A} \qquad \overset{\mathcal{D}_2}{\vdash \Gamma_2, B}}{\vdash \Gamma_1, \Gamma_2, A \otimes B} \qquad \dfrac{\overset{\mathcal{E}}{\vdash \Delta, \overline{A}, \overline{B}}}{\vdash \Delta, \overline{A} \parr \overline{B}}}{\vdash \Gamma_1, \Gamma_2, \Delta}
\quad \rightsquigarrow \quad
\dfrac{\overset{\mathcal{D}_1}{\vdash \Gamma_1, A} \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash \Gamma_2, B} \qquad \overset{\mathcal{E}}{\vdash \Delta, \overline{A}, \overline{B}}}{\vdash \Gamma_2, \Delta, \overline{A}}}{\vdash \Gamma_1, \Gamma_2, \Delta}
$$

Principal cases – or maybe

$$
\genfrac{}{}{1.5pt}{}{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash \Gamma_1, A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \vdash \Gamma_2, B\end{matrix}}{\vdash \Gamma_1, \Gamma_2, A \otimes B} \qquad \begin{matrix}\mathcal{E}\\ \vdash \Delta, \overline{A} \parr \overline{B}\end{matrix}}{\vdash \Gamma_1, \Gamma_2, \Delta}
\quad \rightsquigarrow
$$

$$
\genfrac{}{}{1.5pt}{}{\begin{matrix}\mathcal{D}_1\\ \vdash \Gamma_1, A\end{matrix} \qquad \genfrac{}{}{1.5pt}{}{\begin{matrix}\mathcal{D}_2\\ \vdash \Gamma_2, B\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \vdash \Delta, \overline{A} \parr \overline{B}\end{matrix}}{\vdash \Delta, \overline{A}, \overline{B}}\ \parr^{-1}}{\vdash \Gamma_2, \Delta, \overline{A}}}{\vdash \Gamma_1, \Gamma_2, \Delta}
$$

Lexicographic induction

$$P, Q, \ldots ::= a \mid A \otimes B \mid 1 \mid A \oplus B \mid 0$$
$$N, M, \ldots ::= \bar{a} \mid A \parr B \mid \bot \mid A \& B \mid \top$$

Lexicographic induction

$$P, Q, \ldots ::= a \mid A \otimes B \mid 1 \mid A \oplus B \mid 0$$
$$N, M, \ldots ::= \overline{a} \mid A \parr B \mid \bot \mid A \,\&\, B \mid \top$$

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Gamma, P} \qquad \vdash \Delta, \overline{P}}{\vdash \Gamma, \Delta}$$

Lexicographic induction

$$P, Q, \ldots ::= a \mid A \otimes B \mid 1 \mid A \oplus B \mid 0$$
$$N, M, \ldots ::= \bar{a} \mid A \parr B \mid \bot \mid A \& B \mid \top$$

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Gamma, P} \quad \vdash \Delta, \overline{P}}{\vdash \Gamma, \Delta}$$

The inductive hypothesis may be used whenever:

1. either the size of P (= rank) is smaller, or
2. the rank is the same and the height of $\mathcal{D}$ is smaller.

Encoding an inference rule

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

Encoding an inference rule

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash \Gamma \quad (\Gamma = \Gamma', A) \qquad \vdash \Delta \quad (\Delta = \Delta', B)}{\vdash (\Gamma', \Delta'), A \otimes B}$$

Encoding an inference rule

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash \Gamma \quad (\Gamma = \Gamma', A) \qquad \vdash \Delta \quad (\Delta = \Delta', B)}{\vdash (\Gamma', \Delta'), A \otimes B}$$

$$\frac{\vdash \Gamma \qquad\qquad \vdash \Delta}{\vdash ((\Gamma \setminus A), (\Delta \setminus B)), A \otimes B}$$

Principal cases

$$\frac{\vdash \Gamma \qquad\qquad \vdash \Delta}{\vdash (\Gamma \setminus A), (\Delta \setminus \overline{A})}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash \Gamma_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \vdash \Gamma_2\end{matrix}}{\vdash ((\Gamma_1 \setminus A), (\Gamma_2 \setminus B)), A \otimes B} \qquad \begin{matrix}\mathcal{E}\\ \vdash \Delta\end{matrix}}{\vdash ((\Gamma_1 \setminus A), (\Gamma_2 \setminus B)), (\Delta \setminus \overline{A} \parr \overline{B})} \quad \rightsquigarrow$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash \Gamma_1\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ \vdash \Gamma_2\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \vdash \Delta\end{matrix}}{\vdash ((\Delta \setminus \overline{A} \parr \overline{B}), \overline{A}), \overline{B}}\ \parr^{-1}}{\vdash (\Gamma_2 \setminus B), ((\Delta \setminus \overline{A} \parr \overline{B}), \overline{A})}}{\vdash ((\Gamma_2 \setminus B), (\Delta \setminus \overline{A} \parr \overline{B})), \overline{A}}\ =}{\dfrac{\vdash (\Gamma_1 \setminus A), ((\Gamma_2 \setminus B), (\Delta \setminus \overline{A} \parr \overline{B}))}{\vdash ((\Gamma_1 \setminus A), (\Gamma_2 \setminus B)), (\Delta \setminus \overline{A} \parr \overline{B})}\ =}$$

A multiset theory

- $\Gamma, \Delta = \Delta, \Gamma$
- $(\Gamma, \Delta), \Omega = (\Gamma, \Delta), \Omega$
- $(\Gamma \setminus A) \setminus B = (\Gamma \setminus B) \setminus A$
- $(\Gamma, \Delta) \setminus A = \begin{cases} (\Gamma \setminus A), \Delta \\ \Gamma, (\Delta \setminus A) \end{cases}$

Formalizing multisets

```
Define adj : olist -> o -> olist -> prop by
  adj nil A (A :: nil) ;
  adj (B :: L) A (B :: K) := adj L A K.

Define merge : olist -> olist -> olist -> prop by
  merge nil nil nil ;
  merge J K L :=
    exists A JJ LL, adj JJ A J /\ adj LL A L
      /\ merge JJ K LL ;
  merge J K L :=
    exists A KK LL, adj KK A K /\ adj LL A L
      /\ merge J KK LL.

Define perm : olist -> olist -> prop by
  perm J K := merge J nil K.
```

Multiset properties (example)

- $(\Gamma, \Delta), \Omega = \Gamma, (\Delta, \Omega)$

```
Theorem merge_assoc : forall G D W GD DW GDW,
  merge G D GD -> merge GD W GDW ->
  merge D W DW -> merge G DW GDW.

Theorem merge_assoc_perm : forall G D W GD DW U V,
  merge G D GD -> merge GD W U ->
  merge D W DW -> merge G DW V ->
  perm U V.
```

Formalizing Formulas

```
Kind atm type.

Type atom, natom   atm -> o.
Type tens, par     o -> o -> o.
Type one,  bot     o.
Type wth,  plus    o -> o -> o.
Type top,  zero    o.

Define dual : o -> o -> prop by
; dual (atom A) (natom A)
; dual (tens A B) (par AA BB) := dual A AA /\ dual B BB
; dual one bot
; dual (plus A B) (wth AA BB) := dual A AA /\ dual B BB
; dual zero top.
```

Formalizing MALL

```
Define mall : olist -> prop by
  % init
; mall L :=
    exists A, adj (natom A :: nil) (atom A) L

  % tensor
; mall L :=
    exists A B LL, adj LL (tens A B) L /\
    exists JJ KK, merge JJ KK LL /\
    exists J, adj JJ A J /\ mall J /\
    exists K, adj KK B K /\ mall K.

; mall (one :: nil)

  % par
; mall L :=
    exists A B LL, adj LL (par A B) L /\
    exists J, adj LL A J /\
    exists K, adj J B K /\
```

Meta-theory: inversion

$$\frac{\vdash \Gamma, A \parr B}{\vdash \Gamma, A, B} \parr^{-1}$$

```
Theorem par_inv : forall G A B D,
  mall G -> adj D (tens A B) G ->
  exists U V, adj D A U /\ adj U B V /\ mall V.
```

Meta-theory: cut

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash (\Gamma \setminus A), (\Delta \setminus \overline{A})}$$

```
Theorem cut_admit : forall A B G GG D DD W,
  dual A B ->
  mall G -> adj GG A G ->
  mall D -> adj DD B D ->
  merge GG DD W -> mall W.
induction on 1. induction on 2.
  ...
```

Exponentials

$$\frac{}{\vdash \Xi; a, \bar{a}} \qquad \frac{\vdash \Xi; \Gamma, A \quad \vdash \Xi; \Delta, B}{\vdash \Xi; \Gamma, \Delta, A \otimes B} \qquad \frac{}{\vdash \Xi; 1} \qquad \frac{\vdash \Xi; \Gamma, A, B}{\vdash \Xi; \Gamma, A \parr B} \qquad \frac{\vdash \Xi; \Gamma}{\vdash \Xi; \Gamma, \bot}$$

$$\frac{\vdash \Xi; \Gamma, A \quad \vdash \Xi; \Gamma, B}{\vdash \Xi; \Gamma, A \,\&\, B} \qquad \frac{}{\vdash \Xi; \Gamma, \top} \qquad \frac{\vdash \Xi; \Gamma, A}{\vdash \Xi; \Gamma, A \oplus B} \qquad \frac{\vdash \Xi; \Gamma, B}{\vdash \Xi; \Gamma, A \oplus B}$$

. . . .

$$\frac{\vdash \Xi; A}{\vdash \Xi; !A} \qquad \frac{\vdash \Xi, A; \Gamma}{\vdash \Xi; \Gamma, ?A} \qquad \frac{\vdash \Xi, A; \Gamma, A}{\vdash \Xi, A; \Gamma}$$

Exponentials: lexicographic cuts

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \Xi; P\end{array} \quad \vdash \Xi, \overline{P}; \Gamma}{\vdash \Xi; \Gamma}\ \text{hcut} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \Xi; \Gamma, P\end{array} \quad \vdash \Xi; \Gamma, \overline{P}}{\vdash \Xi; \Gamma, \Delta}\ \text{lcut}$$

The inductive hypothesis may be used whenever:

1. either the size of P (= rank) is smaller; or
2. the rank is the same and the height of $\mathcal{D}$ is smaller; or
3. a lcut is reduced to a hcut of the same rank and $\mathcal{D}$-height.

Encoding dyadic sequents

```
Define mell : olist -> olist -> prop by
; mell QL L :=
    exists A, adj (natom A :: nil) (atom A) L

; mell QL L :=
    exists A QK, adj QK A QL /\
    exists J, adj L A J /\
    mell QL J

; mell QL L :=
    exists A B LL, adj LL (tens A B) L /\
    exists JJ KK, merge JJ KK LL /\
      (exists J, adj JJ A J /\ mell QL J) /\
      (exists K, adj KK B K /\ mell QL K)

; mell QL (one :: nil)

; mell QL L :=
    exists A B LL, adj LL (par A B) L /\
```

Meta-theory: lexicographic cuts

```
Kind weight type.
Type heavy, light weight.

Define is_weight : weight -> prop by
; is_weight light
; is_weight heavy := is_weight light.

Theorem cut_admit: forall A B G Wt,
   dual A B -> mell QL G -> is_weight Wt ->
      (Wt = heavy /\ G = A :: nil /\
       forall QK, adj QL B QK -> mell QK D ->
       mell QL D)
   \/ (Wt = light /\
       forall GG D DD W, adj GG A G ->
       mell QL D -> adj DD B D ->
       merge GG DD W -> mell QL W).
induction on 1. induction on 2. induction on 3.
  ...
```

Focusing

$$\frac{}{\vdash_{\mathsf{F}} \ [a]\,,\bar{a}} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, [A] \qquad \vdash_{\mathsf{F}} \Delta, [B]}{\vdash_{\mathsf{F}} \Gamma, \Delta, [A \otimes B]} \qquad \frac{}{\vdash_{\mathsf{F}} \ [1]} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, A, B}{\vdash_{\mathsf{F}} \Gamma, A \parr B} \qquad \frac{\vdash_{\mathsf{F}} \Gamma}{\vdash_{\mathsf{F}} \Gamma, \bot}$$

$$\frac{\vdash_{\mathsf{F}} \Gamma, A \qquad \vdash_{\mathsf{F}} \Gamma, B}{\vdash_{\mathsf{F}} \Gamma, A \,\&\, B} \qquad \frac{}{\vdash_{\mathsf{F}} \Gamma, \top} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, [A]}{\vdash_{\mathsf{F}} \Gamma, [A \oplus B]} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, [B]}{\vdash_{\mathsf{F}} \Gamma, [A \oplus B]}$$

Focusing

$$\frac{}{\vdash_F [a], \bar{a}} \qquad \frac{\vdash_F \Gamma, [A] \quad \vdash_F \Delta, [B]}{\vdash_F \Gamma, \Delta, [A \otimes B]} \qquad \frac{}{\vdash_F [1]} \qquad \frac{\vdash_F \Gamma, A, B}{\vdash_F \Gamma, A \parr B} \qquad \frac{\vdash_F \Gamma}{\vdash_F \Gamma, \bot}$$

$$\frac{\vdash_F \Gamma, A \quad \vdash_F \Gamma, B}{\vdash_F \Gamma, A \,\&\, B} \qquad \frac{}{\vdash_F \Gamma, \top} \qquad \frac{\vdash_F \Gamma, [A]}{\vdash_F \Gamma, [A \oplus B]} \qquad \frac{\vdash_F \Gamma, [B]}{\vdash_F \Gamma, [A \oplus B]}$$

....

$$\frac{\vdash_F \Gamma, [P] \quad (\Gamma \text{ all neutral})}{\vdash_F \Gamma, P} \qquad \frac{\vdash_F \Gamma, N}{\vdash_F \Gamma, [N]}$$

$$\begin{aligned} P, Q, \ldots &::= a \mid A \otimes B \mid 1 \mid A \oplus B \mid 0 \\ N, M, \ldots &::= \bar{a} \mid A \parr B \mid \bot \mid A \,\&\, B \mid \top \\ \text{neutral} &::= \bar{a} \mid P \end{aligned}$$

Focusing example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\vdash \bar{a}, a} \qquad \dfrac{}{\vdash \bar{b}, b}
    }{
      \vdash \bar{a}, \bar{b}, a \otimes b
    }
  }{
    \vdash \bar{a} \parr \bar{b}, a \otimes b
  }
  \qquad
  \dfrac{
    \dfrac{}{\vdash \bar{c}, c}
  }{
    \vdash \bar{a} \oplus \bar{c}, c
  }
}{
  \vdash \bar{a} \parr \bar{b}, \bar{a} \oplus \bar{c}, a \otimes b \otimes c
}
\quad \Rightarrow \quad
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{\vdash_{\mathsf{F}} \bar{a}, [a]} \qquad \dfrac{}{\vdash_{\mathsf{F}} \bar{b}, [b]} \qquad \dfrac{}{\vdash_{\mathsf{F}} \bar{c}, [c]}
          }{
            \vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{c}, [a \otimes b \otimes c]
          }
        }{
          \vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{c}, a \otimes b \otimes c
        }
      }{
        \vdash_{\mathsf{F}} \bar{a}, \bar{b}, [\bar{c}], a \otimes b \otimes c
      }
    }{
      \vdash_{\mathsf{F}} \bar{a}, \bar{b}, [\bar{a} \oplus \bar{c}], a \otimes b \otimes c
    }
  }{
    \vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{a} \oplus \bar{c}, a \otimes b \otimes c
  }
}{
  \vdash_{\mathsf{F}} \bar{a} \parr \bar{b}, \bar{a} \oplus \bar{c}, a \otimes b \otimes c
}
$$

Meta-theory of focusing

- The main theorem is:

$$(\vdash \Gamma) \qquad \text{iff} \qquad (\vdash_{\mathsf{F}} \Gamma)$$

- Many ways to prove this.
- One way is via cut-elimination

$$\frac{\vdash_{\mathsf{F}} \Gamma, P \qquad \vdash_{\mathsf{F}} \Delta, \overline{P}}{\vdash_{\mathsf{F}} \Gamma, \Delta} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, [A] \qquad \vdash_{\mathsf{F}} \Delta, \overline{A}}{\vdash_{\mathsf{F}} \Gamma, \Delta}$$

Meta-theory of focusing

- The main theorem is:

$$(\vdash \Gamma) \qquad \text{iff} \qquad (\vdash_{\mathsf{F}} \Gamma)$$

- Many ways to prove this.
- One way is via cut-elimination

$$\frac{\vdash_{\mathsf{F}} \Gamma, P \qquad \vdash_{\mathsf{F}} \Delta, \overline{P}}{\vdash_{\mathsf{F}} \Gamma, \Delta} \qquad \frac{\vdash_{\mathsf{F}} \Gamma, [A] \qquad \vdash_{\mathsf{F}} \Delta, \overline{A}}{\vdash_{\mathsf{F}} \Gamma, \Delta}$$

$$\frac{\vdash_{\mathsf{F}} \Gamma, [A] \qquad \vdash_{\mathsf{F}} \Delta, \overline{A}, [B]}{\vdash_{\mathsf{F}} \Gamma, \Delta, [B]}$$

Polarized focusing

$$
\begin{aligned}
P, Q, \ldots &::= a \mid P \otimes Q \mid 1 \mid P \oplus Q \mid 0 \mid {\downarrow} N \\
N, M, \ldots &::= \bar{a} \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid {\uparrow} P \\
\text{neutral} &::= \bar{a} \mid {\uparrow} P
\end{aligned}
$$

$$
\frac{}{\vdash_{\mathsf{P}} [a], \bar{a}} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma, [P] \qquad \vdash_{\mathsf{P}} \Gamma, [Q]}{\vdash_{\mathsf{P}} \Gamma, \Delta, [P \otimes Q]} \qquad
\frac{}{\vdash_{\mathsf{P}} [1]} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma, N, M}{\vdash_{\mathsf{P}} \Gamma, N \parr M} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma}{\vdash_{\mathsf{P}} \Gamma, \bot}
$$

$$
\frac{\vdash_{\mathsf{P}} \Gamma, N \qquad \vdash_{\mathsf{P}} \Gamma, M}{\vdash_{\mathsf{P}} \Gamma, N \,\&\, M} \qquad
\frac{}{\vdash_{\mathsf{P}} \Gamma, \top} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma, [P]}{\vdash_{\mathsf{P}} \Gamma, [P \oplus Q]} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma, [Q]}{\vdash_{\mathsf{P}} \Gamma, [P \oplus Q]}
$$

. . . .

$$
\frac{\vdash_{\mathsf{P}} \Gamma, [P] \qquad (\Gamma \text{ all neutral})}{\vdash_{\mathsf{P}} \Gamma, {\uparrow} P} \qquad
\frac{\vdash_{\mathsf{P}} \Gamma, N}{\vdash_{\mathsf{P}} \Gamma, [{\downarrow} N]}
$$

Synthetic focusing

$$\frac{}{a \in a} \qquad \frac{}{{\downarrow}N \in {\downarrow}N} \qquad \frac{\Omega_1 \in P \quad \Omega_2 \in Q}{\Omega_1, \Omega_2 \in P \otimes Q} \qquad \frac{}{\cdot \in \mathsf{1}}$$

$$\frac{\Omega \in P}{\Omega \in P \oplus Q} \qquad \frac{\Omega \in Q}{\Omega \in P \oplus Q}$$

. . . .

$$\frac{}{\vdash_{\mathsf{S}} \overline{a} : [a]} \qquad \frac{}{\vdash_{\mathsf{S}} \cdot : [\cdot]} \qquad \frac{\vdash_{\mathsf{S}} \Gamma_1 : [\Omega_1] \quad \vdash_{\mathsf{S}} \Gamma_2 : [\Omega_2]}{\vdash_{\mathsf{S}} \Gamma_1, \Gamma_2 : [\Omega_1, \Omega_2]}$$

. . . .

$$\frac{(\Omega \in P) \quad \vdash_{\mathsf{S}} \Gamma : [\Omega]}{\vdash_{\mathsf{S}} \Gamma, {\uparrow}P} \qquad \frac{\left\{ \vdash_{\mathsf{S}} \Gamma, \overline{\Omega} \right\}_{\Omega \in P}}{\vdash_{\mathsf{S}} \Gamma : [{\downarrow}\overline{P}]}$$

Synthetic focusing

$$\frac{}{a \in a} \qquad \frac{}{\downarrow N \in \downarrow N} \qquad \frac{\Omega_1 \in P \quad \Omega_2 \in Q}{\Omega_1, \Omega_2 \in P \otimes Q} \qquad \frac{}{\cdot \in 1}$$

$$\frac{\Omega \in P}{\Omega \in P \oplus Q} \qquad \frac{\Omega \in Q}{\Omega \in P \oplus Q} \qquad \frac{\Omega \in [t/x]P}{\Omega \in \exists x.P}$$

$$\frac{}{\vdash_S \bar{a} : [a]} \qquad \frac{}{\vdash_S \cdot : [\cdot]} \qquad \frac{\vdash_S \Gamma_1 : [\Omega_1] \quad \vdash_S \Gamma_2 : [\Omega_2]}{\vdash_S \Gamma_1, \Gamma_2 : [\Omega_1, \Omega_2]}$$

$$\frac{(\Omega \in P) \quad \vdash_S \Gamma : [\Omega]}{\vdash_S \Gamma, \uparrow P} \qquad \frac{\left\{ \vdash_S \Gamma, \overline{\Omega} \right\}_{\Omega \in P}}{\vdash_S \Gamma : [\downarrow \overline{P}]}$$

Synthetic focusing example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash_{\mathsf{F}} \bar{a}, [a]} \quad \cfrac{}{\vdash_{\mathsf{F}} \bar{b}, [b]} \quad \cfrac{}{\vdash_{\mathsf{F}} \bar{c}, [c]}}{\vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{c}, [a \otimes b \otimes c]}}{\vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{c}, a \otimes b \otimes c}}{\vdash_{\mathsf{F}} \bar{a}, \bar{b}, [\bar{c}], a \otimes b \otimes c}}{\vdash_{\mathsf{F}} \bar{a}, \bar{b}, [\bar{a} \oplus \bar{c}], a \otimes b \otimes c}}{\vdash_{\mathsf{F}} \bar{a}, \bar{b}, \bar{a} \oplus \bar{c}, a \otimes b \otimes c}}{\vdash_{\mathsf{F}} \bar{a} \parr \bar{b}, \bar{a} \oplus \bar{c}, a \otimes b \otimes c}
\quad \Rightarrow \quad
\cfrac{\cfrac{\cfrac{}{\vdash_{\mathsf{S}} \bar{a}, \bar{b}, \bar{c}, \uparrow(a \otimes b \otimes c)}}{\vdash_{\mathsf{S}} \bar{a}, \bar{b}, \uparrow(\bar{a} \oplus \bar{c}), \uparrow(a \otimes b \otimes c)}}{\vdash_{\mathsf{S}} \downarrow\uparrow(\bar{a} \parr \bar{b}), \uparrow(\bar{a} \oplus \bar{c}), \uparrow(a \otimes b \otimes c)}
$$

Formalizing synthetics

```
Kind foc type.
Type fatom  atm -> foc.
Type fshift nf -> foc.
Type fjoin  foc -> foc -> foc.
Type femp   foc.

Define subf : foc -> pf -> prop by
; subf (fatom A) (atom A)
; subf (fshift N) (shp N)
; subf (fjoin F1 F2) (tens P Q) :=
    subf F1 P /\ subf F2 Q
; subf F (oplus P Q) := subf F P
; subf F (oplus P Q) := subf F Q
; nabla x, subf (F x) (fex P) :=
    nabla x, subf (F x) (P x)
; subf femp one.
```

Formalizing synthetic focusing

```
Define
  mall : olist -> prop,
  mallfoc : olist -> foc -> prop
by
; mall L :=
    exists P LL, adj LL P L /\
    exists F,    subf F P /\
                 mallfoc LL F

; mallfoc (natom A :: nil) (fatom A)

; mallfoc L (fshift N) :=
    exists P,  dual P N /\
    forall F,  subf F P ->
    exists LL, extend L F LL /\
               mall LL

; mallfoc L (fjoin F1 F2) :=
    exists J K, merge J K L /\
    mallfoc J F1 /\ mallfoc K F2

; mallfoc nil femp.
```

Synthetic meta-theory

$$\frac{\vdash \Gamma, \uparrow P \quad \vdash \Delta, \uparrow\downarrow \overline{P}}{\vdash \Gamma, \Delta}$$

```
Theorem cut_admit : forall P N J JP K L,
   dual P N ->
   adj J (shn P) JP        -> mall JP ->
   adj K (shn (shp N)) KN -> mall KN ->
   merge J K L             -> mall L.
```

Synthetic meta-theory: but really!

Principal:

$$\frac{(\Omega \in P) \quad \vdash \Gamma : [\Omega] \quad \vdash \Delta : [\downarrow \overline{P}]}{\vdash \Gamma, \Delta}$$

Commutative:

$$\frac{\vdash \Gamma, \uparrow P : [\Omega] \quad \vdash \Delta : [\downarrow \overline{P}]}{\vdash \Gamma, \Delta : [\Omega]}$$

Some stats

`https://github.com/meta-logic/abella-reasoning`

	ordinary	synthetic
Prop. MALL	774	—
Prop. MELL	694	—
Prop. LL	1230	—
$\forall$MALL	—	490
$\forall$LL	—	—

(unfiltered lines of text)

Abella's induction

So far we have used Abella's *implicit* heights in our inductive proofs.

```
Theorem trivial : forall x, nat x -> lt x (s x).
induction on 1. intros. case H1.
  search.
  apply IH to H2. search.
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IH : forall x, nat x * -> lt x (s x)
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 forall x, nat x @ -> lt x (s x)
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============================
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```

```
Variables: X
IH : forall x, nat x * -> lt x (s x)
H2 : nat X *
============================
 lt (s X) (s (s X))
```

Explicit heights in sequents

```
Define mallh : nat -> olist -> prop by
  % init
; mallh (s H) L :=
    exists A, adj (natom A :: nil) (atom A) L

  % tensor
; mallh (s H) L :=
    exists A B LL, adj LL (tens A B) L /\
    exists JJ KK, merge JJ KK LL /\
    exists J, adj JJ A J /\ mallh H J /\
    exists K, adj KK B K /\ mallh H K.

; mallh (s H) (one :: nil)

  % par
; mallh (s H) L :=
    exists A B LL, adj LL (par A B) L /\
    exists J, adj LL A J /\
    exists K, adj J B K /\ mallh H K

; mallh (s H) L :=
    exists LL, adj LL bot L /\ mallh H LL
...
```

Equivalence

Abella's induction is strong enough to prove the following theorems.

```
Theorem equiv_ltr: forall H L, mallh H L -> mall L.

Theorem equiv_rtl: forall L, mall L -> exists H, nat H /\ mallh H L.
```

Equivalence

Abella's induction is strong enough to prove the following theorems.

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Theorem equiv_ltr: forall H L, mallh H L -> mall L.

Theorem equiv_rtl: forall L, mall L -> exists H, nat H /\ mallh H L.
```

This allows us to drop down to explicit heights if we ever need them, but to continue without explicit heights in general.

A mystery

- I initially thought that we could do the same thing for the synthetic version of the calculus.
- To my surprise:
  - I couldn't show the equivalence between the two formalisms
  - I couldn't even prove cut-elimination by itself in the version with explicit heights
- The culprit is the following rule:

$$\frac{\left\{ \vdash \Gamma, \overline{\Omega} \right\}_{\Omega \in P}}{\vdash \Gamma : [\downarrow \overline{P}]}$$

Synthetic focusing with explicit heights

For H a natural number, write

$H \vdash \Gamma$	neutral sequent of max. deriv. height H
$H \vdash \Gamma : [\Omega]$	focused sequent of max. deriv. height H

$$\frac{}{a \in a} \qquad \frac{}{\downarrow N \in \downarrow N} \qquad \frac{\Omega_1 \in P \quad \Omega_2 \in Q}{\Omega_1, \Omega_2 \in P \otimes Q} \qquad \frac{}{\cdot \in 1}$$

$$\frac{\Omega \in P}{\Omega \in P \oplus Q} \qquad \frac{\Omega \in Q}{\Omega \in P \oplus Q} \qquad \frac{\Omega \in [t/x]P}{\Omega \in \exists x.P}$$

$$\frac{}{H \vdash \overline{a} : [a]} \qquad \frac{}{H \vdash \cdot : [\cdot]} \qquad \frac{H \vdash \Gamma_1 : [\Omega_1] \quad H \vdash \Gamma_2 : [\Omega_2]}{H \vdash \Gamma_1, \Gamma_2 : [\Omega_1, \Omega_2]}$$

$$\frac{(\Omega \in P) \quad H \vdash \Gamma : [\Omega]}{(H+1) \vdash \Gamma, \uparrow P} \qquad \frac{\left\{ H \vdash \Gamma, \overline{\Omega} \right\}_{\Omega \in P}}{(H+1) \vdash \Gamma : [\downarrow \overline{P}]}$$

Cut rules with explicit heights

Principal:

$$\frac{(\Omega \in P) \quad H_1 \vdash \Gamma : [\Omega] \quad H_2 \vdash \Delta : [\downarrow\overline{P}]}{H_3 \vdash \Gamma, \Delta}$$

Commutative:

$$\frac{H_1 \vdash \Gamma, \uparrow P : [\Omega] \quad H_2 \vdash \Delta : [\downarrow\overline{P}]}{H_3 \vdash \Gamma, \Delta : [\Omega]}$$

H_1, H_2, H_3 unrelated, or maybe $H_1 \; H_2 \geq H_3 \geq H_1 + H_2$.

Commutative case in detail

$$\dfrac{H_1 \vdash \Gamma, \uparrow P : [\Omega] \qquad \dfrac{\left\{ H_2 \vdash \Delta, \overline{\Omega'} \right\}_{\Omega' \in P}}{H_2 + 1 \vdash \Delta : [\downarrow \overline{P}]}}{H_3 \vdash \Gamma, \Delta : [\Omega]}$$

$$\begin{aligned} IH: &\quad \forall \Omega' \in P.\ \exists H_3.\ H_3 \vdash \Gamma, \Delta : [\Omega] \\ \text{goal}: &\quad \exists H_3'.\ \forall \Omega' \in P.\ H_3' \vdash \Gamma', \Delta, \overline{\Omega'} \end{aligned}$$

Strong continuity?

This is reminiscent of the so called *strong continuity principle*, which is a consequence of Brouwer's fan theorem:

$$(\forall s \in \mathbb{N}^*.\, \exists p.\, A(s,p)) \supset (\exists f \in K_0.\, \forall s.\, A(s, f(s)))$$

where K_0 is the class of all computable functions that depend on only a finite prefix of its input.

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TODO: figure out if Abella's induction has enough strength to do some kinds of bar induction (cf. recent work by Vincent Rahli, Bob Constable, *et al*), and then to see if the SCP or a suitable variant can be derived.