Polarized games

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Abstract

We generalize the intuitionistic Hyland-Ong games to a notion of polarized games *allowing games with plays starting by proponent moves. The usual constructions on games are adjusted to fit this setting yielding a game model for* polarized linear logic *with a definability result. As a consequence this gives a complete game model for various classical systems:* LC, *$\lambda\mu$-calculus, ... for both call-by-name and call-by-value evaluations.*

1. Introduction

Game semantics has been used to interpret both logical systems and programming languages. The logical step has often been a preliminary step towards the study of game models for programming languages. Moreover Linear Logic (LL) has taken a very important place in this first step. We can classify these models of linear logic along two main constraints: some of them are restricted to linear fragments (without exponential connective) of LL, such as MLL [1] or MALL [5], the others are restricted to intuitionistic fragments [20, 2, 24]. In a different spirit, a model of MELL is given in [6] but introduces non deterministic strategies to model a deterministic language. On the computer science side, games have been developed to model different kinds of languages (PCF [2, 18], μPCF [19], Idealized Algol [4], ...). These games are based on call-by-name (CBN) computation which corresponds to the technical property that plays only start by opponent moves which is also the constraint appearing in games for Intuitionistic Linear Logic (ILL). The idea of replacing opponent starting games by proponent starting games leads to a model of call-by-value (CBV) computation [17] (embedded into opponent starting games in [3]).

One of our goals is to liberalize these starting conditions in order to recover a real symmetry between the two players. This is extremely natural in the spirit of LL, where duality (lost in intuitionistic systems) plays a key role, but it is known to be a difficult problem: in Blass's work [7], composition is not associative, and non determinism is required in [6] where full completeness is lost. Our solution is to put together opponent starting and proponent starting games but to refuse plays starting by both players in the same game. The introduction of two families of games: positive (proponent starting) and negative (opponent starting) corresponds to the notion of polarity developed by Girard for his system of classical logic LC [14] and studied by the author in Polarized Linear Logic (LLP).

As is clearly the case for game semantics, full LL is a difficult system to deal with. The problem is to find a more simple fragment of LL which is expressive enough. The main proposition has been ILL, but it refuses the linear negation connective which may be considered as the main connective of LL since it gives duality. From an expressiveness view point, ILL is a good system for the study of intuitionistic logic but the translations of classical logic into ILL are in fact $\neg\neg$-translations. Using Girard's idea of polarization for classical logic, the system LLP [21] gives another possibility. It is obtained from LL by restricting to polarized formulas and by generalizing structural rules to any negative formula (instead of only ?-formulas) to get classical features. The study of this system is easier than for LL (proof-nets, ...), and the current presentation will enforce this view point by giving a game model. Translations of various classical systems into LLP have been developed, generalizing Girard's translations of intuitionistic logic into LL ($A \to B \rightsquigarrow\ !A \multimap B$ and $A \to B \rightsquigarrow\ !(A \multimap B)$) without any $\neg\neg$-feature. Call-by-name systems are translated by the first translation: $\lambda\mu$-calculus [27], LKT [11], λc-calculus, ... and call-by-value systems are translated with a generalization of the second translation for classical logic ($A \to B \rightsquigarrow\ !(A \multimap ?B)$): $\lambda\mu_V$-calculus [26], LKQ [11], ...

In this way, LLP appears as the part of LL corresponding to classical logic. Moreover it allows us to use one system only for both CBN and CBV (this idea has also been independently carried out by Levy [23], without linear logic). LLP is therefore a language in the spirit of a duality between these two evaluation paradigms [28, 9] appearing here as positive/negative (or focalization/reversibility).

We are going to describe the notion of *polarized games* containing both proponent starting and opponent starting games. They are presented as a model of LLP. Turning them into a model of programming languages with control operators would be a further (and probably easier) step. The choice of LLP as a logical setting seems quite natural since it contains many other classical systems through translations and takes an explicit account of polarities (opponent/proponent) of games. Moreover the linear level has shown to give interesting intuitions for game semantics.

We can consider three points of view about these polarized games:

Polarized approach. In order to get a model of LLP, it is natural to put together two kinds of games corresponding to positive and negative formulas. Connectives have then to be defined between games of the corresponding polarity. For example, we have to use the usual Hyland-Ong [18] (cartesian) product (denoted by $\&$) between negative games and the Honda-Yoshida [17] (tensor) product (denoted by $\otimes$) between positive games. From the use of polarities in ludics [15], we get the idea of introducing two new *lifting* connectives $\downarrow$ and $\uparrow$ allowing a "linear" change of polarity (they have also been introduced by Lamarche [20], and used recently in [10]). These two connectives act on games by adding a new move at the beginning of each play, in such a way that $\downarrow$ (resp. $\uparrow$) turns a negative (resp. positive) game into a positive (resp. negative) one. The introduction of this large collection of connectives allows to go one step further than in LL in the decomposition of the classical connectives and allows to give a precise analysis of the structure of games.

In particular, the separation between positive and negative games allows to solve the Blass's problem [7] of composing strategies. The introduction of the lifting connectives gives a solution to McCusker's problem with well-openness for defining the ! construction [24] and leads to a decomposition of the main LL isomorphism $!A \otimes !B \simeq !(A \& B)$.

Finally, to obtain a model of LLP, we have to endow games with some monoidal structure to interpret the structural rules. This is done by introducing the notion of *multiple games* which is preserved by the constructions.

The polarized approach makes explicit the unification we are doing between the interpretation of call-by-name and call-by-value. We are able to define a single polarized game model without any particular choice between CBN and CBV. We then get a model of a particular evaluation paradigm by choosing the corresponding interpretation: negative for call-by-name and positive for call-by-value.

Negative approach. If we hide positive games and consider only the negative ones, we mainly get back a classical version of the Hyland-Ong model. More precisely, with respect to Laird's model of innocent strategies for control [19], we are able to interpret a richer type language (with the difficult point of disjunction) and, using the notion of multiple games, we are not restricting control primitives to ground types. That is, we get a model of the extended $\lambda\mu$-calculus described by Selinger [28] and our multiple negative games give one of the first concrete examples (as far as we know) of a *control category*.

Using some other translations, we can get denotational game models for various call-by-name systems.

Negative games will have a prominent role at some places of the paper since they correspond to the usual notion of game.

Positive approach. Positive games have been considered by Honda-Yoshida [17] in a purely functional setting. We get here an extension of the typing language we are able to interpret and a way of interpreting control primitives. These two points were suggested in their paper but not described.

From a $\lambda\mu$ point of view, we give a model of the call-by-value $\lambda\mu$-calculus [26] (and of the extended version of Selinger). Moreover we build a *co-control category* of multiple positive games.

Using some other translations, we can get denotational game models for various call-by-value systems.

Games are not only used because they allow to define models for a large class of systems but also because they lead to *full completeness* results, see [1, 2, 5, 18, 20] for example. We end our study of polarized games with a full completeness (or definability) theorem with respect to LLP (without atom): a strategy on a polarized game is the interpretation of a proof of LLP. And, as a consequence, we get the same result for both CBN and CBV $\lambda\mu$-calculi.

2. From Hyland-Ong games to polarized games

We introduce our notion of *polarized games* by extending the usual definitions of Hyland-Ong games [18] with plays possibly starting by proponent moves. We are following McCusker's presentation [24], but we change the notations for some constructions in order to have a precise correspondence between games constructions and LLP connectives. We use his definition of exponentials and show how to get rid of the problems with dereliction by a stronger use of well-openness with lifts.

2.1. Definitions

Definition 1 (Polarized arena)
A *polarized arena* is a tuple $A = (\pi_A, M_A, \lambda_A, \vdash_A)$

where:

- $\pi_A \in \{O, P\}$ is the polarity of the arena, an O-arena (resp. P-arena) is also called *negative* (resp. *positive*);
- M_A is the set of *moves*;
- λ_A is the *labelling function* from M_A to $\{O, P\}$, we use the notation $m^{\lambda_A(m)}$ to make explicit the label of a move m;
- $\vdash_A$ is the *enabling relation*, that is a relation on $M_A \times M_A$ denoted by $m \vdash_A n$ and a subset of M_A which are the *initial moves* of the arena denoted by $\vdash_A m$. This relation has to satisfy:
  - $\vdash_A m \Rightarrow \lambda_A(m) = \pi_A \wedge \forall n \in M_A, n \nvdash_A m$
  - $m \vdash_A n \Rightarrow \lambda_A(m) \neq \lambda_A(n)$

We denote by $\overline{\pi_A}$ (resp. $\overline{\lambda_A}$) the opposite of π_A (resp. λ_A) and by M_A^i (resp. M_A^{ni}) the initial (resp. non initial) moves of A.

Notations:

- A^* is the set of the finite sequences of moves of A.
- ε is the empty sequence.
- $\leq$ is the prefix order on sequences.
- The P-prefix order is defined by $s \leq^P t$ if $s \leq t$ and s ends by a P-move (including $s = \varepsilon$ in a negative arena).
- $cl^P(.)$ is the P-prefix closure of a set of sequences.

Definition 2 (Justified sequence)
A *justified sequence* s on A is a sequence of moves of A with, for each non initial move n, a pointer to an earlier move m such that $m \vdash_A n$, we say that m *justifies* n in s.

If there exists a sequence $n_0, \ldots, n_k$ of moves of s such that n_i justifies n_{i+1}, we say that n_0 *hereditarily justifies* n_k in s.

Definition 3 (View)
Let s be a justified sequence, we define a sub-sequence called the *proponent view* $\ulcorner s \urcorner$ by:

- $\ulcorner \varepsilon \urcorner = \varepsilon$
- $\ulcorner s m^P \urcorner = \ulcorner s \urcorner m^P$
- $\ulcorner s m^O \urcorner = m^O$ if m is initial
- $\ulcorner s m t n^O \urcorner = \ulcorner s \urcorner m n^O$ if m justifies n

The *opponent view* $\llcorner s \lrcorner$ is defined exactly as the proponent view by exchanging the two players.

Definition 4 (Legal position)
A justified sequence s is a *legal position* if:

- $tmn \leq s \Rightarrow \lambda(m) \neq \lambda(n)$
- $tm^P \leq s \Rightarrow m$ points in $\ulcorner t \urcorner$ if m is not initial
- $tm^O \leq s \Rightarrow m$ points in $\llcorner t \lrcorner$ if m is not initial

L_A is the set of legal positions of A.

A legal position s is *well opened* if the only initial move in s is the first one.

Definition 5 (Projection)
Let s be a legal position in A and I be a set of initial moves of s, the *projection* $s \upharpoonright_I$ of s on I is the sub-sequence of s of the moves hereditarily justified by a move of I, it is a legal position of A.

Definition 6 (Polarized game)
A *polarized game* is a tuple $A = (\pi_A, M_A, \lambda_A, \vdash_A, \mathcal{P}_A)$ where $(\pi_A, M_A, \lambda_A, \vdash_A)$ is a polarized arena and $\mathcal{P}_A$ is a non empty prefix-closed set of legal positions such that if $s \in \mathcal{P}_A$ and I is a set of initial moves, $s \upharpoonright_I \in \mathcal{P}_A$[1].

A game is *well opened* if all its plays are well opened.

We denote by $\mathcal{P}_A^P$ the plays ending by P-moves.

We define a collection of constructions on polarized arenas and games, mainly coming from intuitionistic games.

2.2. Arenas constructions

Sum of arenas. Let A and B be two arenas of the same polarity, we define the arena $A + B$ by:

- $\pi_{A+B} = \pi_A = \pi_B$
- $M_{A+B} = M_A + M_B$ (disjoint sum)
- $\lambda_{A+B} = [\lambda_A, \lambda_B]$
- $\vdash_{A+B} m \iff \vdash_A m \vee \vdash_B m$
- $m \vdash_{A+B} n \iff m \vdash_A n \vee m \vdash_B n$

If s is a legal position of $A + B$, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the sub-sequence of s containing the moves of A (resp. B), it's a legal position of A (resp. B).

Product of arenas. If A and B have the same polarity, $A \times B$ is defined by:

- $\pi_{A\times B} = \pi_A = \pi_B$
- $M_{A\times B} = M_A^i \times M_B^i + M_A^{ni} + M_B^{ni}$
- $\lambda_{A\times B}(m_1, m_2) = \lambda_A(m_1) = \lambda_B(m_2)$
- $\lambda_{A\times B}(m) = [\lambda_A, \lambda_B](m)$ if $m \in M_A^{ni} + M_B^{ni}$
- $\vdash_{A\times B} (m_1, m_2)$
- $(m_1, m_2) \vdash_{A\times B} n \iff m_1 \vdash_A n \vee m_2 \vdash_B n$
- $m \vdash_{A\times B} n \iff m \vdash_A n \vee m \vdash_B n$ if m is not initial

If s is a well opened legal position of $A \times B$, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the sub-sequence of s containing the moves of A (resp. B) thus the first (resp. second) component of the initial move. It's a legal position of A (resp. B).

Remark: Defining the notion of projection on a component for a non well opened position of $A \times B$ would be more complex, this is why we will restrict ourselves to this particular case which is sufficient for what we want here.

2.3. Games constructions

We turn now to the description of the various constructions of polarized games we are interested in.

[1]This condition is used to obtain $\mathcal{P}_A \subset \mathcal{P}_{\sharp A}$ and is immediate for well opened games.

In the sequel, we will use N, M, L, … for negative games (or formulas) and P, Q, R, … for positive ones. A, B, C, … denote games (or formulas) of any polarity.

Dual. $A^\perp = (\overline{\pi_A}, M_A, \overline{\lambda_A}, \vdash_A, \mathcal{P}_A)$

Top. $\top = (O, \emptyset, \emptyset, \emptyset, \{\varepsilon\})$

Bottom. $\perp = (O, \{*\}, \lambda_\perp(*) = O, \vdash_A *, \{\varepsilon, *\})$

Negative tensor. If M and N are negative, the arena of $M \odot N$ is $M + N$ and $\mathcal{P}_{M \odot N} = \{s \in L_{M+N} \mid s \upharpoonright_M \in \mathcal{P}_M \wedge s \upharpoonright_N \in \mathcal{P}_N\}$.

Implication. If M and N are negative, $M \rightarrowtriangle N = (O, M_M + M_N, [\overline{\lambda_M}, \lambda_N], \vdash_{M \rightarrowtriangle N}, \mathcal{P}_{M \rightarrowtriangle N})$ with:

- $\vdash_{M \rightarrowtriangle N} m \iff \vdash_N m$
- $m \vdash_{M \rightarrowtriangle N} n \iff m \vdash_M n \vee m \vdash_N n \vee (\vdash_N m \wedge \vdash_M n)$
- $\mathcal{P}_{M \rightarrowtriangle N} = \{s \in L_{M \rightarrowtriangle N} \mid s \upharpoonright_M \in \mathcal{P}_M \wedge s \upharpoonright_N \in \mathcal{P}_N\}$

With. If M and N are negative, the arena of $M \,\&\, N$ is $M + N$ and $\mathcal{P}_{M \& N} = \mathcal{P}_M \cup \mathcal{P}_N$ (the only common play is ε).

Par. If M and N are well opened negative games, the arena of $M \parr N$ is $M \times N$ and $\mathcal{P}_{M \parr N} = \{s \in L_{M \times N} \mid s \upharpoonright_M \in \mathcal{P}_M \wedge s \upharpoonright_N \in \mathcal{P}_N\}$.

Sharp. If N is negative, $\sharp N$ has the same arena as N and $\mathcal{P}_{\sharp N} = \{s \in L_N \mid \forall m \text{ initial}, s \upharpoonright_m \in \mathcal{P}_N\}$.

The constraints on the first move and on projections are sufficient to automatically get the usual *switching conditions* for $\odot$, $\parr$, $\sharp$, … That is, only one player is allowed to switch between the components of a game during a play, which is opponent for a “conjunctive” connective ($\odot$, $\&$, $\sharp$, …) and proponent for a “disjunctive” connective ($\parr$, $\rightarrowtriangle$, …).

Lift. If P is positive, $\uparrow P$ is the negative game obtained from P by adding a new *initial* opponent move $*$ with:

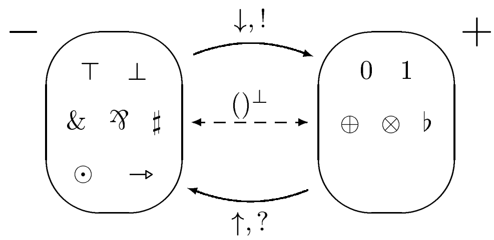
- $* \vdash_{\uparrow P} m \iff \vdash_P m$
- $m \vdash_{\uparrow P} n \iff m \vdash_P n$ if $m \in M_P$
- $\mathcal{P}_{\uparrow P} = *.\mathcal{P}_P + \{\varepsilon\}$

Positive constructions. The positive constructions are obtained by duality: $0 = \top^\perp$, $1 = \perp^\perp$, $P \otimes Q = (P^\perp \parr Q^\perp)^\perp$, $P \oplus Q = (P^\perp \,\&\, Q^\perp)^\perp$, $\flat P = (\sharp P^\perp)^\perp$ and $\downarrow N = (\uparrow N^\perp)^\perp$.

Linear implication. (just notation) $P \multimap N = P^\perp \parr N$

Exponentials. (just notations) $!N = \downarrow\sharp N$ and $?P = \uparrow\flat P = (!P^\perp)^\perp$

Remark: Some of these constructions ($\top$, $\odot$, $\rightarrowtriangle$, $\&$, $\sharp$) are the same as in [24] but used in a slightly different way. The $\parr$-construction is a variant of $\otimes$ defined in [17]. The lifting constructions $\downarrow$ and $\uparrow$ already appeared in Lamarche’s games [20] but their use here is much in the spirit of Girard’s ludics [15]. The novelty is to put these constructions together with an important place given to the lifts:



Remark: $\perp$, $\top$ and $\uparrow P$ are well opened and if M and N are well opened then $N^\perp$, $M \rightarrowtriangle N$, $M \,\&\, N$, $M \parr N$ are well opened. This will allow to define dereliction properly. For the interpretation of logic, we will only use such games however constructions like $M \odot N$ and $\sharp M$ may appear as intermediary steps.

We introduce a strong notion of isomorphism of games without any reference to strategies (defined later). The idea is to represent the main properties of logical connectives by structural properties of games in a simpler way than the usual use of categorical isomorphisms.

Definition 7 (Play isomorphism)
A *p-isomorphism* between two games A and B of the same polarity is a bijective function f between $\mathcal{P}_A$ and $\mathcal{P}_B$ which preserves the length and such that:

- if $s \leq t$ then $f(s) \leq f(t)$,
- if the i^{th} move of s points to the j^{th} move of s then the i^{th} move of $f(s)$ points to the j^{th} move of $f(s)$.

A game has a given p-property if the underlying isomorphism is a p-isomorphism.

Proposition 1 (Structure of constructions)
We have the following properties:

- *all the binary connectives are p-commutative and p-associative,*
- $\perp$ *is p-neutral for* $\parr$, $\top$ *is p-neutral for* $\odot$ *and* $\&$,
- $\parr$ *is p-distributive over* $\&$, $\top$ *is p-absorbing for* $\parr$,
- $M \rightarrowtriangle N \simeq_p \uparrow M^\perp \parr N$,
- $\downarrow(M \odot N) \simeq_p \downarrow M \otimes \downarrow N$,
- $\sharp(M \,\&\, N) \simeq_p \sharp M \odot \sharp N$,
- $!(M \,\&\, N) \simeq_p !M \otimes !N$,
- $\top = \sharp\top$ *and* $1 = \downarrow\top$ *thus* $1 = !\top$.

Remarks:

- The introduction of polarities gives the p-associativity of the $\oplus$ which is stronger than the

result obtained for the corresponding negative connective of [24]. This negative construction can be decomposed by the positive one $\oplus$ into $\uparrow(\downarrow M \oplus \downarrow N)$.
- The decomposition of the ! connective into two distinct operations gives rise to a decomposition of the main LL isomorphism $!(M \,\&\, N) \simeq_p !M \otimes !N$ through $\downarrow(M \odot N) \simeq_p \downarrow M \otimes \downarrow N$ and $\sharp(M \,\&\, N) \simeq_p \sharp M \odot \sharp N$.
- $M \multimap N \simeq_p \uparrow M^\perp \parr N$ can be interpreted as a linear version of the usual translation of the intuitionistic implication in LL: $M \to N = ?M^\perp \parr N$.

Remark: A natural direction in game semantics is to move constraints on plays to constraints on strategies (in order to use arenas instead of games for example). In our setting, this would be problematic for two reasons:
- The two games $M \,\&\, N$ and $M \odot N$ are based on the same arena, as for N and $\sharp N$. Without plays, we can't describe anymore the equations of proposition 1.
- Working with well opened strategies would require to modify the composition and would avoid comparison with the usual intuitionistic setting.

2.4. Strategies

We are now going to introduce the notion of strategy. They will be used to interpret proofs and programs. From a categorical view point, strategies correspond to morphisms.

Definition 8 (Strategy)
A *strategy* σ on the game A, denoted by $\sigma : A$, is a non empty P-prefix closed subset of sequences in $\mathcal{P}_A^P$. Moreover we require some additional properties:
- *determinism*: if $sa^P \in \sigma$ and $sb^P \in \sigma$ then $a = b$;
- *innocence*: if $sab^P \in \sigma$, $t \in \sigma$, $ta \in \mathcal{P}_A$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$ then $tab \in \sigma$.
- *totality*: if $s \in \sigma$ and $sa^O \in \mathcal{P}_A$ then $\exists b, sab \in \sigma$ (moreover, if A is positive $\exists b, b \in \sigma$).

A total deterministic innocent strategy can be represented as a function from proponent views ending by O-moves to P-moves. We define the size $|\sigma|$ of a strategy σ to be the sum of the lengths of the proponent views in its graph.
- σ is *finite* if $|\sigma|$ is finite.

A strategy $\sigma : M \multimap N$ is *central*, denoted by $\sigma : M \overset{\bullet}{\multimap} N$, if each play of σ has its first proponent move in M.

In the sequel, a strategy will always be a *finite total deterministic innocent* strategy.

Lemma 1
Let σ be a strategy on a game A and f be a p-isomorphism from A to B, $f(\sigma)$ is a strategy on B.

Definition 9 (Identity)
Let N be a negative game, the *identity* id_N on $N \overset{\bullet}{\multimap} N$ is the central strategy given by $id_N = \{s \in \mathcal{P}^P_{N_1 \multimap N_2} \mid \forall t \leq^P s, t \restriction_{N_1} = t \restriction_{N_2}\}$ (the indexes are only used to distinguish the two occurrences of N).

Definition 10 (Composition)
Let $\sigma : L \multimap M$ and $\tau : M \multimap N$ be two strategies. The *composition* $\sigma;\tau$ is the strategy on $L \multimap N$ defined by:

$$\begin{aligned}\sigma;\tau = \{ & s\restriction_{L \multimap N} \in \mathcal{P}^P_{L \multimap N} \mid \\ & s \in \mathrm{int}(L,M,N) \wedge s\restriction_{L\multimap M} \in \sigma \wedge s\restriction_{M \multimap N} \in \tau\}\end{aligned}$$

where $\mathrm{int}(L,M,N)$ is the set of sequences s of $(L+M+N)^*$ with a pointer for each move (except those initial in N) such that $s\restriction_{L+M} \in \mathcal{P}_{L\multimap M}$ and $s\restriction_{M+N} \in \mathcal{P}_{M\multimap N}$. $s\restriction_{L\multimap N}$ is obtained by replacing the pointer of the L-moves pointing in M by the justifier of the M-move that must be an initial N-move.

Composition can be generalized to obtain a strategy on N from a strategy $\sigma : M$ and a strategy $\tau : M \multimap N$ since $M \simeq_p \bot \parr M \simeq_p \top \multimap M$.

Proposition 2 (HO category)
Negative games with strategies on $M \multimap N$ as morphisms give a symmetric monoidal closed category with finite products.

Definition 11 ($\parr$ of strategies)
Let $\sigma : M_1 \overset{\bullet}{\multimap} N_1$ and $\tau : M_2 \multimap N_2$ be two strategies, the strategy $\sigma \parr \tau$ is $\{s \in \mathcal{P}^P_{M_1 \parr M_2 \multimap N_1 \parr N_2} \mid s\restriction_{M_1 \multimap N_1} \in \sigma \wedge s\restriction_{M_2 \multimap N_2} \in \tau\} : M_1 \parr M_2 \multimap N_1 \parr N_2$.

We denote $id_N \parr \tau$ by $N \parr \tau$, which is defined for any strategy $\tau : L \multimap M$ since id_N is a central strategy on $N \overset{\bullet}{\multimap} N$.

Lemma 2
$\parr$ is a bifunctor in the category of negative games with central *strategies on $M \overset{\bullet}{\multimap} N$ as morphisms.*

Remark: This result is false for general strategies because the $\parr$ is only defined if one of the strategies is central. The $\parr$ is not bifunctorial in the full category which corresponds to the premonoidal structure of control categories of P. Selinger [28]. All this has also to be linked with the problem of constructions on strategies for Blass games [7], solved here by adding the centrality constraint.

3. Polarized Linear Logic (linear case)

Polarized Linear Logic has been introduced as a subsystem of Linear Logic with more structure. It is easier to study but expressive enough to interpret classical logic. The main deterministic classical systems have translations into LLP. Moreover LLP allows to interpret both *call-by-name* and *call-by-value* classical logics by pointing out negative or positive formulas.

Polarized games give a denotational model of LLP. As a first step, we will consider only the fragment MALLP of LLP without exponential.

3.1. MALLP

This calculus is a linear fragment (without contraction and weakening) of polarized linear logic, the full system will be studied in section 5. In this linear setting, exponentials are replaced by *lifting* operators used to change the polarity.

Linear polarized formulas.

$$\begin{array}{lccccc} P & ::= & X^\perp & \mid & 1 & \mid & 0 \\ & \mid & P \otimes P & \mid & P \oplus P & \mid & {\downarrow}N \\ N & ::= & X & \mid & \perp & \mid & \top \\ & \mid & N \parr N & \mid & N \,\&\, N & \mid & {\uparrow}P \end{array}$$

Rules

$$\frac{}{\vdash N, N^\perp}\mathit{ax} \qquad \frac{\vdash \Gamma, N \quad \vdash \Delta, N^\perp}{\vdash \Gamma, \Delta}\mathit{cut}$$

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, {\downarrow}N}{\downarrow} \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, {\uparrow}P}{\uparrow}$$

where $\mathcal{N}$ contains only negative formulas.

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\otimes \qquad \frac{\vdash \Gamma, M, N}{\vdash \Gamma, M \parr N}\parr$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\oplus_2$$

$$\frac{\vdash \Gamma, M \quad \vdash \Gamma, N}{\vdash \Gamma, M \,\&\, N}\&$$

$$\frac{}{\vdash 1}1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\perp \qquad \frac{}{\vdash \Gamma, \top}\top$$

with at most one positive formula in Γ for the $\top$-rule.

Proposition 3
If $\vdash \Gamma$ is provable in MALLP*, Γ contains at most one positive formula.*

3.2. Interpretation of proofs

A proof π of the sequent $\vdash N_1, \dots, N_k$ is interpreted by a *strategy* σ_π on $N_1 \parr \dots \parr N_k$, and a proof π of $\vdash P, N_1, \dots, N_k$ by a *central strategy* σ_π on $P^\perp \overset{\bullet}{\rightarrow} N_1 \parr \dots \parr N_k$ (with the particular case $N_1 \parr \dots \parr N_k = \perp$ if $k = 0$).

Since games are interpreting formulas and we have used the same notation for the connectives and the corresponding game constructions, we will often use the same notation for a formula and the corresponding game.

Axioms.

- The *ax*-rule introducing $\vdash N, N^\perp$ is interpreted by the central strategy $\mathit{id}_N : N \overset{\bullet}{\rightarrow} N$.
- The 1-rule is interpreted by the central strategy $\mathit{id}_\perp$ on $1^\perp \overset{\bullet}{\rightarrow} \perp$.
- The $\top$-rule is interpreted by the central strategy $\{\varepsilon\}$ on $P^\perp \overset{\bullet}{\rightarrow} \mathcal{N} \parr \top$ if $\Gamma = P, \mathcal{N}$, and by the strategy $\{\varepsilon\}$ on $\mathcal{N} \parr \top$ if Γ doesn't contain any positive formula.

Cut rule. The interpretation of the two premises gives a strategy $\sigma : \mathcal{N} \parr N$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \mathcal{N} \parr N$) if $\Gamma = \mathcal{N}$ (resp. $\Gamma = P, \mathcal{N}$) and a central strategy $\tau : N \overset{\bullet}{\rightarrow} \Delta$. The *cut*-rule is interpreted by composition: $\sigma; (\mathcal{N} \parr \tau) : \mathcal{N} \parr \Delta$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \mathcal{N} \parr \Delta$).

Lifts.

- ${\uparrow}$: by lemma 1, applying the p-isomorphism ${\uparrow}P \parr \Gamma \simeq_p P^\perp \multimap \Gamma$ of proposition 1 to a strategy σ on $P^\perp \overset{\bullet}{\rightarrow} \Gamma$ gives a strategy ${\uparrow}\sigma$ on ${\uparrow}P \parr \Gamma$.
- ${\downarrow}$: if σ is a strategy on $N \parr \mathcal{N}$, we obtain the strategy ${\downarrow}\sigma : {\uparrow}N^\perp \overset{\bullet}{\rightarrow} \mathcal{N}$ defined by ${\downarrow}\sigma = \mathit{cl}^P(\{m{*}ns \mid (n,m)s \in \sigma\})$.

Multiplicatives.

- $\perp$: by lemma 1, applying the p-isomorphism $\Gamma \simeq_p \Gamma \parr \perp$ of proposition 1 to a strategy on Γ (resp. $P^\perp \overset{\bullet}{\rightarrow} \Gamma$) gives a strategy on $\Gamma \parr \perp$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \Gamma \parr \perp$).
- $\parr$: this rule doesn't modify the interpretation.
- $\otimes$: if $\sigma : P^\perp \overset{\bullet}{\rightarrow} \Gamma$ and $\tau : Q^\perp \overset{\bullet}{\rightarrow} \Delta$, we obtain the strategy $\sigma \parr \tau : (P \otimes Q)^\perp \overset{\bullet}{\rightarrow} \Gamma \parr \Delta$ (definition 11).

Additives.

- $\&$: if σ is the strategy on $\Gamma \parr M$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \Gamma \parr M$) and τ is the strategy on $\Gamma \parr N$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \Gamma \parr N$), we use the strategy $\sigma \cup \tau$ on $\Gamma \parr (M \,\&\, N)$ (resp. $P^\perp \overset{\bullet}{\rightarrow} \Gamma \parr (M \,\&\, N)$).
- $\oplus_i$: if $\sigma : P_i^\perp \overset{\bullet}{\rightarrow} \Gamma$, we obtain the strategy $\sigma : (P_1 \oplus P_2)^\perp \overset{\bullet}{\rightarrow} \Gamma$.

Remark: The MIX-rule cannot be interpreted in a natural way:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\mathrm{MIX}$$

if $g_1 g_2 \in \sigma : \Gamma$ and $d_1 d_2 \in \tau : \Delta$ we want to build plays in $\Gamma \parr \Delta$ in a symmetric way but after the move (g_1, d_1) we have to make a choice between g_2 and d_2 and we cannot choose the two moves if we want a deterministic strategy. This corresponds again to the non bifunctoriality of $\parr$.

3.3. Cut elimination

Cut elimination $\pi \to \pi'$ for MALLP is defined in the natural way coming from the LL cut elimination procedure [13].

Lemma 3 (Maximality of total strategies)
Let σ and τ be two strategies on A such that $\sigma \subset \tau$, we have $\sigma = \tau$ by totality.

Theorem 1 (Correctness)
If $\pi \to \pi'$ then $\sigma_\pi = \sigma_{\pi'}$.

PROOF: Polarized games may be define on proof-nets [22] (with boxes for $\downarrow$) instead of sequent calculus, this is why we will only look at the needed cut-elimination steps and not at all the commutative ones. Since they are easy to reconstruct, we omit the pointers in the plays.

- Axiom cut: if the cut-formula is negative in the axiom, it's just composition with the identity; if this formula is positive, we use $id_N \parr \Gamma = id_{N \parr \Gamma}$ by lemma 2.
- $\downarrow - \uparrow$: let $\sigma : N \parr \Gamma$ and $\tau : N \overset{\bullet}{\rightarrow} \Delta$ be two strategies, we have to prove $\sigma; (\tau \parr \Gamma) = \uparrow\tau; (\Delta \parr \downarrow\sigma) : \Delta \parr \Gamma$. Let s be a play in $\sigma; (\tau \parr \Gamma)$, it's the projection on $\Delta \parr \Gamma$ of a sequence $s_1 = (d,g)(n,g)s'_1$ in $((N \parr \Gamma) + (\Delta \parr \Gamma))^*$. We define $s_2 = (d,g)(d,*)ns'_1$ in $((\Delta \parr \uparrow N^\perp) + (\Delta \parr \Gamma))^*$, we have $s_2 \upharpoonright_{\Delta \parr \uparrow N^\perp} \in \uparrow\tau$ because $s_1 \upharpoonright_{N \rightarrow \Delta} \in \tau$ by definition of $\tau \parr \Gamma$ and $s_2 \in \Delta \parr \uparrow\sigma$ because $s_1 \upharpoonright_{N \parr \Gamma} \in \sigma$, moreover $s_2 \upharpoonright_{\Delta \parr \Gamma} = s_1 \upharpoonright_{\Delta \parr \Gamma} = s$ thus $s \in \uparrow\tau; (\Delta \parr \downarrow\sigma)$. Equality is given by lemma 3.
- Commutative $\downarrow$: Let $\sigma : N \parr \Gamma \parr M$ and $\tau : N \overset{\bullet}{\rightarrow} \Delta$ be two strategies, we have to show that $\downarrow(\sigma; (\tau \parr \Gamma \parr M)) = \downarrow\sigma; (\tau \parr \Gamma)$. If s is a play of $\downarrow(\sigma; (\tau \parr \Gamma \parr M))$, by definition of $\downarrow$, $s = (d,g)*ms'$ with $s_1 = (d,g,m)s' \in \sigma; (\tau \parr \Gamma \parr M)$. By definition of composition, s_1 is the projection on $\Delta \parr \Gamma \parr M$ of a sequence $s_0 = (d,g,m)(n,g,m)s'_0$ in $((N \parr \Gamma \parr M) + (\Delta \parr \Gamma \parr M))^*$, let $s_2 = (d,g)(n,g)*ms'_0$ in $(\uparrow M^\perp + (N \parr \Gamma) + (\Delta \parr \Gamma))^*$, we have $s_2 \upharpoonright_{\uparrow M^\perp \rightarrow N \parr \Gamma} \in \downarrow\sigma$ because $s_0 \upharpoonright_{M \parr N \parr \Gamma} \in \sigma$ and we have $s_2 \upharpoonright_{N \rightarrow \Delta} \in \tau$. Moreover $s_2 \upharpoonright_{\uparrow M^\perp \rightarrow \Gamma \parr \Delta} = s$ thus $s \in \downarrow\sigma; (\tau \parr \Gamma)$. We conclude by lemma 3.
- $\otimes - \parr$: By lemma 2, if $\sigma : M \overset{\bullet}{\rightarrow} \Gamma$ and $\tau : N \overset{\bullet}{\rightarrow} \Delta$, we have $\sigma \parr \tau \parr \Sigma = \sigma \parr N \parr \Sigma; \Gamma \parr \tau \parr \Sigma$.
- Additive steps are basically proved as in [18].
- $1 - \perp$: as for *ax* (the 1-rule gives an identity).
- $\top$: the strategy $\{\varepsilon\}$ composed with any strategy gives the strategy $\{\varepsilon\}$ (because strategies aren't empty). $\square$

In fact this result may be extended to a focalized calculus for MALL (see [15] for example) by replacing the constraint of a negative context in the $\downarrow$-rule by a *focalization* constraint (stoup [14] or η-constraint [12] for example), even if provable sequents in these systems may contain several positive formulas.

4. Exponentials

In order to interpret classical logic, we have to restrict to some particular games allowing us to define structural rules: contraction and weakening.

4.1. Games

Definition 12 (Multiple game)
A game A is a *multiple game* if it is well opened and:

- if $s \in \mathcal{P}_A$, $\vdash_A m$ and $m \vdash_A n$ then $s \upharpoonright_{mn} \in \mathcal{P}_A$ where $s \upharpoonright_{mn}$ is the sub-sequence of s containing m, n and the moves hereditarily justified by n.
- if $s \in L_A$ is a well opened position with an initial move m, $I + J$ is a partition of the moves justified by m in s and $s \upharpoonright_{mI}, s \upharpoonright_{mJ} \in \mathcal{P}_A$ then $s \in \mathcal{P}_A$.

Proposition 4 (Multiple constructions)
Multiplicity is preserved by the following constructions:

- $1, 0, \top$ *and* $\perp$ *are multiple games.*
- $!N$ *and* $?P$ *are multiple games.*
- *If P, Q, M and N are multiple then $P \otimes Q$, $P \oplus Q$, $M \parr N$ and $M \,\&\, N$ are multiple games.*

4.2. Strategies

Dereliction. id_N is a strategy on $\sharp N \overset{\bullet}{\rightarrow} N$ ($\simeq_p !N \multimap N$) since any play of N is a play of $\sharp N$ by definition 6. We define $d_N = \uparrow id_N : ?N^\perp \parr N$.

Contraction. Let N be a multiple negative game, if t is a play in $N_1 \parr N_2 \rightarrow N$ (where the indexes are just used to distinguish the occurrences), we denote by t_i the sub-sequence of t containing the moves in N_i and the moves in N before which the last move in $N_1 \parr N_2$ is in N_i. We define the strategy $c_N = \{s \in \mathcal{P}^P_{N_1 \parr N_2 \rightarrow N} \mid \forall t \leq^P s, t_i \in id_N, i = 1,2\} : N_1 \parr N_2 \overset{\bullet}{\rightarrow} N$.

Weakening. w_N is the strategy on $\perp \overset{\bullet}{\rightarrow} N$ defined by $w_N = \{\varepsilon\} \cup \{m* \mid m \in M^i_N\}$.

5. Polarized Linear Logic (exponential case)

To get a really expressive system corresponding to classical logic, we go from MALLP to the full system LLP.

5.1. LLP

Polarized formulas. We replace the lifted formulas of MALLP by the corresponding exponential version.

$$
\begin{array}{rclcccc}
P & ::= & X^\perp & \mid & 1 & \mid & 0 \\
 & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \\
N & ::= & X & \mid & \perp & \mid & \top \\
 & \mid & N \parr N & \mid & N \,\&\, N & \mid & ?P
\end{array}
$$

Rules. The two lifting rules are replaced by *promotion* and *dereliction*:

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}?d$$

where $\mathcal{N}$ contains only negative formulas. And we add structural rules.

$$\frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N}?w$$

Instead of the usual LL structural rules on ?-formulas, LLP allows structural rules on any negative formula N. So that LLP is obtained from LL by first restricting linear formulas to polarized ones and then by using the properties of the induced system to generalize structural rules.

5.2. Interpretation of proofs

We will now interpret negative formulas by multiple negative games and proofs by strategies. The interpretation of axioms, cuts, multiplicatives and additives is the same as for the linear case (section 3.2).

Exponentials.

- !: If $\sigma : \mathcal{N} ⅋ N$, we define $!\sigma = \{s \in \mathcal{P}^P_{?N^\perp \multimap \mathcal{N}} \mid \forall m \text{ initial and } \vdash_N n, s\restriction_{m*n} \in \downarrow\sigma\} : ?N^\perp \overset{\bullet}{\multimap} \mathcal{N}$ where $s\restriction_{m*n}$ is the sub-sequence of s containing m, $*$, n and the moves hereditarily justified by n.
- ?*d*: If $\sigma : P^\perp \overset{\bullet}{\multimap} \Gamma$ then $d_{P^\perp}; ?P ⅋ \sigma$ is a strategy on $?P ⅋ \Gamma$.
- ?*c*: If $\sigma : \Gamma ⅋ N ⅋ N$ (resp. $P^\perp \overset{\bullet}{\multimap} \Gamma ⅋ N ⅋ N$), we compose it with $\Gamma ⅋ c_N$ to obtain a strategy on $\Gamma ⅋ N$ (resp. $P^\perp \overset{\bullet}{\multimap} \Gamma ⅋ N$).
- ?*w*: If $\sigma : \Gamma$ (resp. $P^\perp \overset{\bullet}{\multimap} \Gamma$), we compose it with $\Gamma ⅋ w_N$ to obtain a strategy on $\Gamma ⅋ N$ (resp. $P^\perp \overset{\bullet}{\multimap} \Gamma ⅋ N$), using $\Gamma \simeq_p \Gamma ⅋ \perp$.

Theorem 2 (Correctness)
If $\pi \to \pi'$ then $\sigma_\pi = \sigma_{\pi'}$.

PROOF: This is an adaptation of the usual result for Hyland-Ong games [18, 24, 16] where the lifts are treated like in the proof of theorem 1. □

6. Propositional definability

Now, we prove a definability result for LLP without atom, showing that every strategy is the interpretation of an LLP proof.

Lemma 4 (Additive type)
Let N be a game corresponding to a negative formula of LLP *without atom, there exist some negative formulas $N_1, \ldots, N_n$ such that $N \simeq_p \underset{1\le i\le n}{\&} ?N_i^\perp$. Moreover this isomorphism is definable.*

Lemma 5 (Bang lemma)
If $\sigma : ?M^\perp \overset{\bullet}{\multimap} \mathcal{N}$ then $\sigma = !(d_M; \sigma ⅋ M)$.

Lemma 6 (Plus lemma)
If $\sigma : (\underset{1\le j\le m}{\&} ?M_j^\perp) \overset{\bullet}{\multimap} ?N^\perp$ then there exists $1 \le j_0 \le m$ such that $\sigma : ?M_{j_0}^\perp \overset{\bullet}{\multimap} ?N^\perp$.

The definition of the interpretation of a proof of LLP as a strategy depends on its conclusion sequent: if it contains a positive formula, we obtain a central strategy on a game of the shape $P^\perp \overset{\bullet}{\multimap} \mathcal{N}$ and if it doesn't, we obtain a strategy on a game $\mathcal{N}$. Since we want to show a completeness result (that is a converse of this interpretation), and since we want to be as precise as possible, we get two cases in the following theorem:

Theorem 3 (Definability)
Let $P, \mathcal{N}$ be formulas without atom,

- *If σ is a central strategy on $P^\perp \overset{\bullet}{\multimap} \mathcal{N}$, σ is the interpretation of a proof of $\vdash P, \mathcal{N}$ in* LLP.
- *If σ is a strategy on $\mathcal{N}$, σ is the interpretation of a proof of $\vdash \mathcal{N}$ in* LLP.

PROOF: By lemma 4, we can restrict ourselves to the case of types $\underset{1\le i\le n}{\&} ?N_i^\perp$ and $(\underset{1\le j\le m}{\&} ?M_j^\perp) \multimap \underset{1\le i\le n}{\&} ?N_i^\perp$. We prove the result by induction on the pair $(|\lceil\sigma\rceil|, |P| + |\mathcal{N}|)$ where the size $|A|$ of a formula A is its number of symbols. We first reduce the cases $n \neq 1$ or $m \neq 1$ to the case $n = 1$ and $m = 1$:

- If $n = 0$, the game is empty and σ also, that corresponds to a $\top$-rule.
- If $n > 1$, then $\sigma_i = \sigma\restriction_{?N_i^\perp}$ (resp. $\sigma\restriction_{(\&_{1\le j\le m} ?M_j^\perp) \multimap ?N_i^\perp}$) is a definable strategy by induction hypothesis with $\sigma = \bigcup_{1\le i\le n} \sigma_i$, which corresponds to a $\&$-rule.
- If $n = 1$ and $m = 0$, σ cannot be central and total on $\top \multimap ?N_1^\perp$.
- If $n = 1$ and $m > 1$, by the plus lemma, σ is a strategy on $?M_{j_0}^\perp \overset{\bullet}{\multimap} ?N_1^\perp$ and is definable by induction hypothesis. The strategy σ is obtained on $P^\perp \overset{\bullet}{\multimap} \mathcal{N}$ by $\oplus$-rules.

We now prove the cases of formulas $?N^\perp$ or $?M^\perp \multimap ?N^\perp$. For the second one, by the bang lemma, we just have to prove the definability of $d_N; \sigma ⅋ M$. This is a smaller strategy on $?N^\perp ⅋ M$ thus definable by induction hypothesis.

If σ is a strategy on $?N^\perp$, either each play has only one move justified by the initial one and $\sigma = d_N; ?N^\perp ⅋ \sigma'$, this corresponds to a dereliction rule on a strategy of the same size on a smaller formula (thus definable). Or there exists a play with two moves justified by the initial one. We define the strategy σ_1 on $?N_1^\perp ⅋ ?N_2^\perp$ (the indexes are just used to distinguish the occurrences) by: if s is a play in σ, the play in $?N_1^\perp \parr ?N_2^\perp$, obtained by putting the first proponent move and the moves justified by it in $?N_1^\perp$ and the other ones in $?N_2^\perp$, is a play in σ_1. We have $\sigma = \sigma_1; c_{?N^\perp}$. It's easy to see that $\sigma_1 = d_N; ?N^\perp \parr \sigma_2$ where σ_2 is a strategy on $N \overset{\bullet}{\multimap} ?N^\perp$. By applying the p-isomorphism of lemma 4 to N and the plus lemma, we get a strategy σ_3 on a game $?M^\perp \overset{\bullet}{\multimap} ?N^\perp$. Finally, we apply the bang lemma and we obtain a strategy σ_4 on $?N^\perp \parr M$ which is smaller than σ. This last step is a bit complicated because if $N = ?N'^\perp$ we may have $|\sigma| = |\sigma_1| = |\sigma_2| = |\sigma_3|$. □

Using the usual techniques of game semantics and the notion of uniform families of strategies, dinatural transformations, ... the definability result can certainly be extended to formulas with atoms.

7. From logic to programming languages

We have described a game model for Polarized Linear Logic claiming that it gives a model for many other systems by translation in LLP. We give some details about the translations of the $\lambda\mu$-calculus into LLP and the consequences we get.

7.1. Call by name

The translation of the call-by-name $\lambda\mu$-calculus into LLP is obtained by translating types by negative formulas:

$$\begin{array}{ccc} X & \rightsquigarrow & X \\ \mathrm{T} & \rightsquigarrow & \top \\ \mathrm{F} & \rightsquigarrow & \bot \\ A \wedge B & \rightsquigarrow & A^- \,\&\, B^- \\ A \vee B & \rightsquigarrow & A^- \parr B^- \\ A \to B & \rightsquigarrow & !A^- \multimap B^- \end{array}$$

the judgment $\Gamma \vdash \Delta$ is translated as $\vdash ?(\Gamma^-)^\perp, \Delta^-$.

The translation of terms is then easy to construct, see [22] for a detailed study of this translation. One of its main properties is that reduction in the $\lambda\mu$-calculus is simulated by cut-elimination in LLP.

Theorem 2 entails that we obtain a denotational model of the CBN $\lambda\mu$-calculus. This can be expressed through Selinger's categories [28]:

Proposition 5 (Control category of games)
The category of multiple negative games with morphisms given by strategies on $!M \multimap N$ ($\simeq_p \sharp M \multimap N$ used in intuitionistic games) is a control category.

We can apply the definability result to the $\lambda\mu$-calculus:

Proposition 6 (Full completeness)
Let A be a type without variable and σ be a strategy on A^-, there exists a $\lambda\mu$-term u of type A such that σ is the call-by-name interpretation of u.

7.2. Call by value

The translation of the call-by-value $\lambda\mu$-calculus into LLP is obtained by translating types by positive formulas:

$$\begin{array}{ccc} X & \rightsquigarrow & X^\perp \\ \mathrm{T} & \rightsquigarrow & 1 \\ \mathrm{F} & \rightsquigarrow & 0 \\ A \wedge B & \rightsquigarrow & A^+ \otimes B^+ \\ A \vee B & \rightsquigarrow & A^+ \oplus B^+ \\ A \to B & \rightsquigarrow & !(A^+ \multimap ?B^+) \end{array}$$

the judgment $\Gamma \vdash \Delta$ is translated as $\vdash {\Gamma^+}^\perp, ?\Delta^+$. The definition of the translation of terms is easy from that.

By applying some properties of this translation (in particular simulation of reduction) and the results about polarized games and LLP, we obtain:

Proposition 7 (Co-control category of games)
The category of multiple positive games with morphisms from P to Q given by strategies on $P^\perp \parr ?Q$ is a co-control category.

Proposition 8 (Full completeness)
Let A be a type without variable and σ be a strategy on A^+, there exists a $\lambda\mu$-term u of type A such that σ is the call-by-value interpretation of u.

This shows that polarized games give a tool for building models of call-by-name and call-by-value programming languages with control operators. We can easily interpret `call/cc`, `catch`, ...

8. Further considerations

Other exponentials. Our interpretation of exponentials is based on Hyland-Ong games which are adapted for definability but we can also define a model with Abramsky-Jagadeesan-Malacaria exponentials. It would be interesting to investigate the other possibilities (sequential algorithms [8], ...).

Polarized structures. Polarized games give a way to conciliate ILL and LLP through the various constructions we have described: $\&$, $\parr$, $\odot$, $\sharp$, $\uparrow$, ... This is actually the only kind of model in which we can define all these constructions, it is natural to try to find some other structures with the same property.

μPCF. Using these polarized games and the definability result, R. Montelatici [25] has shown a completeness theorem for non total strategies with respect to an extension of the $\lambda\mu$-calculus with fix points. Adding ground types shouldn't be difficult and would lead to a completeness result for μPCF, extending Laird's result by a richer type language and control at any type. The next step is to study some other programming languages with control.

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