Focusing in Orthologic*

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—— **Abstract** ——

We propose new sequent calculus systems for *orthologic* (also known as *minimal quantum logic*) which satisfy the cut elimination property. The first one is a very simple system relying on the involutive status of negation. The second one incorporates the notion of *focusing* (coming from linear logic) to add constraints on proofs and thus to facilitate proof search. We demonstrate how to take benefits from the new systems in automatic proof search for orthologic.

Classical (propositional) logic can be used to reason about facts in classical mechanics and is related with the lattice structure of Boolean algebras. On its side, quantum (propositional) logic has been introduced to represent observable facts in quantum mechanics. It is provided as an axiomatization of the lattice structure of the closed subspaces of Hilbert spaces. This corresponds to the structure of so-called orthomodular lattices. Among the properties of these lattices, and thus of quantum logic, one finds the orthomodularity law ($a \leq b \implies b \leq a \vee (\neg a \wedge b)$) which is a very weak form of distributivity. Removing this law gives the notion of *ortholattice* and leads to the associated *orthologic* (also called minimal quantum logic, as it can be defined as quantum logic without orthomodularity). In the description and reasoning about quantum properties, quantum logic is more accurate than orthologic. Nevertheless a formula valid in orthologic is also valid in quantum logic, and thus provides a valid quantum property. In the current state of the art, orthologic benefits of much better logical properties than plain quantum logic (in proof theory in particular) and, since it also corresponds to a nice class of lattices, many authors focus on it [9, 14, 10, 13, 4]. From the point of view of lattice theory, ortholattices are bounded lattices with an involutive negation such that $p \vee \neg p = \top$. As a consequence they can be understood as Boolean lattices without distributivity, and indeed distributive ortholattices are exactly Boolean lattices.

The main topic of the present work is the study of the proof theory of orthologic, from the sequent calculus point of view. Sound and complete sequent calculi satisfying the cut-elimination property already occur in the literature (see for example [14, 13, 6]). Our first result is another such calculus which is particularly simple: each sequent has exactly two formulas and only seven rules are required. It relies on ideas of J.-Y. Girard in linear logic [7] for the representation of systems with an involutive negation, and shows how orthologic can be seen as an extension of the additive fragment of linear logic with one new contraction-weakening rule. The second and main contribution of this paper lies in the development of a "second-level proof-theory" for orthologic by investigating the notion of focusing in this setting.

Focusing, introduced in linear logic by J.-M. Andreoli [1], is a constraint on the structure of proofs which requires connectives sharing some structural properties (like reversibility) to be grouped together. The key point is that this restriction is sound and complete: focused proofs are proofs and any provable sequent admits a focused proof. Together with cut elimination, focusing can be used as a strong tool in proof search and proof study since it reduces the search space to focused proofs. Focusing has also been used to define new logical systems [8].

In the case of orthologic, we show that focusing can be defined and interacts particularly well with the 2-formulas sequents. In particular, not only logical rules associated with connectives are constrained but also structural rules can be organised. The exchange rule can be hidden easily in the specific focusing rules and the contraction-weakening rule becomes precisely constrained. As a consequence, we obtain a bound on the height of all focused proofs of a given sequent, which is rarely the case in the presence of a contraction rule. Starting from this remark, we experiment proof search strategies for orthologic based on our focused system.

In Section 1, we recall the definition of ortholattice and orthologic with the main results from the literature on sequent calculus and cut-elimination for orthologic. In Section 2, we introduce the sequent calculus OL (inspired by additive linear logic) with a few properties. Section 3 gives the two-steps construction of the focused system $\mathsf{OL_f}$. We explain how focusing is applied to orthologic and we prove soundness, completeness and cut-elimination. The last Section 4 is dedicated to the application of $\mathsf{OL_f}$ in (backward and forward) proof search for orthologic. This is based on upper bounds on the height of proofs and on additional structural properties of focused cut-free proofs.

1   Ortholattices and Orthologic

Orthologic or *minimal quantum logic* is the logic associated with the order relation of *ortholattices* (for some results about ortholattices, see for example [2]).

▶ **Definition 1** (Ortholattice)**.** An *ortholattice* $\mathcal{O}$ is a bounded lattice (a lattice with smallest and biggest elements $\bot$ and $\top$) with an order-reversing involution $p \mapsto \neg p$ (also often denoted $p^\perp$ in the literature), called *orthocomplement*, satisfying $p \vee \neg p = \top$ (for all p in $\mathcal{O}$).

In particular the following properties hold for any two elements p and q of any ortholattice: $p \leq q \implies \neg q \leq \neg p$, $\neg\neg p = p$, $\neg\bot = \top$, $\neg(p \vee q) = \neg p \wedge \neg q$, $p \wedge \neg p = \bot$, as well as the other De Morgan's laws, but there is no distributivity law between $\wedge$ and $\vee$.

Orthologic is the logic associated with the class of ortholattices, or conversely ortholattices are the algebras associated with orthologic. Formulas in orthologic are built using connectives corresponding to the basic operations of ortholattices:

$$A ::= X \mid A \wedge A \mid A \vee A \mid \top \mid \bot \mid \neg A$$

where X ranges over elements of a given countable set $\mathcal{X}$ of *variables*.

We want then $A \vdash B$ to be derivable in orthologic if and only if $A \leq B$ is true in any ortholattice $\mathcal{O}$ (for every interpretation of variables as elements of $\mathcal{O}$, and with connectives in A and B interpreted through the corresponding operations of $\mathcal{O}$). In particular, the Lindenbaum algebra associated with orthologic over the set $\mathcal{X}$ is the free ortholattice over $\mathcal{X}$ (which is infinite as soon as $\mathcal{X}$ contains at least two elements [3]).

If we adopt a sequent calculus style presentation, an (sound and complete) axiomatization of orthologic can be given by the following axioms and rules (in the spirit of [9]):

$$\frac{}{A \vdash A} \qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

$$\frac{}{A \wedge B \vdash A} \qquad \frac{}{A \wedge B \vdash B} \qquad \frac{C \vdash A \qquad C \vdash B}{C \vdash A \wedge B} \qquad \frac{}{C \vdash \top}$$

$$\frac{}{A \vdash A \vee B} \qquad \frac{}{B \vdash A \vee B} \qquad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C} \qquad \frac{}{\bot \vdash C}$$

$$\frac{A \vdash B}{\neg B \vdash \neg A} \qquad \frac{}{A \vdash \neg\neg A} \qquad \frac{}{\neg\neg A \vdash A} \qquad \frac{}{\top \vdash A \vee \neg A}$$

The first line corresponds to an (pre) order relation. The second and third lines correspond to a bounded inf semi-lattice and bounded sup semi-lattice (thus together they provide us the structure of a bounded lattice). The fourth line adds the missing ortholattice ingredients related with the orthocomplement $\neg A$.

▶ **Example 2.** If one wants to prove that for any p and q in an ortholattice, we have: $\top \leq ((p \wedge q) \vee \neg p) \vee \neg q$. We can either use algebraic properties of ortholattices (which have to be proved as well): $((p \wedge q) \vee \neg p) \vee \neg q = (p \wedge q) \vee (\neg p \vee \neg q) = (p \wedge q) \vee \neg(p \wedge q) = \top$ or we can use, on the logic side, a derivation with conclusion the corresponding sequent $\top \leq ((X \wedge Y) \vee \neg X) \vee \neg Y$. This requires us to use most of the rules above.

The axiomatization proposed above is a direct translation of the order-theoretic definition of ortholattices. From a proof-theoretic point of view, it has strong defects such has the impossibility of eliminating the cut rule:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C} \; cut$$

(which encodes the transitivity of the order relation). Example 2 could not be derived without this rule for example. A reason for trying to avoid the cut rule is that when studying a property like $A \vdash C$, the cut rule tells us that we may need to invent some arbitrary B (unrelated with A and C). This may lead us to difficulties, undecidability, etc. In the opposite, cut-free systems usually satisfy the *sub-formula property* stating that every formula appearing in a proof of a given sequent is a sub-formula of a formula of this sequent. The idea of finding presentations of the logic associated with lattices in such a way that cut (or transitivity) could be eliminated goes back to Whitman [15] with applications to the theory of lattices. In the case of ortholattices, one can find such an axiomatization in [14] under the name OCL+ (also called GOL in [5]):

OCL+

$$\frac{}{A \vdash A} \; ax \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \; wL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \; wR$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; \wedge_1 L \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; \wedge_2 L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \; \wedge R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \; \vee_1 R \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \; \vee_2 R \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \; \vee L$$

$$\frac{}{\Gamma \vdash \top, \Delta} \; \top R \qquad \frac{}{\Gamma, \bot \vdash \Delta} \; \bot L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \; \neg R \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \; \neg L$$

where sequents $\Gamma \vdash \Delta$ are given from two finite *sets* Γ and Δ of formulas such that the size of Γ plus the size of Δ is *at most* 2 (and the comma denotes set union).

▶ **Example 3.** We can prove in OCL+ the sequent of Example 2:

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\dfrac{\dfrac{\overline{X \vdash X}\ ax}{\vdash X, \neg X}\ \neg R}{\vdash X, (X \wedge Y) \vee \neg X}\ \vee_2 R
}{\vdash X, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_1 R
\quad
\dfrac{\dfrac{\dfrac{\overline{Y \vdash Y}\ ax}{\vdash Y, \neg Y}\ \neg R}{\vdash Y, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_2 R
}{}
}{\vdash X \wedge Y, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \wedge R
}{\vdash (X \wedge Y) \vee \neg X, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_1 R
}{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_1 R
}{\top \vdash ((X \wedge Y) \vee \neg X) \vee \neg Y}\ wL
$$

The following key properties of OCL+ are proved in [14]:

▶ **Theorem 4** (Cut Elimination in OCL+). *The cut rule* $\dfrac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$ *is admissible in* OCL+.

▶ **Theorem 5** (Soundness and Completeness of OCL+). OCL+ *is sound and complete for orthologic.*

By looking at the structure of the rules, one can see there is an important symmetry between $\vee$ on the left and $\wedge$ on the right, $\wedge$ on the left and $\vee$ on the right, $\bot$ on the left and $\top$ on the right, etc. This is not very surprising in a context where negation is an involution, and this is an incarnation of De Morgan's duality between $\wedge$ and $\vee$ and $\top$ and $\bot$. J.-Y. Girard has shown for linear logic [7] how to simplify sequent calculi in the presence of an involutive negation by restricting negation to variables and by considering one-sided sequents only. This idea has been partly applied in [6] where they define formulas for orthologic as:

$$A ::= X \mid A \wedge A \mid A \vee A \mid \top \mid \bot \mid \neg X$$

and negation is then extended to all formulas by induction (it is not a true connective anymore):

$$\neg(\neg X) := X \quad \neg(\bot) := \top \quad \neg(\top) := \bot \quad \neg(A \vee B) := \neg A \wedge \neg B \quad \neg(A \wedge B) := \neg A \vee \neg B$$

so that we obtain $\neg\neg A = A$ for any A. However the system proposed in [6] does not really take benefits from this encoded involutive negation on formulas, since they use two-sided sequents. One can also note that no remark is given in [6] regarding the number of formulas in sequents. However one can see that, in their system, $\Gamma \vdash \Delta$ is provable if and only if $\bigwedge \Gamma \vdash \bigvee \Delta$ is provable, and that a proof of a sequent $\Gamma \vdash \Delta$ with at most one formula in Γ and at most one formula in Δ contains only sequents satisfying this property.

We propose to go further in this direction of involutive negation to target a simpler sequent calculus system for orthologic.

2 One-Sided Orthologic

In order to clarify the analysis and to be closer to an implementation, we prefer to consider sequents based on lists rather than sets or multi-sets. The main difference with respect to OCL+ is the necessity to use an explicit contraction rule and an explicit exchange rule. We thus consider two kinds of sequents: $\vdash A, B$ and $\vdash A$. As a notation, Π corresponds to 0

or 1 formula so that $\vdash A, \Pi$ is a common notation for both kinds of sequents. Like in [6], formulas are built with negation on variables only:

$$A ::= X \mid A \wedge A \mid A \vee A \mid \top \mid \bot \mid \neg X$$

and, by moving to a one-sided list-based system, we obtain derivation rules like:

$$\frac{}{\vdash \neg A, A} \; ax \qquad \frac{\vdash A, B}{\vdash B, A} \; ex \qquad \frac{\vdash A, A}{\vdash A} \; c \qquad \frac{\vdash A}{\vdash A, B} \; w \qquad \frac{\vdash A, \Pi}{\vdash A \vee B, \Pi} \; \vee_1 \qquad \cdots$$

But we can optimise these rules. First, we can assume Π not to be empty since the case of an empty Π is derivable from the non-empty case. For example, for the $(\vee_1)$ rule:

$$\frac{\dfrac{\vdash A}{\vdash A, A \vee B} \; w}{\dfrac{\vdash A \vee B, A \vee B}{\vdash A \vee B} \; c} \; \vee_1$$

Second, once we thus consider only logical rules with two formulas in sequents, the only rule with a premise with only one formula is the (w) rule and the only rule with a conclusion with only one formula is the (c) rule. This means that in a proof of a sequent with two formulas, (c) and (w) rules always come together, one above the other, and we can group them. Finally a sequent $\vdash A$ can always be encoded as $\vdash A, A$ since one is provable if and only if the other is (thanks to the rules (c) and (w)). We thus focus on sequents $\vdash A, B$ only, and on the following rules:

OL

$$\frac{}{\vdash \neg A, A} \; ax \qquad\qquad \frac{\vdash A, B}{\vdash B, A} \; ex \qquad\qquad \frac{\vdash A, A}{\vdash A, B} \; cw$$

$$\frac{\vdash A, C}{\vdash A \vee B, C} \; \vee_1 \qquad \frac{\vdash B, C}{\vdash A \vee B, C} \; \vee_2 \qquad \frac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C} \; \wedge \qquad \frac{}{\vdash \top, C} \; \top$$

This sequent calculus with 7 rules (6 rules in its multi-set-based and set-based versions) does not seem to occur in the literature and looks simpler than all the sound and complete calculi for orthologic we have found. We call it OL. Relying on the remarks above, we have:

▶ **Theorem 6** (Soundness and Completeness of OL). *$\vdash \neg A, B$ is provable in OL if and only if $A \vdash B$ is provable in OCL+, so that OL is sound and complete for orthologic.*

Proof. To be completely precise, we have to recall that formulas of OL are all formulas of OCL+. While the converse is not true, there is a canonical mapping of formulas of OCL+ into formulas of OL obtained by unfolding the definition of $\neg$. For soundness, we use Theorem 4. Concerning completeness, we prove simultaneously that $A \vdash B$ in OCL+ entails $\vdash \neg A, B$ in OL, $A \vdash$ in OCL+ entails $\vdash \neg A, \neg A$ in OL and $\vdash B$ in OCL+ entails $\vdash B, B$ in OL. ◀

For readers familiar with linear logic [7], this calculus OL can be seen as one-sided additive linear logic extended with the (cw) rule, if we replace $\vee$ by $\oplus$, $\wedge$ by $\&$ and $\bot$ by 0.

▶ **Example 7.** We can prove in OL the sequent of Example 2 in its one-sided version:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\vdash \neg X, X}\ ax}{\vdash (X \wedge Y) \vee \neg X, X}\ \vee_2}{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y, X}\ \vee_1}{\vdash X, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ ex \qquad \cfrac{\cfrac{\cfrac{\overline{\vdash \neg Y, Y}\ ax}{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y, Y}\ \vee_2}{\vdash Y, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ ex}{}}{\cfrac{\cfrac{\cfrac{\vdash X \wedge Y, ((X \wedge Y) \vee \neg X) \vee \neg Y}{\vdash (X \wedge Y) \vee \neg X, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_1}{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ \vee_1}{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y, \bot}\ cw}\ \wedge}{\vdash \bot, ((X \wedge Y) \vee \neg X) \vee \neg Y}\ ex$$

We now describe a few properties of OL which will be used later. First, the cut rule $\cfrac{\vdash A, B \qquad \vdash \neg B, C}{\vdash A, C}$ is admissible (see Proposition 19 for an indirect proof). Also:

▶ **Proposition 8** (Axiom expansion for OL). *If we restrict the axiom rule of* OL *to its variable case* $\overline{\vdash \neg X, X}\ ax_v$, *the general rule (ax) is derivable.*

▶ **Lemma 9** (Reversibility of $\wedge$). $\vdash A \wedge B, C$ *is provable iff both* $\vdash A, C$ *and* $\vdash B, C$ *are.*

▶ **Lemma 10** (Reversing). *If we restrict the (cw) rule to formulas of the shape* $A_1 \vee A_2$:

$$\cfrac{\vdash A_1 \vee A_2, A_1 \vee A_2}{\vdash A_1 \vee A_2, B}\ cw_\vee$$

where moreover B is neither $\top$ nor a $\wedge$, the general rule (cw) is admissible.

Proof. This is done in two steps, first by proving the restriction on A (by induction on A for an arbitrary B) and then the restriction on B (by induction on B, with $A = A_1 \vee A_2$). ◀

3   Focused Orthologic

Relying on the strong relation between the sequent calculus OL and linear logic, we import the idea of *focusing* [1]. This constraint on the structure of proofs is based on an analysis of the polarity of connectives, by separating those which are reversible and those which are not. By reducing the space of proofs of each formula, it is a strong tool for accelerating proof search. In orthologic, the connectives $\wedge$ and $\top$ are reversible: the conclusion of their introduction rule implies its premises (see Lemma 9 for example). Such connectives are also called *asynchronous* or negative. Their dual connectives are called *synchronous* or positive. Following this pattern, we separate formulas into synchronous and asynchronous ones according to their main connective: X, $\bot$ and $A \vee B$ are synchronous, and $\neg X$, $\top$ and $A \wedge B$ are asynchronous. So that A is synchronous if and only if $\neg A$ is asynchronous. The choice for variables is in fact arbitrary, as soon as we preserve this dual polarity between X and $\neg X$ for each of them.

3.1   A First Focused System OL_f^0

Dealing with variables in focused systems is delicate, so we recommend the reader not very familiar with focusing to concentrate on the other aspects of the system first.

A key result will be to prove the focused system to be as expressive as OL (and thus sound and complete for orthologic). In order to make this as simple and clear as possible, we will work in two steps. Indeed some optimisations (to be introduced later on in Section 3.4) would make a direct translation more difficult.

Our first focused system OL_f^0 is based on four kinds of sequents. For each of them, we give an *informal explanation* based on how we can find a proof of such a sequent, thus from the point of view of a bottom-up reading of proofs and rules:

- In a sequent $\vdash\ \Uparrow A, B$, all the asynchronous connectives at the roots of A and B (in formulas A and B seen as trees) will be deconstructed and after that, A and B will be synchronous (or negation of a variable) and allowed to move to the left of $\Uparrow$. In fact we first work on A and then we move to a sequent $\vdash A \Uparrow B$.
- In a sequent $\vdash A \Uparrow B$, A is synchronous or is the negation of a variable. The asynchronous connectives at the root of B will be deconstructed and after that, B will be synchronous (or negation of a variable) and allowed to move to the left of $\Uparrow$.
- In a sequent $\vdash A, B \Uparrow$, A and B are synchronous or the negation of a variable. We have to select a synchronous formula and start decomposing its synchronous connectives at the root, in a sequent $\vdash A \Downarrow B$. Before that, we can apply contraction-weakening rules to A and B. This is the main place where choices have to be made during proof search.
- In a sequent $\vdash A \Downarrow B$, A is synchronous or is the negation of a variable. The synchronous connectives at the root of B will be deconstructed and after that, B will be asynchronous (and we will start decomposing its asynchronous connectives at the root in a sequent $\vdash A \Uparrow B$). Choices concerning the decomposition of $\vee$ will have to be made here.

Note, sequents $\vdash\ \Uparrow A, B$ are crucial for the comparison with other systems but play a weak role inside this system. Indeed they occur only in proofs of sequents of the same shape and only at the bottom part of such a proof. As soon as we reach a sequent $\vdash A \Uparrow B$ (in the bottom-up reading of a proof), we will not find any other sequent $\vdash\ \Uparrow A, B$ above.

Let us be more formal now with the explicit list of the rules of the system OL_f^0:

OL_f^0

$$\dfrac{\vdash\ \Uparrow A, C \qquad \vdash\ \Uparrow B, C}{\vdash\ \Uparrow A \wedge B, C}\ \Uparrow\wedge \qquad \dfrac{}{\vdash\ \Uparrow \top, C}\ \Uparrow\top \qquad \dfrac{\vdash A \Uparrow C}{\vdash\ \Uparrow A, C}\ \Uparrow R$$

$$\dfrac{\vdash C \Uparrow A \qquad \vdash C \Uparrow B}{\vdash C \Uparrow A \wedge B}\ \wedge\Uparrow \qquad [(\mathrm{s})\ \text{or}\ (\mathrm{n})]\ \dfrac{}{\vdash A \Uparrow \top}\ \top\Uparrow \qquad \dfrac{\vdash C, A \Uparrow}{\vdash C \Uparrow A}\ R\Uparrow$$

$$[(\mathrm{s})\ \text{or}\ (\mathrm{n})]\ \dfrac{\vdash C, C \Uparrow}{\vdash C, A \Uparrow}\ cw_1 \qquad\qquad [(\mathrm{s})\ \text{or}\ (\mathrm{n})]\ \dfrac{\vdash C, C \Uparrow}{\vdash A, C \Uparrow}\ cw_2$$

$$[(\mathrm{s})]\ \dfrac{\vdash C \Downarrow A}{\vdash A, C \Uparrow}\ \mathrm{D}_1 \qquad [(\mathrm{s})]\ \dfrac{\vdash C \Downarrow A}{\vdash C, A \Uparrow}\ \mathrm{D}_2$$

$$\dfrac{}{\vdash \neg X \Downarrow X}\ ax_v \qquad \dfrac{\vdash C \Downarrow A}{\vdash C \Downarrow A \vee B}\ \vee_1 \qquad \dfrac{\vdash C \Downarrow B}{\vdash C \Downarrow A \vee B}\ \vee_2 \qquad [(\mathrm{a})]\ \dfrac{\vdash C \Uparrow A}{\vdash C \Downarrow A}\ R\Downarrow$$

with the following side conditions written between square brackets [_]:

(a) A is asynchronous   (s) A is synchronous   (n) A is the negation of a variable.

One could have been more explicit by asking [(s) or (n)] as side condition in the ($\Uparrow R$) and (R$\Uparrow$) rules but the following lemma proves these two side conditions to be redundant.

▶ **Lemma 11.** *If* $\vdash A \Uparrow C$ *or* $\vdash A, B \Uparrow$ *or* $\vdash A \Downarrow C$ *is provable then A and B are synchronous or the negation of a variable.*

▶ **Example 12.** The sequent $\vdash (X \vee A) \vee B, (C \vee (D \vee \neg X)) \wedge \top$ has many proofs in the systems of the previous sections, in particular in OL. However the corresponding sequent $\vdash\ \Uparrow (X \vee A) \vee B, (C \vee (D \vee \neg X)) \wedge \top$ has a unique proof in OL_f^0:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\vdash \neg X \Downarrow X}\ ax_v}{\vdash \neg X \Downarrow X \vee A}\ \vee_1}{\vdash \neg X \Downarrow (X \vee A) \vee B}\ \vee_1}{\vdash (X \vee A) \vee B, \neg X \Uparrow}\ D_1}{\vdash (X \vee A) \vee B \Uparrow \neg X}\ R\Uparrow}{\vdash (X \vee A) \vee B \Downarrow \neg X}\ R\Downarrow}{\vdash (X \vee A) \vee B \Downarrow D \vee \neg X}\ \vee_2}{\vdash (X \vee A) \vee B \Downarrow C \vee (D \vee \neg X)}\ \vee_2}{\vdash (X \vee A) \vee B, C \vee (D \vee \neg X) \Uparrow}\ D_2}{\vdash (X \vee A) \vee B \Uparrow C \vee (D \vee \neg X)}\ R\Uparrow \qquad \cfrac{}{\vdash (X \vee A) \vee B \Uparrow \top}\ \top\Uparrow}{\cfrac{\vdash (X \vee A) \vee B \Uparrow (C \vee (D \vee \neg X)) \wedge \top}{\vdash\ \Uparrow (X \vee A) \vee B, (C \vee (D \vee \neg X)) \wedge \top}\ \Uparrow R}\ \wedge\Uparrow}$$

One can prove the soundness of OL_f^0 with respect to orthologic by translation into OL.

▶ **Proposition 13** (Soundness of OL_f^0). *If $\vdash\ \Uparrow A, B$ or $\vdash A \Uparrow B$ or $\vdash A, B \Uparrow$ or $\vdash A \Downarrow B$ is provable in OL_f^0 then $\vdash A, B$ is provable in OL.*

To conclude this section, here are a few simple facts which will be useful later and which can be obtained by simple induction on proofs:

- $\vdash X, Y \Uparrow$, $\vdash \neg X, \neg Y \Uparrow$ and $\vdash \bot, \bot \Uparrow$ are not provable (both if $X = Y$ or $X \neq Y$);
- if $\vdash A, B \Uparrow$ is provable then $\vdash B, A \Uparrow$ as well (and with a proof of the same size);
- if $\vdash A, A \Uparrow$ is provable then the proof contains a proof of $\vdash A \Downarrow A$.

3.2 Cut Elimination in OL_f^0

Due to the very rigid structure of proofs in focused systems, the possibility of enriching them with admissible cut rules is often used in their study [8, 11] (in particular for expressiveness analysis). It is the tool we are going to use here in order to prove the completeness of OL_f^0 with respect to orthologic.

▶ **Theorem 14** (Cut Elimination in OL_f^0). *The following cut rules are admissible in OL_f^0:*

$$\cfrac{\ \ C \text{ synchronous}\ \ \\ \vdash X \Downarrow A \qquad \vdash \neg X \Downarrow C}{\vdash C \Downarrow A}\ v\text{-}cut_2 \qquad\qquad \cfrac{\ \ C \text{ synchronous}\ \ \\ \vdash X \Uparrow A \qquad \vdash \neg X \Downarrow C}{\vdash C \Uparrow A}\ v\text{-}cut_3$$

$$\cfrac{\vdash A, X \Uparrow \qquad \vdash C, \neg X \Uparrow}{\vdash A, C \Uparrow}\ v\text{-}cut_1 \qquad\qquad \cfrac{\ \ B \text{ asynchronous or variable}\ \ \\ \vdash A \Uparrow B \qquad \vdash C \Uparrow \neg B}{\vdash A, C \Uparrow}\ cut_1$$

$$\cfrac{\vdash A \Uparrow B \qquad \vdash C, \neg B \Uparrow}{\vdash A, C \Uparrow}\ cut_2 \qquad\qquad \cfrac{\ \ B \text{ asynchronous}\ \ \\ \vdash A \Uparrow B \qquad \vdash \neg B \Downarrow C}{\vdash A \Downarrow C}\ cut_3$$

$$\cfrac{\ \ B \text{ asynchronous or variable}\ \ \\ \vdash A \Uparrow B \qquad \vdash C \Downarrow \neg B}{\vdash A, C \Uparrow}\ cut_4 \qquad\qquad \cfrac{\vdash A \Uparrow B \qquad \vdash \neg B \Uparrow C}{\vdash A \Uparrow C}\ cut_5$$

$$\cfrac{\vdash\ \Uparrow A, B \qquad \vdash\ \Uparrow C, \neg B}{\vdash\ \Uparrow A, C}\ cut_0 \qquad\qquad \cfrac{\vdash\ \Uparrow A, B \qquad \vdash C \Uparrow \neg B}{\vdash C \Uparrow A}\ cut_0'$$

Proof. This is a proof involving many cases which require a precise management of the four kinds of sequents. We try to explain the key ingredients which work in successive steps.

- We prove simultaneously the admissibility of (v-cut_2) and (v-cut_3) by induction on the size of the left premise.
- We deduce the admissibility of (v-cut_1) by induction on the size of the left premise. For example:

$$\dfrac{\dfrac{\vdash X \Downarrow A}{\vdash A, X \Uparrow}\ \mathrm{D}_1 \quad \dfrac{\vdash \neg X \Downarrow C}{\vdash C, \neg X \Uparrow}\ \mathrm{D}_1}{\vdash A, C \Uparrow}\ v\text{-}cut_1 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash X \Downarrow A \quad \vdash \neg X \Downarrow C}{\vdash C \Downarrow A}\ v\text{-}cut_2}{\vdash A, C \Uparrow}\ \mathrm{D}_1$$

since A and C are synchronous.

- Using the previous steps, we prove simultaneously the admissibility of (cut_1), (cut_2), (cut_3), (cut_4) and (cut_5) by induction on the pair (f, p) where f is the size of the cut-formula B and p is the size of the right premise. The crucial cases are the following:
  - Starting from:

  $$\dfrac{\dfrac{\vdash A \Uparrow B_1 \quad \vdash A \Uparrow B_2}{\vdash A \Uparrow B_1 \wedge B_2}\ \wedge\!\Uparrow \quad \dfrac{\vdash C \Downarrow \neg B_1}{\vdash C \Downarrow \neg B_1 \vee \neg B_2}\ \vee_1}{\vdash A, C \Uparrow}\ cut_4$$

  we can apply the induction hypothesis by means of (cut_4) with a smaller cut formula.
  - If B is asynchronous, we have:

  $$\dfrac{\vdash A \Uparrow B \quad \dfrac{\vdash \neg B \Downarrow C}{\vdash C, \neg B \Uparrow}\ \mathrm{D}_1}{\vdash A, C \Uparrow}\ cut_2 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash A \Uparrow B \quad \vdash \neg B \Downarrow C}{\vdash A \Downarrow C}\ cut_3}{\vdash A, C \Uparrow}\ \mathrm{D}_2$$

  otherwise B is a variable so that $\vdash A \Uparrow X$ must come from ($\mathrm{R}\!\Uparrow$) and we apply (v-cut_1).
  - The most tricky case is contraction where we need two induction steps:

  $$\dfrac{\vdash A \Uparrow B \quad \dfrac{\dfrac{\dfrac{\vdash \neg B \Downarrow \neg B}{\vdash \neg B, \neg B \Uparrow}\ \mathrm{D}}{\vdash \neg B, \neg B \Uparrow}\ cw}{\vdash C, \neg B \Uparrow}\ cw_2}{\vdash A, C \Uparrow}\ cut_2 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash A \Uparrow B \quad \dfrac{\vdash A \Uparrow B \quad \vdash \neg B \Downarrow \neg B}{\vdash A \Downarrow \neg B}\ cut_3}{\vdash A, A \Uparrow}\ cut_4}{\vdash A, C \Uparrow}\ cw_1$$

  First we apply (cut_3) with a smaller right premise and then, by transforming one more step the (cut_4), we reach a smaller cut formula.
- We deduce the case (cut'_0) and then (cut_0), by induction on the size of the left premise.

◀

Among the 10 cut rules considered in the theorem above, mainly two will be used now (namely cut_0 and cut'_0). The other rules were however necessary as intermediary steps to prove the admissibility of these two rules.

3.3 Completeness of OL_f^0

We are going to translate proofs of OL into proofs of OL_f^0. We start with some preliminary results about sequents $\vdash\ \Uparrow A, B$ in OL_f^0 which will be the target of sequents of OL.

▶ **Lemma 15.** *The following rules are admissible in OL_f^0:*

$$\dfrac{}{\vdash\ \Uparrow C, \top} \qquad \dfrac{\vdash\ \Uparrow C, A \quad \vdash\ \Uparrow C, B}{\vdash\ \Uparrow C, A \wedge B} \qquad \dfrac{\vdash A \Uparrow C}{\vdash\ \Uparrow C, A} \qquad \dfrac{\vdash\ \Uparrow A, C}{\vdash\ \Uparrow C, A}$$

▶ **Lemma 16.** *In* OL_f^0*, the following rules are admissible (and similarly for $B \vee A$ instead of $A \vee B$):*

A asynchronous	*A synchronous*	*A synchronous*	*A synchronous*
$\dfrac{\vdash C \Uparrow A}{\vdash A \vee B \Uparrow C}$	$\dfrac{\vdash A \Uparrow C}{\vdash A \vee B \Uparrow C}$	$\dfrac{\vdash A, C \Uparrow}{\vdash A \vee B, C \Uparrow}$	$\dfrac{\vdash A \Downarrow C}{\vdash A \vee B \Downarrow C}$

▶ **Proposition 17** (Axiom expansion for OL_f^0). *If A is synchronous or a negation of a variable, $\vdash A \Uparrow \neg A$ is provable.*

This leads us to the completeness of OL_f^0 for orthologic by means of the completeness of OL and the following translation result:

▶ **Theorem 18** (Completeness of OL_f^0). *If $\vdash A, B$ is provable in OL then $\vdash \, \Uparrow A, B$ is provable in OL_f^0.*

Proof. By induction on the proof of $\vdash A, B$ in OL, the main cases are:

- If the last rule is a contraction-weakening rule, we use Lemma 10 to restrict ourselves to the $(cw_\vee)$ case, and by induction hypothesis we have $\vdash \, \Uparrow A_1 \vee A_2, A_1 \vee A_2$. The only way this is provable is by:

$$\dfrac{\dfrac{\dfrac{\vdash A_1 \vee A_2, A_1 \vee A_2 \Uparrow}{\vdash A_1 \vee A_2 \Uparrow A_1 \vee A_2} \, \mathrm{R}\Uparrow}{\vdash \, \Uparrow A_1 \vee A_2, A_1 \vee A_2}}{} \, \Uparrow\mathrm{R}$$

so that we can build:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A_1 \vee A_2, A_1 \vee A_2 \Uparrow}{\vdash A_1 \vee A_2, B \Uparrow} \, cw_1}{\vdash A_1 \vee A_2 \Uparrow B} \, \mathrm{R}\Uparrow}{\vdash \, \Uparrow A_1 \vee A_2, B}}{} \, \Uparrow\mathrm{R}$$

- If the last rule is a $(\vee_1)$ rule, by induction hypothesis we have $\vdash \, \Uparrow A, C$, thus using Lemmas 15 and 16, Proposition 17 and Theorem 14:

$$\dfrac{\dfrac{\dfrac{\vdash \, \Uparrow A, C}{\vdash \, \Uparrow C, A} \quad \overset{\textit{A synchronous}}{\dfrac{\vdash \bar{A} \Uparrow \neg A}{\vdash A \vee B \Uparrow \neg A}}}{\vdash A \vee B \Uparrow C} \, cut'_0}{\vdash \, \Uparrow A \vee B, C} \, \Uparrow\mathrm{R} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\vdash \, \Uparrow A, C}{\vdash \, \Uparrow C, A} \quad \overset{\textit{A asynchronous}}{\dfrac{\vdash \neg A \Uparrow \bar{A}}{\vdash A \vee B \Uparrow \neg A}}}{\vdash A \vee B \Uparrow C} \, cut'_0}{\vdash \, \Uparrow A \vee B, C} \, \Uparrow\mathrm{R}$$

◀

As promised in Section 2, we can deduce cut elimination for OL.

▶ **Proposition 19** (Cut Elimination for OL). *The cut rule is admissible in OL.*

Proof. By Theorem 18, we have $\vdash \, \Uparrow A, B$ and $\vdash \, \Uparrow \neg B, C$ in OL_f^0. By Lemma 15 we deduce $\vdash \, \Uparrow C, \neg B$. Using cut_0 (Theorem 14) we have $\vdash \, \Uparrow A, C$, and by Proposition 13, $\vdash A, C$ in OL. ◀

3.4   A Second Focused System OL_f

If we try to apply a simple bottom-up proof-search procedure in a sequent calculus system, a first obstacle to the finiteness of the search is given by cut rules. If a cut rule cannot be eliminated then a given conclusion leads us to a possibly infinite set of premises. A second obstacle comes from loops, *i.e.* non trivial derivations leading from a sequent to the

same sequent (note however this obstacle can be dealt with by using loop detection during the search, but loops make the proof-search longer). All the systems we have seen so far contain non-trivial loops. Avoiding loops is one of the motivations for looking for a more constrained focused system. Let us analyse loops in OL_f^0. They mainly come from rules acting on sequents of the shape $\vdash \_,\_ \Uparrow$. If we look at derivations in a bottom-up way, we reach such a sequent through a $(R\Uparrow)$ rule:

$$\frac{\vdash C, A \Uparrow}{\vdash C \Uparrow A} \ R\Uparrow$$

then we stay with sequents $\vdash \_,\_ \Uparrow$ by using (upwardly):

$$\frac{\vdash C, C \Uparrow}{\vdash C, A \Uparrow} \ cw_1 \qquad \text{and} \qquad \frac{\vdash C, C \Uparrow}{\vdash A, C \Uparrow} \ cw_2$$

until we reach:

$$\frac{\vdash C \Downarrow A}{\vdash A, C \Uparrow} \ D_1 \qquad \text{or} \qquad \frac{\vdash C \Downarrow A}{\vdash C, A \Uparrow} \ D_2 \ .$$

Globally, this means we start with a sequent $\vdash C \Uparrow A$ and we must end with $\vdash C \Downarrow A$, $\vdash A \Downarrow C$, $\vdash A \Downarrow A$ or $\vdash C \Downarrow C$. This would correspond to four derivable rules:

$$\frac{\vdash C \Downarrow A}{\vdash C \Uparrow A} \qquad\qquad \frac{\vdash A \Downarrow C}{\vdash C \Uparrow A} \qquad\qquad \frac{\vdash A \Downarrow A}{\vdash C \Uparrow A} \qquad\qquad \frac{\vdash C \Downarrow C}{\vdash C \Uparrow A}$$

In the same time we want to try to constrain contraction so that it is applied on $\vee$-formulas only (in the spirit of Lemma 10). Moreover we would like contraction not being applied twice on the same formula. In particular we get read of the fourth rule just above, which would allow C to be contracted (uselessly) many times. All these remarks lead us to the following new focused system called OL_f:

OL_f

$$\frac{\vdash \ \Uparrow A, C \qquad \vdash \ \Uparrow B, C}{\vdash \ \Uparrow A \wedge B, C} \ \Uparrow\wedge \qquad\qquad \frac{}{\vdash \ \Uparrow \top, C} \ \Uparrow\top$$

$$\frac{\vdash C \Uparrow A \qquad \vdash C \Uparrow B}{\vdash C \Uparrow A \wedge B} \ \wedge\Uparrow \qquad\qquad [\text{(s) or (n)}] \ \frac{}{\vdash A \Uparrow \top} \ \top\Uparrow$$

$$\frac{\vdash B \vee C \Downarrow B \vee C}{\vdash \ \Uparrow B \vee C, A} \ \Uparrow cw \qquad\qquad [\text{(s) or (n)}] \ \frac{\vdash B \vee C \Downarrow B \vee C}{\vdash A \Uparrow B \vee C} \ cw\Uparrow$$

$$\frac{}{\vdash \neg X \Downarrow X} \ ax_v \qquad\qquad \frac{\vdash C \Downarrow A}{\vdash C \Downarrow A \vee B} \ \vee_1 \qquad\qquad \frac{\vdash C \Downarrow B}{\vdash C \Downarrow A \vee B} \ \vee_2$$

$$\frac{\vdash A \Uparrow C}{\vdash \ \Uparrow A, C} \ \Uparrow R \quad [\text{(a)}] \ \frac{\vdash C \Uparrow A}{\vdash C \Downarrow A} \ R\Downarrow \quad [\text{(s)}] \ \frac{\vdash C \Downarrow A}{\vdash A \Uparrow C} \ D_1 \quad [\text{(s)}] \ \frac{\vdash C \Downarrow A}{\vdash C \Uparrow A} \ D_2$$

(a) A is asynchronous   (s) A is synchronous   (n) A is the negation of a variable

---

Note, sequents $\vdash A, B \Uparrow$ disappear in this system which relies on three kinds of sequents only: $\vdash A \Uparrow B$, $\vdash A \Downarrow B$ and $\vdash \ \Uparrow A, B$.

▶ **Example 20.** We can prove in OL_f the sequent associated with Example 2:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \overline{\vdash \neg X \Downarrow X} \; ax_v
            }{\vdash X \Uparrow \neg X} \; D_1
          }{\vdash X \Downarrow \neg X} \; R\Downarrow
        }{\vdash X \Downarrow (X \wedge Y) \vee \neg X} \; \vee_2
      }{\vdash X \Downarrow ((X \wedge Y) \vee \neg X) \vee \neg Y} \; \vee_1
    }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Uparrow X} \; D_1
    \qquad
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \overline{\vdash \neg Y \Downarrow Y} \; ax_v
            }{\vdash Y \Uparrow \neg Y} \; D_1
          }{\vdash Y \Downarrow \neg Y} \; R\Downarrow
        }{\vdash Y \Downarrow ((X \wedge Y) \vee \neg X) \vee \neg Y} \; \vee_2
      }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Uparrow Y} \; D_1
    }{}
  }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Uparrow X \wedge Y} \; \wedge\Uparrow
}{}
$$

continued:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Uparrow X \wedge Y
        }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Downarrow X \wedge Y} \; R\Downarrow
      }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Downarrow (X \wedge Y) \vee \neg X} \; \vee_1
    }{\vdash ((X \wedge Y) \vee \neg X) \vee \neg Y \Downarrow ((X \wedge Y) \vee \neg X) \vee \neg Y} \; \vee_1
  }{\vdash \bot \Uparrow ((X \wedge Y) \vee \neg X) \vee \neg Y} \; cw\Uparrow
}{\vdash \; \Uparrow \bot, ((X \wedge Y) \vee \neg X) \vee \neg Y} \; \Uparrow R
$$

The system $\mathsf{OL_f}$ is as expressive as $\mathsf{OL_f^0}$ for sequents $\vdash \; \Uparrow A, B$. In particular:

▶ **Proposition 21** (Expressiveness of $\mathsf{OL_f}$). *If $\vdash \; \Uparrow A, B$ is provable in $\mathsf{OL_f^0}$, it is also provable in $\mathsf{OL_f}$.*

Proof. We prove by induction on the proof π in $\mathsf{OL_f^0}$ the more general statement:

- If $\vdash \; \Uparrow A, B$ in $\mathsf{OL_f^0}$ then $\vdash \; \Uparrow A, B$ in $\mathsf{OL_f}$.
- If $\vdash A \Uparrow B$ in $\mathsf{OL_f^0}$ then either $\vdash A \Uparrow B$ in $\mathsf{OL_f}$ or $A = A_1 \vee A_2$ with $\vdash A \Downarrow A$ in $\mathsf{OL_f}$.
- If $\vdash A \Downarrow B$ in $\mathsf{OL_f^0}$ then either $\vdash A \Downarrow B$ in $\mathsf{OL_f}$ or $A = A_1 \vee A_2$ with $\vdash A \Downarrow A$ in $\mathsf{OL_f}$.
- If $\vdash A, B \Uparrow$ in $\mathsf{OL_f^0}$ then at least one of the following four possibilities holds:
  - B is synchronous and $\vdash A \Downarrow B$ in $\mathsf{OL_f}$;
  - A is synchronous and $\vdash B \Downarrow A$ in $\mathsf{OL_f}$;
  - $A = A_1 \vee A_2$ and $\vdash A \Downarrow A$ in $\mathsf{OL_f}$;
  - $B = B_1 \vee B_2$ and $\vdash B \Downarrow B$ in $\mathsf{OL_f}$.

We consider each possible last rule for π. Interesting cases are:

- For the two contraction rules, we have $\vdash C, C \Uparrow$ in $\mathsf{OL_f^0}$ thus, by induction hypothesis, $\vdash C \Downarrow C$ in $\mathsf{OL_f}$ with C synchronous and we are done since $\vdash \bot \Downarrow \bot$ and $\vdash X \Downarrow X$ are not provable thus $C = C_1 \vee C_2$.
- For $(\vee_1)$, by induction hypothesis, we have either $\vdash C \Downarrow A$ or $\vdash C_1 \vee C_2 \Downarrow C_1 \vee C_2$ in $\mathsf{OL_f}$ with $C = C_1 \vee C_2$. In the first case, we apply the corresponding rule. In the second case, we are immediately done.
- For $(\Uparrow R)$, by induction hypothesis we have $\vdash A \Uparrow C$ or $A = A_1 \vee A_2$ with $\vdash A_1 \vee A_2 \Downarrow A_1 \vee A_2$, we can build: $\quad \cfrac{\vdash A \Uparrow C}{\vdash \; \Uparrow A, C} \; \Uparrow R \qquad$ or $\qquad \cfrac{\vdash A_1 \vee A_2 \Downarrow A_1 \vee A_2}{\vdash \; \Uparrow A_1 \vee A_2, C} \; \Uparrow cw$
- For $(R\Uparrow)$, we apply the induction hypothesis and we obtain four possible cases:
  - If $\vdash C \Downarrow A$ in $\mathsf{OL_f}$ with A synchronous, we have: $\quad \cfrac{\vdash C \Downarrow A}{\vdash C \Uparrow A} \; D_2$
  - If $\vdash A \Downarrow C$ in $\mathsf{OL_f}$ with C synchronous, we have: $\quad \cfrac{\vdash A \Downarrow C}{\vdash C \Uparrow A} \; D_1$
  - If $\vdash C_1 \vee C_2 \Downarrow C_1 \vee C_2$ ($C = C_1 \vee C_2$) in $\mathsf{OL_f}$, we are done.
  - If $\vdash A_1 \vee A_2 \Downarrow A_1 \vee A_2$ ($A = A_1 \vee A_2$) in $\mathsf{OL_f}$, we have: $\quad \cfrac{\vdash A_1 \vee A_2 \Downarrow A_1 \vee A_2}{\vdash C \Uparrow A_1 \vee A_2} \; cw\Uparrow$

◀

▶ **Proposition 22** (Soundness of $\mathsf{OL_f}$). *If $\vdash \; \Uparrow A, B$ is provable in $\mathsf{OL_f}$ then $\vdash A, B$ is provable in OL.*

From Propositions 19, 21 and 22, and Theorem 18, we can deduce the admissibility of the following cut rule in $\mathsf{OL_f}$: $\dfrac{\vdash\ \Uparrow A, B \qquad \vdash\ \Uparrow \neg B, C}{\vdash\ \Uparrow A, C}\ cut$

We have thus built yet another sound and complete system for orthologic. This one has very strong constraints on the structure of proofs. A key property of this new system (which holds in none of the previous ones) is the termination of the naive bottom-up proof search strategy (Proposition 23).

4 Proof Search in $\mathsf{OL_f}$

We first develop a few properties of $\mathsf{OL_f}$ on which we will rely for proof search. In a second time, we will compare with other algorithms from the literature.

4.1 Backward Proof Search

The basic idea of backward proof search in a cut-free sequent calculus system is to start from the sequent to be proved, to look in a bottom-up manner at each possible instance of a rule with this sequent as conclusion and to continue recursively with the premises of these instances until axioms are reached. Given a sequent, we are going to bound the length of branches of its proofs in $\mathsf{OL_f}$. Let us first define the following measure on formulas:

$$\varphi(X) = \varphi(\neg X) = \varphi(\bot) = \varphi(\top) = 1 \quad \varphi(A \wedge B) = \varphi(A) + \varphi(B) \quad \varphi(A \vee B) = 2\varphi(A) + 2\varphi(B)$$

As a bound on φ, we have $\varphi(A) < 2^{|A|}$ where $|A|$ is the size (number of symbols) of A.

▶ **Proposition 23** (Finiteness of Branches in $\mathsf{OL_f}$). *Given two formulas A and B, $2\varphi(A) + 2\varphi(B)$ is a bound on the length of the branches of any proof of $\vdash\ \Uparrow A, B$ in $\mathsf{OL_f}$.*

Proof. We define the measure ψ of a sequent, according to its shape:

$$\psi(\vdash\ \Uparrow A, B) = 2\varphi(A) + 2\varphi(B) \qquad\qquad \psi(\vdash A \Uparrow B) = \varphi(A) + 2\varphi(B)$$

$$\psi(\vdash A \Downarrow B) = \begin{cases} \varphi(A) + \varphi(B) & \text{if } B \text{ is synchronous} \\ \varphi(A) + 2\varphi(B) + 1 & \text{if } B \text{ is asynchronous} \end{cases}$$

We now prove for each rule of $\mathsf{OL_f}$: if S_1 is a sequent premise of the rule and S_2 is the sequent conclusion of the rule, then $\psi(S_1) < \psi(S_2)$. For example:

$(\vee_1)$ with A synchronous $\qquad\qquad\qquad$ $(\vee_1)$ with A asynchronous

$$\begin{aligned} \psi(\vdash C \Downarrow A) &= \varphi(C) + \varphi(A) \\ &< \varphi(C) + 2\varphi(A) + 2\varphi(B) \\ &= \varphi(C) + \varphi(A \vee B) \\ &= \psi(\vdash C \Downarrow A \vee B) \end{aligned} \qquad \begin{aligned} \psi(\vdash C \Downarrow A) &= \varphi(C) + 2\varphi(A) + 1 \\ &< \varphi(C) + 2\varphi(A) + 2\varphi(B) \\ &= \varphi(C) + \varphi(A \vee B) \\ &= \psi(\vdash C \Downarrow A \vee B) \end{aligned}$$

Thus for any sequent S, $\psi(S)$ is a bound on the height of the branches of the proofs of S. ◀

Since rules of $\mathsf{OL_f}$ are finitely branching, this bound on the length of branches ensures (the absence of loops and) the termination of the backward proof search. Moreover, thanks to the sub-formula property, we know every sequent appearing in a proof of $\vdash\ \Uparrow A, B$ is made of two formulas which are sub-formulas of A or B. Since we have three different kinds of sequents, there are at most $3(|A| + |B|)^2$ such sequents. We have just proved a sequent cannot appear twice in a branch of a proof, so we can deduce a tighter bound than ψ on the height of branches: $3(|A| + |B|)^2$. We thus have an upper bound $2^{3(|A|+|B|)^2+1}$ on the size of proofs since rules have arity at most 2.

4.2   Single Formula Proof Search

As we have seen in Section 2, in systems with exactly two formulas in sequents presented in this paper, the provability of a formula A in orthologic is encoded as the provability of a sequent of the shape $\vdash A, A$ or $\vdash \Uparrow A, A$. Since we are often interested in the provability of a single formula, these sequents play a specific role. We can give some optimisation on the bottom structure of proofs of sequents $\vdash \Uparrow A, A$.

▶ **Proposition 24** (Diagonal Sequent). *The following properties hold in $\mathsf{OL_f}$:*
- *$\vdash \Uparrow X, X$, $\vdash \Uparrow \neg X, \neg X$ and $\vdash \Uparrow \bot, \bot$ are not provable.*
- *$\vdash \Uparrow \top, \top$ is provable.*
- *$\vdash \Uparrow B \wedge C, B \wedge C$ is provable if and only if both $\vdash \Uparrow B, B$ and $\vdash \Uparrow C, C$ are provable.*
- *$\vdash \Uparrow B \vee C, B \vee C$ is provable if and only if $\vdash B \vee C \Downarrow B \vee C$ is provable.*

Proof. We consider the last two statements only. For $\wedge$, we move back and forth to OL thanks to Theorem 18 and Propositions 21 and 22. In OL, we use Lemma 9 and:

$$\frac{\dfrac{\vdash B, B}{\vdash B, B \wedge C} \, cw \qquad \dfrac{\vdash C, C}{\vdash C, B \wedge C} \, cw}{\vdash B \wedge C, B \wedge C} \, \wedge$$

For $\vee$, the only possible last rules are:

$$\frac{\vdash B \vee C \Downarrow B \vee C}{\vdash \Uparrow B \vee C, B \vee C} \, \Uparrow cw \qquad\qquad \frac{\vdash B \vee C \Uparrow B \vee C}{\vdash \Uparrow B \vee C, B \vee C} \, \Uparrow \mathrm{R}$$

and for a proof of $\vdash B \vee C \Uparrow B \vee C$, the only possible last rules are:

$$\frac{\vdash B \vee C \Downarrow B \vee C}{\vdash B \vee C \Uparrow B \vee C} \, cw{\Uparrow} \qquad \frac{\vdash B \vee C \Downarrow B \vee C}{\vdash B \vee C \Uparrow B \vee C} \, \mathrm{D}_1 \qquad \frac{\vdash B \vee C \Downarrow B \vee C}{\vdash B \vee C \Uparrow B \vee C} \, \mathrm{D}_2$$

so that $\vdash B \vee C \Downarrow B \vee C$ must be provable for $\vdash \Uparrow B \vee C, B \vee C$ to be provable. In the other direction we directly use $(\Uparrow cw)$.   ◀

This means in particular that any sequent $\vdash A, A$ is equivalent to a finite family of sequents $\vdash B_1 \vee C_1 \Downarrow B_1 \vee C_1, \dots, \vdash B_n \vee C_n \Downarrow B_n \vee C_n$ (with each $B_i \vee C_i$ sub-formula of A) or clearly not provable.

4.3   Forward Proof Search

Forward proof search consists in building, in a top-down way, proof-trees which are candidates to be sub-proof-trees of proofs of a given sequent. Clearly the sub-formula property can be used to control the sequents to be considered inside the proof-trees. We use here even stronger constraints. Let us fix a formula A. We want to study sub-proof-trees of proofs of $\vdash \Uparrow A, A$ in $\mathsf{OL_f}$. We do not consider the more general case $\vdash \Uparrow A, B$ here.

▶ **Proposition 25** (Strengthened Sub-Formula Property). *If $\vdash C \Downarrow B$ or $\vdash C \Uparrow B'$ or $\vdash \Uparrow D, E$ appears in a proof of $\vdash \Uparrow A, A$ in $\mathsf{OL_f}$, these are sub-formulas of A and moreover:*
- *if B is asynchronous, it appears inside A just below a $\vee$ connective;*
- *if B' is synchronous, it is equal to A or it appears inside A just below a $\wedge$ connective;*
- *if $B' = A$ then $C = A$ or C appears inside A below $\wedge$ connectives only;*
- *if C is synchronous, it is equal to A or it appears inside A just below a $\wedge$ connective;*
- *$E = A$, and $D = A$ or D appears inside A below $\wedge$ connectives only.*

Proof. Since $\vdash \Uparrow A$, A satisfies the conclusion of the statement, we prove for each rule that if the conclusion satisfies it, then all its premises as well. For example, for the $(R\Downarrow)$ rule, the formula in position B' in the premise must be asynchronous. Moreover we cannot have $B' = A$, since the property for the conclusion gives B' below a $\vee$ connective inside A. ◀

This proposition provides us constraints on the meaningful sequents to be considered during forward proof search. This means we can restrict the application of rules in the algorithm for forward proof search to the case where they generate sequents satisfying the properties given by Proposition 25.

4.4 Benchmark

We want to compare our proof-search procedures with procedures from the literature. We consider some formulas from [12] and [5] as well as some random formulas in the language of orthologic:

$$\mathtt{E}_1 = ((\neg X \vee Y) \wedge X) \ \vee \ ((X \wedge \neg Y) \ \vee \ ((\neg X \wedge ((X \vee \neg Y) \wedge (X \vee Y)))$$
$$\vee \ (\neg X \wedge ((\neg X \wedge Y) \vee (\neg X \wedge \neg Y))))))$$
$$\mathtt{E}_2 = X \ \vee \ ((\neg X \wedge ((X \vee \neg Y) \wedge (X \vee Y))) \ \vee \ (\neg X \wedge ((\neg X \wedge Y) \vee (\neg X \wedge \neg Y))))$$
$$\mathtt{E}_3 = (((X \vee \neg Y) \wedge (X \vee Y)) \wedge (\neg X \vee (X \wedge \neg Y))) \ \vee \ (\neg X \vee Y)$$
$$\Phi_0 = X \vee \neg X \qquad \Phi_{n+1} = ((X_n \wedge Y_n) \wedge (X_n \wedge Z_n)) \vee (((\neg X_n \wedge \Phi_n) \vee \neg Y_n) \vee \neg Z_n)$$
$$\Psi_0^1 = \top \qquad \Psi_0^2 = \bot \qquad \Psi_{n+1}^1 = \Psi_n^1 \wedge X_n \qquad \Psi_{n+1}^2 = \Psi_n^2 \vee Y_n$$
$$\Psi_n^3 = (X \vee (Y \wedge \Psi_n^2)) \wedge \Psi_n^1 \qquad \Psi_n^4 = (Y \wedge (X \vee \Psi_n^1)) \vee \Psi_n^2 \qquad \Psi_n = \neg \Psi_n^3 \vee \Psi_n^4$$

The formulas $\mathtt{E}_2$, $\mathtt{E}_3$ and Φ_n are provable, while $\mathtt{E}_1$ and Ψ_n are not.

We compare four algorithms: cf is prove-cf from [5], fw is the forward algorithm from [5], $\mathsf{bw_f}$ is the backward algorithm based on $\mathsf{OL_f}$, and $\mathsf{fw_f}$ is the forward algorithm based on $\mathsf{OL_f}$. The implementations are done in `OCaml` in the most naive way (except that we use some memoization), so that running time (*time*, in seconds) should not be taken too seriously. As an alternative measure which depends less on the particular implementation, we also count the number of rule occurrences (*rules*) applied during search.

time	cf	$\mathsf{bw_f}$	fw	$\mathsf{fw_f}$
$\mathtt{E}_1$	0.00	0.00	0.04	0.03
$\mathtt{E}_2$	0.00	0.00	0.02	0.01
$\mathtt{E}_3$	0.00	0.00	0.02	0.02
Φ_5	0.07	0.00	15.00	3.56
Φ_{10}	0.34	0.00	368.86	88.60
Ψ_5	0.22	0.00	1.43	0.13
Ψ_{20}	—	0.00	161.84	4.92

rules	cf	$\mathsf{bw_f}$	fw	$\mathsf{fw_f}$
$\mathtt{E}_1$	2 305	132	47	64
$\mathtt{E}_2$	210	104	33	49
$\mathtt{E}_3$	42	144	47	49
Φ_5	6 094	384	724	338
Φ_{10}	12 344	774	2 639	1 023
Ψ_5	244 055	308	343	123
Ψ_{20}	—	2 603	2 083	723

rules	cf	$\mathsf{bw_f}$	fw	$\mathsf{fw_f}$
Average on some random formulas of size 20	971	28	163	29
Average on some random formulas of size 100	129 428	264	2 753	734

This is really a minimalist benchmarking. Deeper experiments must be done to obtain more precise comparison informations. Focusing-based algorithms look really competitive. Forward algorithms are sometimes more efficient than backward ones concerning the number of rules applied, but require more management of data structures so take longer execution time.

5   Conclusion

We have presented new sequent-calculus proof-systems for orthologic, mainly: OL which is the simplest such system we know, and OL$_f$ which is based on focusing to constrain the structure of proofs. With some complementary analysis on the structure of proofs in OL$_f$ we have proposed efficient proof search algorithms for orthologic which look quicker than the state of the art [5] (but additional studies in this direction must be done to obtain fully convincing evaluations).

Our new systems open the door for additional proof-theoretical studies of orthologic (and the possibility of extracting counter-models from proof-search failures should be investigated). We hope also this will lead to results in the theory of ortholattices (free ones in particular) in the spirit of Whitman's work [15]. The present work could be extended to second-order quantifiers on the logic side in relation with complete ortholattices. We plan also to work on the application of focusing to other lattice-related logics [14].

Finally, the proof theory of orthologic seems to be mature enough to try to develop some Curry-Howard correspondence aiming at exhibiting the computational content of orthologic.

Additional Material

A `Coq` development formalising the main proofs of the paper is available at:

> https://hal.archives-ouvertes.fr/hal-01306132/file/olf.v.txt

The `OCaml` code for the benchmark of Section 4.4 is available at:

> https://hal.archives-ouvertes.fr/hal-01306132/file/olf.ml.txt

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