A syntactic introduction to intersection types

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Abstract

We give an incremental presentation of the invariance of types through reduction in some intersection type systems with subtyping.

1 The λ-calculus

Terms are the usual λ-terms with λ as binder for λ-variables (x, y, ...):

$$t \quad ::= \quad x \mid \lambda x.t \mid t\,t$$

We use the notation $x \notin t$ for x *not free in* t. The *syntactic substitution* of x by u in t is denoted $t\{{}^{u}/_{x}\}$. It makes possible the capture of free variables of the substituting term u by λs of the substituted term t. Except when this syntactic substitution is directly involved (which will occur only in a few places in the paper), we consider λ-terms up to α-conversion of bound variables. We denote the *capture-free substitution* of x by u in t as $t[{}^{u}/_{x}]$.

The β-reduction relation $t \to_\beta u$ is the congruence generated by $(\lambda x.t)\,u \to_{\beta_0} t[{}^{u}/_{x}]$ (see Table 1).

The η-reduction relation $t \to_\eta u$ is the congruence generated by $\lambda x.(t\,x) \to_{\eta_0} t$ if $x \notin t$ (see Table 2).

2 The simply typed λ-calculus

Base types are denoted by X, Y, ... and types are built from base types by means of the binary operation $\to$:

$$A \quad ::= \quad X \mid A \to A$$

Typing judgments are of the shape $\Gamma \vdash t : A$ where Γ is a finite set of pairs of λ-variables and types $(x : A)$ in which each λ-variable occurs at most once, and all the free variables of t are declared in Γ.

$$\frac{}{(\lambda x.t)\,u \to_{\beta_0} t[{}^{u}/_{x}]} \qquad \frac{t \to_{\beta_0} u}{t \to_\beta u} \qquad \frac{t \to_\beta u}{\lambda x.t \to_\beta \lambda x.u} \qquad \frac{t \to_\beta u}{t\,v \to_\beta u\,v} \qquad \frac{t \to_\beta u}{v\,t \to_\beta v\,u}$$

Table 1: β-reduction rules

$$\frac{}{\lambda x.(t\,x) \to_{\eta_0} t}\; x \notin t \qquad \frac{t \to_{\eta_0} u}{t \to_\eta u} \qquad \frac{t \to_\eta u}{\lambda x.t \to_\eta \lambda x.u} \qquad \frac{t \to_\eta u}{t\,v \to_\eta u\,v} \qquad \frac{t \to_\eta u}{v\,t \to_\eta v\,u}$$

Table 2: η-reduction rules

$$\frac{}{\Gamma, x : A \vdash x : A}\; var \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\; abs \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\; app$$

Table 3: Typing rules

The typing system obtained from the previously defined terms, types and typing rules is called ST.

Statement (Monotonicity [Monot])
If $\Gamma \vdash t : A$ *and* $\Delta \supseteq \Gamma$ *then* $\Delta \vdash t : A$ *(where* $\Delta \supseteq \Gamma$ *means that each typing declaration* $x : B$ *in* Γ *appears in* Δ*).*

Lemma 1 (Monotonicity for ST)
Monot *holds for* ST.

Proof: By induction on the derivation of $\Gamma \vdash t : A$. We consider each possible last rule from Table 3:

- **(*var*)** If $t = x$, we have $x : A \in \Gamma$ thus $\Delta \vdash x : A$.
- **(*abs*)** If $t = \lambda x.t'$ with $A = A' \to A''$ and $\Gamma, x : A' \vdash t' : A''$, by induction hypothesis, we have $\Delta, x : A' \vdash t' : A''$ thus $\Delta \vdash \lambda x.t' : A$.
- **(*app*)** If $t = t'\,t''$ with $\Gamma \vdash t' : A' \to A$ and $\Gamma \vdash t'' : A'$, by induction hypothesis, we have $\Delta \vdash t' : A' \to A$ and $\Delta \vdash t'' : A'$. So that $\Delta \vdash t'\,t'' : A$. □

Statement (Non-free variables [NFVar])
If $x \notin t$ *and* $\Gamma, x : B \vdash t : A$ *then* $\Gamma \vdash t : A$.

Lemma 2 (Non-free variables for ST)
NFVar *holds for* ST.

Proof: By induction on the derivation of $\Gamma, x : B \vdash t : A$. We consider each possible last rule from Table 3:

- **(*var*)** If $t = y \neq x$ then $y : A \in \Gamma$ and $\Gamma \vdash y : A$.
- **(*abs*)** If $t = \lambda y.t'$ and $A = A' \to A''$ with $\Gamma, x : B, y : A' \vdash t' : A''$ then, by induction hypothesis, $\Gamma, y : A' \vdash t' : A''$ and thus $\Gamma \vdash \lambda y.t' : A$.
- **(*app*)** If $t = t'\,t''$ with $\Gamma, x : B \vdash t' : A' \to A$ and $\Gamma, x : B \vdash t'' : A'$ then, by induction hypothesis, $\Gamma \vdash t' : A' \to A$ and $\Gamma \vdash t'' : A'$ thus $\Gamma \vdash t'\,t'' : A$. □

Statement (General substitution [GSubst])
Assume that $\Gamma \vdash t\{^v/_x\} : A$ *and for all* Δ *and* B, $\Gamma, \Delta \vdash v : B$ *implies* $\Gamma, \Delta \vdash u : B$, *then* $\Gamma \vdash t\{^u/_x\} : A$.

Lemma 3 (General substitution for ST)
GSubst *holds for* ST.

PROOF: By induction on the derivation of $\Gamma \vdash t\{^v/_x\} : A$. If $t = x$, we have $t\{^v/_x\} = v$ with $\Gamma \vdash v : A$ and we conclude by hypothesis since $t\{^u/_x\} = u$. Otherwise we consider each possible last rule of the derivation of $\Gamma \vdash t\{^v/_x\} : A$ from Table 3:

- ***(var)*** If $t = y \neq x$, we have $t\{^v/_x\} = y = t\{^u/_x\}$.
- ***(abs)*** If $t = \lambda y.t'$ ($y = x$ or $y \neq x$) with $A = A' \to A''$ and $\Gamma, y : A' \vdash t'\{^v/_x\} : A''$ then, by induction hypothesis, $\Gamma, y : A' \vdash t'\{^u/_x\} : A''$ thus $\Gamma \vdash t\{^u/_x\} : A$.
- ***(app)*** If $t = t'\,t''$ with $\Gamma \vdash t'\{^v/_x\} : A' \to A$ and $\Gamma \vdash t''\{^v/_x\} : A'$ then, by induction hypothesis, $\Gamma \vdash t'\{^u/_x\} : A' \to A$ and $\Gamma \vdash t''\{^u/_x\} : A'$ thus $\Gamma \vdash t\{^u/_x\} : A$. □

Statement (Variable inversion [INVVAR])
If $\Gamma \vdash x : A$ *then* $x : A \in \Gamma$.

Lemma 4 (Variable inversion for ST)
INVVAR *holds for* ST.

PROOF: The only possible last rule for deriving $\Gamma \vdash x : A$ is (*var*) and thus $x : A \in \Gamma$. □

Statement (Application inversion [INVAPP])
If $\Gamma \vdash t\,u : A$, *there exists a type* B *such that* $\Gamma \vdash t : B \to A$ *and* $\Gamma \vdash u : B$.

Lemma 5 (Application inversion for ST)
INVAPP *holds for* ST.

PROOF: The only possible last rule for deriving $\Gamma \vdash t\,u : A$ is (*app*) and thus there exists a type B such that $\Gamma \vdash t : B \to A$ and $\Gamma \vdash u : B$. □

Statement (Abstraction inversion [INVABS])
If $\Gamma \vdash \lambda x.t : A$, *there exist* B *and* C *such that* $A = B \to C$ *and* $\Gamma, x : B \vdash t : C$.

Lemma 6 (Abstraction inversion for ST)
INVABS *holds for* ST.

PROOF: The only possible last rule for deriving $\Gamma \vdash \lambda x.t : A$ is (*abs*) and thus there exist B and C such that $A = B \to C$ and $\Gamma, x : B \vdash t : C$. □

Statement (Implicative abstraction inversion [INVABSIMP])
If $\Gamma \vdash \lambda x.t : A \to B$ *then* $\Gamma, x : A \vdash t : B$.

Lemma 7 (Implicative abstraction inversion)
INVABS $\Longrightarrow$ INVABSIMP.

PROOF: Immediate. □

Statement (Substitution [SUBST])
If $\Gamma, x : A \vdash t : B$ *and* $\Gamma \vdash u : A$ *then* $\Gamma \vdash t[^u/_x] : B$.

Lemma 8 (Substitution)
GSUBST $\wedge$ INVVAR $\wedge$ MONOT $\wedge$ NFVAR $\Longrightarrow$ SUBST.

PROOF: Note first that x is not declared in Γ (otherwise $\Gamma, x : A$ is not a valid context) and thus x is not free in u.

Up to α-conversion in t, we can assume that x is not bound in t and that no free variable of u is bound in t. As a consequence $t[^u/_x] = t\{^u/_x\}$.

We have $\Gamma, x : A \vdash t\{^x/_x\} : B$. If $\Gamma, x : A, \Delta \vdash x : C$ then $A = C$ by (INVVAR), and by (MONOT) we obtain $\Gamma, x : A, \Delta \vdash u : A$. It is thus possible to apply (GSUBST) to deduce $\Gamma, x : A \vdash t\{^u/_x\} : B$. Finally, since $t\{^u/_x\} = t[^u/_x]$ and x is not free in $t[^u/_x]$, we can apply (NFVAR) to conclude $\Gamma \vdash t[^u/_x] : B$. □

Statement (Subject reduction for β_0 [βSubjRed$_0$])
If $\Gamma \vdash t : A$ *and* $t \to_{\beta_0} u$ *then* $\Gamma \vdash u : A$.

Lemma 9 (Subject reduction for β_0)
Subst $\wedge$ InvApp $\wedge$ InvAbsImp $\Longrightarrow$ βSubjRed$_0$.

Proof: If $\Gamma \vdash (\lambda x.t)\, u : A$, by (InvApp), there exists B such that $\Gamma \vdash \lambda x.t : B \to A$ and $\Gamma \vdash u : B$. By (InvAbsImp), $\Gamma, x : B \vdash t : A$, and by (Subst), $\Gamma \vdash t[^u/_x] : A$. $\square$

Statement (Subject reduction for β [βSubjRed])
If $\Gamma \vdash t : A$ *and* $t \to_\beta u$ *then* $\Gamma \vdash u : A$.

Lemma 10 (Subject reduction)
GSubst $\wedge$ βSubjRed$_0$ $\Longrightarrow$ βSubjRed.

Proof: If $\Gamma \vdash t : A$ and $t \to_\beta u$ then $t = c\{^{t'}/_x\}$ and $u = c\{^{u'}/_x\}$ with $t' \to_{\beta_0} u'$. Assume that $\Gamma, \Delta \vdash t' : B$ then by (βSubjRed$_0$) we have $\Gamma, \Delta \vdash u' : B$, thus by (GSubst) we obtain $\Gamma \vdash u : A$. $\square$

Theorem 1 (Subject reduction for ST)
βSubjRed *holds for* ST.

Proof: By Lemma 3 we have (GSubst). By Lemma 1 we have (Monot). By Lemma 2 we have (NFVar). By Lemma 4 we have (InvVar). By Lemma 5 we have (InvApp). By Lemma 6 we have (InvAbs).

By Lemma 8 we deduce (Subst). By Lemma 7 we deduce (InvAbsImp). By Lemma 9 we deduce (βSubjRed$_0$). By Lemma 10 we deduce (βSubjRed). $\square$

Statement (Subject reduction for η_0 [ηSubjRed$_0$])
If $\Gamma \vdash t : A$ *and* $t \to_{\eta_0} u$ *then* $\Gamma \vdash u : A$.

Lemma 11 (Subject reduction for η_0)
NFVar $\wedge$ InvVar $\wedge$ InvApp $\wedge$ InvAbs $\Longrightarrow$ ηSubjRed$_0$.

Proof: If $\Gamma \vdash \lambda x.(t\, x) : A$, by (InvAbs), there exist B and C such that $A = B \to C$ and $\Gamma, x : B \vdash t\, x : C$. By (InvApp), there exists D such that $\Gamma, x : B \vdash t : D \to C$ and $\Gamma, x : B \vdash x : D$. By (InvVar), we have $B = D$. By (NFVar), we conclude $\Gamma \vdash t : B \to C$ since $x \notin t$. $\square$

Statement (Subject reduction for η [ηSubjRed])
If $\Gamma \vdash t : A$ *and* $t \to_\eta u$ *then* $\Gamma \vdash u : A$.

Lemma 12 (Subject reduction for η)
GSubst $\wedge$ ηSubjRed$_0$ $\Longrightarrow$ ηSubjRed.

Proof: If $\Gamma \vdash t : A$ and $t \to_\eta u$ then $t = c\{^{t'}/_x\}$ and $u = c\{^{u'}/_x\}$ with $t' \to_{\eta_0} u'$. Assume that $\Gamma, \Delta \vdash t' : B$ then by (ηSubjRed$_0$) we have $\Gamma, \Delta \vdash u' : B$, thus by (GSubst) we obtain $\Gamma \vdash u : A$. $\square$

Theorem 2 (Subject reduction for η for ST)
ηSubjRed *holds for* ST.

Proof: By Lemma 3 we have (GSubst). By Lemma 2 we have (NFVar). By Lemma 4 we have (InvVar). By Lemma 5 we have (InvApp). By Lemma 6 we have (InvAbs).

By Lemma 11 we deduce (ηSubjRed$_0$). By Lemma 12 we deduce (ηSubjRed). $\square$

$$\frac{}{\Gamma, x : A \vdash x : A}\ var \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ abs \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\ app$$

$$\frac{\Gamma \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B} \leq$$

Table 4: Typing rules with subtyping

2.1 Additional results

Statement (Co non-free variables [coNFVar])
If $\Gamma \vdash t : A$ *with* x *not declared in* Γ *then* $x \notin t$ *and* $\Gamma, x : B \vdash t : A$.

Lemma 13 (Co non-free variables)
Monot $\Longleftrightarrow$ coNFVar.

Proof: First direction: $x \notin t$ by definition of typing judgments since x is not declared in Γ. (Monot) gives $\Gamma, x : B \vdash t : A$.

Second direction: by induction on the context $\Delta \backslash \Gamma$, noting that all its elements correspond to declarations of variables not free in t. □

Statement (Co variable inversion [coInvVar])
If $x : A \in \Gamma$ *then* $\Gamma \vdash x : A$.

Lemma 14 (Co variable inversion)
(*var*) $\Longleftrightarrow$ coInvVar.

Proof: Immediate. □

Statement (Co application inversion [coInvApp])
If $\Gamma \vdash t : B \to A$ *and* $\Gamma \vdash u : B$ *then* $\Gamma \vdash t\,u : A$.

Lemma 15 (Co application inversion)
(*app*) $\Longleftrightarrow$ coInvApp.

Proof: Immediate. □

Statement (Co abstraction inversion [coInvAbs])
If $A = B \to C$ *and* $\Gamma, x : B \vdash t : C$ *then* $\Gamma \vdash \lambda x.t : A$.

Lemma 16 (Co abstraction inversion)
(*abs*) $\Longleftrightarrow$ coInvAbs.

Proof: Immediate. □

3 The simply typed λ-calculus with subtyping

3.1 General case

The system $\mathsf{ST}_{\leq}$ is obtained from ST by replacing the typing rules of Table 3 by those from Table 4 where the relation $\leq$ between types in any relation satisfying the rules of Table 5 (thus any reflexive and transitive relation).

$$\frac{}{A \leq A}\ refl \qquad \frac{A \leq B \quad B \leq C}{A \leq C}\ trans$$

Table 5: Minimal subtyping rules

Statement (Monotonicity with $\leq$ [MONOT$_\leq$])
If $\Gamma \vdash t : A$, $\Delta \leq \Gamma$ and $A \leq B$ then $\Delta \vdash t : B$ (where $\Delta \leq \Gamma$ means that for each typing declaration $x : C$ in Γ there is a declaration $x : D$ with $D \leq C$ in Δ).

Lemma 17 (Monotonicity for $\mathsf{ST}_\leq$)
MONOT$_\leq$ *holds for* $\mathsf{ST}_\leq$.

PROOF: We first prove the case $A = B$ by induction on the derivation of $\Gamma \vdash t : A$. We consider each possible last rule from Table 4:

- ***(var)*** If $t = x$, let A' be the type of x in Δ, we have $A' \leq A$ and:

$$\frac{\dfrac{}{\Delta \vdash x : A'}\ var \qquad A' \leq A}{\Delta \vdash x : A} \leq$$

- ***(abs)*** If $t = \lambda x.t'$ with $A = A' \to A''$ and $\Gamma, x : A' \vdash t' : A''$, by induction hypothesis, we have $\Delta, x : A' \vdash t' : A''$ thus $\Delta \vdash \lambda x.t' : A$.
- ***(app)*** If $t = t'\,t''$ with $\Gamma \vdash t' : A' \to A$ and $\Gamma \vdash t'' : A'$, by induction hypothesis, we have $\Delta \vdash t' : A' \to A$ and $\Delta \vdash t'' : A'$. So that $\Delta \vdash t'\,t'' : A$.
- ($\leq$) If $A' \leq A$ with $\Gamma \vdash t : A'$ then, by induction hypothesis, we have $\Delta \vdash t : A'$ thus $\Delta \vdash t : A$.

We conclude with:

$$\frac{\Delta \vdash t : A \qquad A \leq B}{\Delta \vdash t : B} \leq$$

□

Lemma 18 (Non-free variables for $\mathsf{ST}_\leq$)
NFVAR *holds for* $\mathsf{ST}_\leq$.

PROOF: By induction on the derivation of $\Gamma, x : B \vdash t : A$. By using the proof of Lemma 2, we only need to consider ($\leq$) as last rule:

- ($\leq$) If $A' \leq A$ with $\Gamma, x : B \vdash t : A'$ then, by induction hypothesis, $\Gamma \vdash t : A'$ thus $\Gamma \vdash t : A$. □

Lemma 19 (General substitution for $\mathsf{ST}_\leq$)
GSUBST *holds for* $\mathsf{ST}_\leq$.

PROOF: By following the proof of Lemma 3, it is enough to consider the case of $\Gamma \vdash t\{{}^{v}/_{x}\} : A$ obtained with a ($\leq$) rule:

- ($\leq$) If $A' \leq A$ with $\Gamma \vdash t\{{}^{v}/_{x}\} : A'$ then, by induction hypothesis, $\Gamma \vdash t\{{}^{u}/_{x}\} : A'$ thus $\Gamma \vdash t\{{}^{u}/_{x}\} : A$. □

Statement (Variable inversion with $\leq$ [$\textsc{InvVar}_{\leq}$])
If $\Gamma \vdash x : A$ *then there exists* B *such that* $B \leq A$ *and* $x : B \in \Gamma$*.*

Lemma 20 (Variable inversion for $\mathsf{ST}_{\leq}$)
$\textsc{InvVar}_{\leq}$ *holds for* $\mathsf{ST}_{\leq}$*.*

PROOF: By induction on the derivation of $\Gamma \vdash x : A$. The only possible last rules are (*var*) and ($\leq$):

- **(*var*)** We have $x : A \in \Gamma$ with $A \leq A$.
- **($\leq$)** If $\Gamma \vdash x : A'$ with $A' \leq A$ then, by induction hypothesis, we have $x : B \in \Gamma$ with $B \leq A'$ thus $B \leq A$. □

Statement (Application inversion with $\leq$ [$\textsc{InvApp}_{\leq}$])
If $\Gamma \vdash t\,u : A$*, there exist* B *and* C *such that* $B \leq A$*,* $\Gamma \vdash t : C \to B$ *and* $\Gamma \vdash u : C$*.*

Lemma 21 (Application inversion for $\mathsf{ST}_{\leq}$)
$\textsc{InvApp}_{\leq}$ *holds for* $\mathsf{ST}_{\leq}$*.*

PROOF: By induction on the derivation of $\Gamma \vdash t\,u : A$. The only possible last rules are (*app*) and ($\leq$):

- **(*app*)** There exists a type C such that $\Gamma \vdash t : C \to A$ and $\Gamma \vdash u : C$ and we have $A \leq A$.
- **($\leq$)** If $\Gamma \vdash t\,u : A'$ with $A' \leq A$ then, by induction hypothesis, there exist B and C such that $B \leq A'$ (thus $B \leq A$), $\Gamma \vdash t : C \to B$ and $\Gamma \vdash u : C$. □

Statement (Abstraction inversion with $\leq$ [$\textsc{InvAbs}_{\leq}$])
If $\Gamma \vdash \lambda x.t : A$*, there exist* B *and* C *such that* $B \to C \leq A$ *and* $\Gamma, x : B \vdash t : C$*.*

Lemma 22 (Abstraction inversion for $\mathsf{ST}_{\leq}$)
$\textsc{InvAbs}_{\leq}$ *holds for* $\mathsf{ST}_{\leq}$*.*

PROOF: By induction on the derivation of $\Gamma \vdash \lambda x.t : A$. The only possible last rules are (*abs*) and ($\leq$):

- **(*abs*)** There exist B and C such that $A = B \to C$ (thus $B \to C \leq A$) and $\Gamma, x : B \vdash t : C$.
- **($\leq$)** If $\Gamma \vdash \lambda x.t : A'$ with $A' \leq A$ then, by induction hypothesis, there exist B and C such that $B \to C \leq A'$ (thus $B \to C \leq A$) and $\Gamma, x : B \vdash t : C$. □

Statement (Implication inversion with $\leq$ [$\textsc{Imp}_{\leq}$])
If $A \to B \leq C \to D$ *then* $C \leq A$ *and* $B \leq D$*.*

Statement (Implicative abstraction inversion with $\leq$ [$\textsc{InvAbsImp}_{\leq}$])
If $\Gamma \vdash \lambda x.t : A \to B$*, there exist* A' *and* B' *such that* $A \leq A'$*,* $B' \leq B$ *and* $\Gamma, x : A' \vdash t : B'$*.*

Lemma 23 (Implicative abstraction inversion with $\leq$)
$\textsc{InvAbs}_{\leq} \wedge \textsc{Imp}_{\leq} \Longrightarrow \textsc{InvAbsImp}_{\leq}$*.*

PROOF: If $\Gamma \vdash \lambda x.t : A \to B$, by ($\textsc{InvAbs}_{\leq}$), there exist A' and B' such that $A' \to B' \leq A \to B$ and $\Gamma, x : A' \vdash t : B'$. By ($\textsc{Imp}_{\leq}$), we have $A \leq A'$ and $B' \leq B$. □

Lemma 24 (Substitution)
$\textsc{GSubst} \wedge \textsc{InvVar}_{\leq} \wedge \textsc{Monot}_{\leq} \wedge \textsc{NFVar} \Longrightarrow \textsc{Subst}$.

PROOF: Note first that x is not declared in Γ (otherwise $\Gamma, x : A$ is not a valid context) and thus x is not free in u.

Up to α-conversion in t, we can assume that x is not bound in t and that no free variable of u is bound in t. As a consequence $t[^u/_x] = t\{^u/_x\}$.

We have $\Gamma, x : A \vdash t\{^x/_x\} : B$. If $\Gamma, x : A, \Delta \vdash x : C$ then $A \leq C$ by ($\text{InvVar}_{\leq}$), and by ($\text{Monot}_{\leq}$) we obtain $\Gamma, x : A, \Delta \vdash u : C$. It is thus possible to apply (GSubst) to deduce $\Gamma, x : A \vdash t\{^u/_x\} : B$. Finally, since $t\{^u/_x\} = t[^u/_x]$ and x is not free in $t[^u/_x]$, we can apply (NFVar) to conclude $\Gamma \vdash t[^u/_x] : B$. □

Lemma 25 (Subject reduction for β_0 with $\leq$)
$\text{Subst} \wedge \text{InvApp}_{\leq} \wedge \text{InvAbsImp}_{\leq} \wedge (\leq) \Longrightarrow \beta\text{SubjRed}_0$.

PROOF: If $\Gamma \vdash (\lambda x.t)\, u : A$, by ($\text{InvApp}_{\leq}$), there exist B and C such that $B \leq A$, $\Gamma \vdash \lambda x.t : C \to B$ and $\Gamma \vdash u : C$. By ($\text{InvAbsImp}_{\leq}$), there exist B' and C' such that $C \leq C'$, $B' \leq B$ and $\Gamma, x : C' \vdash t : B'$. By $(\leq)$ we have $\Gamma \vdash u : C'$ and by (Subst), $\Gamma \vdash t[^u/_x] : B'$ thus:

$$\dfrac{\dfrac{\Gamma \vdash t[^u/_x] : B' \qquad B' \leq B}{\Gamma \vdash t[^u/_x] : B}\leq \qquad B \leq A}{\Gamma \vdash t[^u/_x] : A}\leq$$

□

3.1.1 Additional results

Lemma 26 (Subtyping and typing inclusion)
If $\text{InvVar}_{\leq} \wedge (var) \wedge (\leq)$ *then:*

$$((\forall t \forall \Gamma,\ \Gamma \vdash t : A \Rightarrow \Gamma \vdash t : B) \iff A \leq B)$$

PROOF: The first implication is obtained by applying ($\text{InvVar}_{\leq}$) to $x : A \vdash x : B$ (obtained from $x : A \vdash x : A$ by (var)).

The second implication is $(\leq)$. □

Lemma 27 (Co non-free variables with $\leq$)
$\text{Monot}_{\leq} \Longrightarrow \text{coNFVar}$.

PROOF: $x \notin t$ by definition of typing judgments since x is not declared in Γ. ($\text{Monot}_{\leq}$) gives $\Gamma, x : B \vdash t : A$. □

Statement (Co variable inversion with $\leq$ [$\text{coInvVar}_{\leq}$])
If $B \leq A$ *and* $x : B \in \Gamma$ *then* $\Gamma \vdash x : A$.

Lemma 28 (Co variable inversion with $\leq$)
$(var) \wedge (\leq) \Longrightarrow \text{coInvVar}_{\leq}$.

PROOF: Assume $x : B \in \Gamma$:

$$\dfrac{\dfrac{}{\Gamma \vdash x : B}var \qquad B \leq A}{\Gamma \vdash x : A}\leq$$

□

$$\frac{}{A \leq A}\ \mathit{refl} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C}\ \mathit{trans} \qquad \frac{C \leq A \qquad B \leq D}{A \to B \leq C \to D} \to$$

Table 6: Subtyping rules for $\mathsf{ST}^{\to}_{\leq}$

Statement (Co application inversion with $\leq$ [$\textsc{coInvApp}_{\leq}$])
If $\Gamma \vdash t : C \to B$, $\Gamma \vdash u : C$ and $B \leq A$ then $\Gamma \vdash t\, u : A$.

Lemma 29 (Co application inversion with $\leq$)
$(\mathit{app}) \wedge (\leq) \Longrightarrow \textsc{coInvApp}_{\leq}$.

Proof:

$$\frac{\dfrac{\Gamma \vdash t : C \to B \qquad \Gamma \vdash u : C}{\Gamma \vdash t\, u : B}\ \mathit{app} \qquad B \leq A}{\Gamma \vdash t\, u : A} \leq$$

□

Statement (Co abstraction inversion with $\leq$ [$\textsc{coInvAbs}_{\leq}$])
If $B \to C \leq A$ and $\Gamma, x : B \vdash t : C$ then $\Gamma \vdash \lambda x.t : A$.

Lemma 30 (Co abstraction inversion with $\leq$)
$(\mathit{abs}) \wedge (\leq) \Longrightarrow \textsc{coInvAbs}_{\leq}$.

Proof:

$$\frac{\dfrac{\Gamma, x : B \vdash t : C}{\Gamma \vdash \lambda x.t : B \to C}\ \mathit{abs} \qquad B \to C \leq A}{\Gamma \vdash \lambda x.t : A} \leq$$

□

Statement (Co implicative abstraction inversion with $\leq$ [$\textsc{coInvAbsImp}_{\leq}$])
If $A \leq A'$, $B' \leq B$ and $\Gamma, x : A' \vdash t : B'$ then $\Gamma \vdash \lambda x.t : A \to B$.

Lemma 31 (Co implicative abstraction inversion with $\leq$)
$\textsc{Monot}_{\leq} \wedge (\mathit{abs}) \Longrightarrow \textsc{coInvAbsImp}_{\leq}$.

Proof: By ($\textsc{Monot}_{\leq}$) we have $\Gamma, x : A \vdash t : B$ and then:

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ \mathit{abs}$$

□

3.2 Covariant contravariant implication

The system $\mathsf{ST}^{\to}_{\leq}$ is the particular case of $\mathsf{ST}_{\leq}$ where the relation $\leq$ is defined *exactly* by the rules of Table 6.

Statement (Admissibility of the (*trans*) rule [$\textsc{TransElim}$])
If $A \leq B$ is derivable then $A \leq B$ is derivable without the (trans) rule.

Lemma 32 (Transitivity elimination for $\mathsf{ST}_{\leq}^{\rightarrow}$)
TRANSELIM *holds for* $\mathsf{ST}_{\leq}^{\rightarrow}$.

PROOF: Let the size $|d|$ of a derivation d be its number of rules. We first prove by induction on the sum $|d_1| + |d_2|$ that if d_1 is a (*trans*)-free derivation of $A \leq B$ and d_2 is a (*trans*)-free derivation of $B \leq C$, then there exists a (*trans*)-free derivation of $A \leq C$. We look at each possible last rule for d_2:

- ***(refl)*** We have $B = C$ and d_1 is a (*trans*)-free derivation of $A \leq C$.
- **($\rightarrow$)** If $B = B' \rightarrow B''$ and $C = C' \rightarrow C''$, we have (*trans*)-free derivations d_2' of $C' \leq B'$ and d_2'' of $B'' \leq C''$. We consider each possible last rule for d_1:
  - ***(refl)*** We have $A = B$ and d_2 is a (*trans*)-free derivation of $A \leq C$.
  - **($\rightarrow$)** If $A = A' \rightarrow A''$, we have (*trans*)-free derivations d_1' of $B' \leq A'$ and d_1'' of $A'' \leq B''$. By induction hypothesis applied to the derivations d_2' and d_1', and d_1'' and d_2'', we obtain (*trans*)-free derivations of $C' \leq A'$ and $A'' \leq C''$ and we conclude with ($\rightarrow$) that $A \leq C$.

We now prove (TRANSELIM) by induction on the derivation of $A \leq B$. We consider each possible last rule from Table 6:

- ***(refl)*** The derivation is directly without (*trans*).
- **($\rightarrow$)** If $A = A' \rightarrow B'$ and $B = C' \rightarrow D'$ then, by induction hypothesis, we have derivations of $C' \leq A'$ and $B' \leq D'$ without the (*trans*) rule. We thus have:

$$\frac{C' \leq A' \qquad B' \leq D'}{A' \rightarrow B' \leq C' \rightarrow D'} \rightarrow$$

  without the (*trans*) rule.
- ***(trans)*** If $A \leq C$ and $C \leq B$, by induction hypothesis, we have (*trans*)-free derivations of $A \leq C$ and of $C \leq B$. We apply the preliminary result to obtain a (*trans*)-free derivation of $A \leq B$. □

Lemma 33 (Transitivity-free implication inversion for $\mathsf{ST}_{\leq}^{\rightarrow}$)
In $\mathsf{ST}_{\leq}^{\rightarrow}$, TRANSELIM $\Longrightarrow$ IMP$_{\leq}$.

PROOF: By induction on the derivation of $A \rightarrow B \leq C \rightarrow D$. We consider each possible last rule from Table 6 except (*trans*) (thanks to (TRANSELIM)):

- ***(refl)*** $A = C$ and $B = D$ thus $C \leq A$ and $B \leq D$.
- **($\rightarrow$)** We immediately have $C \leq A$ and $B \leq D$. □

Theorem 3 (Subject reduction for $\mathsf{ST}_{\leq}^{\rightarrow}$)
βSUBJRED *holds for* $\mathsf{ST}_{\leq}^{\rightarrow}$.

PROOF: By Lemma 19 we have (GSUBST). By Lemma 17 we have (MONOT$_{\leq}$). By Lemma 18 we have (NFVAR). By Lemma 20 we have (INVVAR$_{\leq}$). By Lemma 21 we have (INVAPP$_{\leq}$). By Lemma 22 we have (INVABS$_{\leq}$). By Lemma 32 we have (TRANSELIM). By Lemma 33 we deduce (IMP$_{\leq}$).

By Lemma 24 we deduce (SUBST). By Lemma 23 we deduce (INVABSIMP$_{\leq}$). By Lemma 25 we deduce (βSUBJRED$_0$). By Lemma 10 we deduce (βSUBJRED). □

Lemma 34 (Subject reduction for η_0 with $\leq$)
NFVAR $\wedge$ INVVAR$_{\leq}$ $\wedge$ INVAPP$_{\leq}$ $\wedge$ INVABS$_{\leq}$ $\wedge$ ($\rightarrow$) $\Longrightarrow$ ηSUBJRED$_0$.

PROOF: If $\Gamma \vdash \lambda x.(t\,x) : A$, by ($\textsc{InvAbs}_{\leq}$), there exist B and C such that $B \to C \leq A$ and $\Gamma, x : B \vdash t\,x : C$. By ($\textsc{InvApp}_{\leq}$), there exist D and E such that $E \leq C$, $\Gamma, x : B \vdash t : D \to E$ and $\Gamma, x : B \vdash x : D$. By ($\textsc{InvVar}_{\leq}$), we have $B \leq D$. By (NF$\textsc{Var}$), we obtain $\Gamma \vdash t : D \to E$ since $x \notin t$. We conclude with:

$$\dfrac{\Gamma \vdash t : D \to E \qquad \dfrac{B \leq D \qquad E \leq C}{D \to E \leq B \to C}\,\to}{\Gamma \vdash t : B \to C}\,\leq$$

□

Theorem 4 (Subject reduction for η for $\mathsf{ST}^{\to}_{\leq}$)
$\eta\textsc{SubjRed}$ holds for $\mathsf{ST}^{\to}_{\leq}$.

PROOF: By Lemma 19 we have (G$\textsc{Subst}$). By Lemma 18 we have (NF$\textsc{Var}$). By Lemma 20 we have ($\textsc{InvVar}_{\leq}$). By Lemma 21 we have ($\textsc{InvApp}_{\leq}$). By Lemma 22 we have ($\textsc{InvAbs}_{\leq}$). ($\to$) holds in $\mathsf{ST}^{\to}_{\leq}$ (Table 6).

By Lemma 34 we deduce ($\eta\textsc{SubjRed}_0$). By Lemma 12 we deduce ($\eta\textsc{SubjRed}$). □

3.2.1 Additional results

Statement (Co implication inversion with $\leq$ [$\textsc{coImp}_{\leq}$])
If $C \leq A$ and $B \leq D$ then $A \to B \leq C \to D$.

Lemma 35 (Co implication inversion with $\leq$)
$(\to) \iff \textsc{coImp}_{\leq}$.

PROOF: Immediate. □

4 The intersection typed λ-calculus with subtyping

Types are now built from base types and the type constant Ω by means of the binary operations $\to$ and $\cap$:

$$A \quad ::= \quad X \mid A \to A \mid \Omega \mid A \cap A$$

In order to enhance readability, we use the notation $\bigcap_{i \in I} A_i$ for a type obtained *in some way* by applying $\cap$ connectives to the types in $(A_i)_{i \in I}$. If $I = \emptyset$, such an empty intersection is a notation for Ω. If I is a singleton $\{i\}$ then it is simply a notation for A_i

4.1 General case

The system $\mathsf{IT}_{\leq}$ is obtained from the typing rules of Table 7 with any relation $\leq$ between types satisfying the rules of Table 8.

Lemma 36 (Monotonicity for $\mathsf{IT}_{\leq}$)
$\textsc{Monot}_{\leq}$ holds for $\mathsf{IT}_{\leq}$.

PROOF: By induction on the derivation of $\Gamma \vdash t : A$. By using the proof of Lemma 17, it is enough to consider the case $A = B$ and ($\cap$) and (Ω) as last rules:

($\cap$) If $A = A' \cap A''$ with $\Gamma \vdash t : A'$ and $\Gamma \vdash t : A''$, by induction hypothesis, we have $\Delta \vdash t : A'$ and $\Delta \vdash t : A''$ thus $\Delta \vdash t : A$.

$$\frac{}{\Gamma, x : A \vdash x : A}\ var \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ abs \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\ app$$

$$\frac{\Gamma \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B} \leq \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B} \cap \qquad \frac{}{\Gamma \vdash t : \Omega}\ \Omega$$

Table 7: Typing rules with subtyping and intersection

$$\frac{}{A \leq A}\ \mathit{refl} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C}\ \mathit{trans}$$

$$\frac{A \leq C}{A \cap B \leq C} \cap_l^1 \qquad \frac{B \leq C}{A \cap B \leq C} \cap_l^2 \qquad \frac{C \leq A \qquad C \leq B}{C \leq A \cap B} \cap_r \qquad \frac{}{C \leq \Omega}\ \Omega_r$$

Table 8: Minimal subtyping rules with intersection

(Ω) If $A = \Omega$ then $\Delta \vdash t : \Omega$. □

Lemma 37 (Non-free variables for $\mathsf{IT}_{\leq}$)
NFVar *holds for* $\mathsf{IT}_{\leq}$.

Proof: By induction on the derivation of $\Gamma, x : B \vdash t : A$. By using the proof of Lemma 18, we only need to consider ($\cap$) and (Ω) as last rules:

- ($\cap$) If $A = A' \cap A''$ with $\Gamma, x : B \vdash t : A'$ and $\Gamma, x : B \vdash t : A''$ then, by induction hypothesis, $\Gamma \vdash t : A'$ and $\Gamma \vdash t : A''$ thus $\Gamma \vdash t : A$.
- (Ω) We have $\Gamma \vdash t : \Omega$. □

Lemma 38 (General substitution for $\mathsf{IT}_{\leq}$)
GSubst *holds for* $\mathsf{IT}_{\leq}$.

Proof: By following the proof of Lemma 19, it is enough to consider the case of $\Gamma \vdash t\{^v/_x\} : A$ obtained with a ($\cap$) or a (Ω) rule:

- ($\cap$) If $A = A' \cap A''$ with $\Gamma \vdash t\{^v/_x\} : A'$ and $\Gamma \vdash t\{^v/_x\} : A''$ then, by induction hypothesis, $\Gamma \vdash t\{^u/_x\} : A'$ and $\Gamma \vdash t\{^u/_x\} : A''$ thus $\Gamma \vdash t\{^u/_x\} : A$.
- (Ω) We have $\Gamma \vdash t\{^u/_x\} : \Omega$. □

Lemma 39 (Variable inversion for $\mathsf{IT}_{\leq}$)
InvVar$_{\leq}$ *holds for* $\mathsf{IT}_{\leq}$.

Proof: By induction on the derivation of $\Gamma \vdash x : A$. By using the proof of Lemma 20, we only need to consider ($\cap$) and (Ω) as last rules:

- ($\cap$) If $A = A' \cap A''$ with $\Gamma \vdash x : A'$ and $\Gamma \vdash x : A''$ then, by induction hypothesis, we have $x : B \in \Gamma$ with $B \leq A'$ and $B \leq A''$ and:

$$\frac{B \leq A' \qquad B \leq A''}{B \leq A} \cap_r$$

- (Ω) If $A = \Omega$ then x must be declared with some type B in Γ and we have:

$$\frac{}{B \leq \Omega}\ \Omega_r$$

□

Statement (Application inversion with ∩ [InvApp$_\cap$])
If $\Gamma \vdash t\,u : A$, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq A$ and for all $i \in I$, $\Gamma \vdash t : C_i \to B_i$ and $\Gamma \vdash u : C_i$.

Lemma 40 (Application inversion for $\mathsf{IT}_\leq$)
InvApp$_\cap$ *holds for* $\mathsf{IT}_\leq$*.*

Proof: By induction on the derivation of $\Gamma \vdash t\,u : A$. We look at the possible last rules:

- **(*app*)** There exists a type C_1 such that $\Gamma \vdash t : C_1 \to A$ and $\Gamma \vdash u : C_1$ and we have $I = \{1\}$ and $B_1 = A \leq A$.
- ($\leq$) If $\Gamma \vdash t\,u : A'$ with $A' \leq A$ then, by induction hypothesis, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq A'$ and for all $i \in I$, $\Gamma \vdash t : C_i \to B_i$ and $\Gamma \vdash u : C_i$. We then deduce:

$$\frac{\bigcap_{i\in I} B_i \leq A' \qquad A' \leq A}{\bigcap_{i\in I} B_i \leq A}\ \mathit{trans}$$

- ($\cap$) If $\Gamma \vdash t\,u : A'$ and $\Gamma \vdash t\,u : A''$ with $A = A' \cap A''$ then, by induction hypothesis, there exist a set I' and a set I'' (we can assume I' and I'' to be disjoint) and families $(B_i)_{i\in I'}$, $(C_i)_{i\in I'}$, $(B_i)_{i\in I''}$ and $(C_i)_{i\in I''}$ such that $\bigcap_{i\in I'} B_i \leq A'$, $\bigcap_{i\in I''} B_i \leq A''$, for all $i \in I' \cup I''$, $\Gamma \vdash t : C_i \to B_i$ and $\Gamma \vdash u : C_i$. We then define $I = I' \cup I''$ and we have:

$$\frac{\dfrac{\bigcap_{i\in I'} B_i \leq A'}{\bigcap_{i\in I} B_i \leq A'}\ \cap_l^1 \qquad \dfrac{\bigcap_{i\in I''} B_i \leq A''}{\bigcap_{i\in I} B_i \leq A''}\ \cap_l^2}{\bigcap_{i\in I} B_i \leq A}\ \cap_r$$

- (Ω) If $A = \Omega$, we choose $I = \emptyset$ and we have $\Omega \leq A$. □

Statement (Abstraction inversion with ∩ [InvAbs$_\cap$])
If $\Gamma \vdash \lambda x.t : A$, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \to C_i \leq A$ and for all $i \in I$, $\Gamma, x : B_i \vdash t : C_i$.

Lemma 41 (Abstraction inversion for $\mathsf{IT}_\leq$)
InvAbs$_\cap$ *holds for* $\mathsf{IT}_\leq$*.*

Proof: By induction on the derivation of $\Gamma \vdash \lambda x.t : A$. We look at the possible last rules:

- **(*abs*)** There exist B_1 and C_1 such that $A = B_1 \to C_1$ (thus $B_1 \to C_1 \leq A$) and $\Gamma, x : B_1 \vdash t : C_1$. We choose $I = \{1\}$.
- ($\leq$) If $\Gamma \vdash \lambda x.t : A'$ with $A' \leq A$ then, by induction hypothesis, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \to C_i \leq A'$ and for all $i \in I$, $\Gamma, x : B_i \vdash t : C_i$. We then deduce:

$$\frac{\bigcap_{i\in I} B_i \to C_i \leq A' \qquad A' \leq A}{\bigcap_{i\in I} B_i \to C_i \leq A}\ \mathit{trans}$$

($\cap$) If $\Gamma \vdash \lambda x.t : A'$ and $\Gamma \vdash \lambda x.t : A''$ with $A = A' \cap A''$ then, by induction hypothesis, there exist a set I' and a set I'' (we can assume I' and I'' to be disjoint) and families $(B_i)_{i\in I'}$, $(C_i)_{i\in I'}$, $(B_i)_{i\in I''}$ and $(C_i)_{i\in I''}$ such that $\bigcap_{i\in I'} B_i \to C_i \leq A'$, $\bigcap_{i\in I''} B_i \to C_i \leq A''$, for all $i \in I' \cup I''$, $\Gamma, x : B_i \vdash t : C_i$. We then define $I = I' \cup I''$ and we have:

$$\dfrac{\dfrac{\bigcap_{i\in I'} B_i \to C_i \leq A'}{\bigcap_{i\in I} B_i \to C_i \leq A'}\cap_l^1 \quad \dfrac{\bigcap_{i\in I''} B_i \to C_i \leq A''}{\bigcap_{i\in I} B_i \to C_i \leq A''}\cap_l^2}{\bigcap_{i\in I} B_i \to C_i \leq A}\cap_r$$

(Ω) If $A = \Omega$, we choose $I = \emptyset$ and we have $\Omega \leq A$. □

Statement (Implication inversion with $\cap$ [Imp$_\cap$])
If $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B$ then there exists $J \subseteq I$ such that for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq B$.

Statement (Implicative abstraction inversion with $\cap$ [InvAbsImp$_\cap$])
If $\Gamma \vdash \lambda x.t : A \to B$, there exist a set I and two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq B$ and for all $i \in I$, $A \leq A_i$ and $\Gamma, x : A_i \vdash t : B_i$.

Lemma 42 (Implicative abstraction inversion with $\cap$)
InvAbs$_\cap$ $\wedge$ Imp$_\cap$ $\Longrightarrow$ InvAbsImp$_\cap$.

Proof: If $\Gamma \vdash \lambda x.t : A \to B$, by (InvAbs$_\cap$), there exist a set I and two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ such that $\bigcap_{i\in I} A_i \to B_i \leq A \to B$ and for all $i \in I$, $\Gamma, x : A_i \vdash t : B_i$. By (Imp$_\cap$), we have $J \subseteq I$ such that for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq B$. □

Lemma 43 (Subject reduction for β_0 with $\cap$)
Subst $\wedge$ InvApp$_\cap$ $\wedge$ InvAbsImp$_\cap$ $\wedge$ $(\leq) \wedge (\cap) \wedge (\Omega) \Longrightarrow \beta$SubjRed$_0$.

Proof: If $\Gamma \vdash (\lambda x.t)\, u : A$, by (InvApp$_\cap$), there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq A$ and for all $i \in I$, $\Gamma \vdash \lambda x.t : C_i \to B_i$ and $\Gamma \vdash u : C_i$.

For each $i \in I$, by (InvAbsImp$_\cap$), there exist a set J_i and two families $(D_i^j)_{j\in J_i}$ and $(E_i^j)_{j\in J_i}$ such that $\bigcap_{j\in J_i} D_i^j \leq B_i$ and for all $j \in J_i$, $C_i \leq E_i^j$ and $\Gamma, x : E_i^j \vdash t : D_i^j$. By $(\leq)$ we have $\Gamma \vdash u : E_i^j$ thus, by (Subst), $\Gamma \vdash t[{}^u/_x] : D_i^j$.

Then we have:

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\dfrac{\cdots \quad \Gamma \vdash t[{}^u/_x] : D_i^j \quad \cdots}{\Gamma \vdash t[{}^u/_x] : \bigcap_{j\in J_i} D_i^j}\cap \qquad \bigcap_{j\in J_i} D_i^j \leq B_i}{\Gamma \vdash t[{}^u/_x] : B_i}\leq \qquad \cdots}{\Gamma \vdash t[{}^u/_x] : \bigcap_{i\in I} B_i}\cap \qquad \bigcap_{i\in I} B_i \leq A}{\Gamma \vdash t[{}^u/_x] : A}\leq$$

We use (Ω) instead of $(\cap)$ if $I = \emptyset$ or if $J_i = \emptyset$ for some $i \in I$. □

Statement (Co-substitution [coSubst])
If $\Gamma \vdash t[{}^u/_x] : B$ with $x \notin u$ and Γ contains declarations for the free variables of u then there exists a type A such that $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash u : A$.

Lemma 44 (Co-substitution for $\mathsf{IT}_\leq$)
coSubst *holds for* $\mathsf{IT}_\leq$.

PROOF: By induction on the derivation of $\Gamma \vdash t[^u/_x] : B$. If $t = x$ then $t[^u/_x] = u$ and we choose $A = B$. We have $\Gamma, x : B \vdash x : B$ and $\Gamma \vdash u : B$. Otherwise we look at the last rule of the derivation of $\Gamma \vdash t[^u/_x] : B$ from Table 7:

(*var*) If we have $t = y \neq x$ and $t[^u/_x] = y$. With $A = \Omega$, we get $\Gamma, x : \Omega \vdash y : B$ (since $y : B \in \Gamma$) and $\Gamma \vdash u : \Omega$.

(*abs*) We have $t = \lambda y.t'$, $t[^u/_x] = \lambda y.(t'[^u/_x])$ and $B = B' \to B''$ with $\Gamma, y : B' \vdash t'[^u/_x] : B''$. By induction hypothesis, there exists A such that $\Gamma, x : A, y : B' \vdash t' : B''$ and $\Gamma \vdash u : A$. We then have $\Gamma, x : A \vdash \lambda y.t' : B$ and we conclude.

(*app*) If $t = t'\,t''$ with $\Gamma \vdash t'[^u/_x] : B' \to B$ and $\Gamma \vdash t''[^u/_x] : B'$ then, by induction hypothesis, there exist A' and A'' such that $\Gamma, x : A' \vdash t' : B' \to B$, $\Gamma \vdash u : A'$, $\Gamma, x : A'' \vdash t'' : B'$ and $\Gamma \vdash u : A''$. By Lemma 36 and using:

$$\dfrac{\dfrac{}{A' \leq A'}\,\textit{refl}}{A' \cap A'' \leq A'}\,\cap_l^1 \qquad \text{and} \qquad \dfrac{\dfrac{}{A'' \leq A''}\,\textit{refl}}{A' \cap A'' \leq A''}\,\cap_l^2$$

we have $\Gamma, x : A' \cap A'' \vdash t' : B' \to B$ and $\Gamma, x : A' \cap A'' \vdash t'' : B'$ and we can derive:

$$\dfrac{\Gamma, x : A' \cap A'' \vdash t' : B' \to B \qquad \Gamma, x : A' \cap A'' \vdash t'' : B'}{\Gamma, x : A' \cap A'' \vdash t'\,t'' : B}\,\textit{app}$$

and

$$\dfrac{\Gamma \vdash u : A' \qquad \Gamma \vdash u : A''}{\Gamma \vdash u : A' \cap A''}\,\cap$$

so that we choose $A = A' \cap A''$.

($\leq$) If $B' \leq B$ with $\Gamma \vdash t[^u/_x] : B'$ then, by induction hypothesis, there exists A such that $\Gamma, x : A \vdash t : B'$ and $\Gamma \vdash u : A$. We can derive:

$$\dfrac{\Gamma, x : A \vdash t : B' \qquad B' \leq B}{\Gamma, x : A \vdash t : B}\,\leq$$

($\cap$) If $B = B' \cap B''$ with $\Gamma \vdash t[^u/_x] : B'$ and $\Gamma \vdash t[^u/_x] : B''$, by induction hypothesis, there exist A' and A'' such that $\Gamma, x : A' \vdash t : B'$, $\Gamma \vdash u : A'$, $\Gamma, x : A'' \vdash t : B''$ and $\Gamma \vdash u : A''$. By Lemma 36, we can build:

$$\dfrac{\Gamma, x : A' \cap A'' \vdash t : B' \qquad \Gamma, x : A' \cap A'' \vdash t : B''}{\Gamma, x : A' \cap A'' \vdash t : B}\,\cap$$

and

$$\dfrac{\Gamma \vdash u : A' \qquad \Gamma \vdash u : A''}{\Gamma \vdash u : A' \cap A''}\,\cap$$

so that we choose $A = A' \cap A''$.

(Ω) If $B = \Omega$, we choose $A = \Omega$ and we have:

$$\dfrac{}{\Gamma, x : \Omega \vdash t : \Omega}\,\Omega \qquad \text{and} \qquad \dfrac{}{\Gamma \vdash u : \Omega}\,\Omega$$

□

Statement (Subject expansion for β_0 [βSUBJEXP$_0$])
If $\Gamma \vdash t : A$ *with* Γ *containing declarations for the free variables of* u *and* $t \leftarrow_{\beta_0} u$ *then* $\Gamma \vdash u : A$.

Lemma 45 (Subject expansion for β_0)
COSUBST $\wedge$ (*abs*) $\wedge$ (*app*) $\Longrightarrow$ βSUBJEXP$_0$.

PROOF: We use (coSubst) and we build:

$$\dfrac{\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\,abs \qquad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.t)\, u : B}\,app$$

□

Statement (Subject expansion for β [βSubjExp])
If $\Gamma \vdash t : A$ with Γ containing declarations for the free variables of u and $t \leftarrow_\beta u$ then $\Gamma \vdash u : A$.

Lemma 46 (Subject expansion)
GSubst $\wedge$ βSubjExp$_0$ $\Longrightarrow$ βSubjExp.

PROOF: If $\Gamma \vdash t : A$ and $t \leftarrow_\beta u$ then $t = c\{^{t'}/_x\}$ and $u = c\{^{u'}/_x\}$ with $t' \leftarrow_{\beta_0} u'$. Assume that $\Gamma, \Delta \vdash t' : B$ then by (βSubjExp$_0$) we have $\Gamma, \Delta \vdash u' : B$, thus by (GSubst) we obtain $\Gamma \vdash u : A$. □

Theorem 5 (Subject expansion for $\mathsf{IT}_{\leq}$)
βSubjExp holds for $\mathsf{IT}_{\leq}$.

PROOF: By Lemma 44 we have (coSubst). By Lemma 38 we have (GSubst).

By Lemma 45 we deduce (βSubjExp$_0$). By Lemma 46 we deduce (βSubjExp). □

4.1.1 Additional results

Statement (Co application inversion with $\cap$ [coInvApp$_\cap$])
If for all $i \in I$, $\Gamma \vdash t : C_i \to B_i$ and $\Gamma \vdash u : C_i$, and $\bigcap_{i \in I} B_i \leq A$ then $\Gamma \vdash t\, u : A$.

Lemma 47 (Co application inversion with $\cap$)
$(app) \wedge (\leq) \wedge (\cap) \wedge (\Omega) \Longrightarrow$ coInvApp$_\cap$.

PROOF: If I is not empty, we have:

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\Gamma \vdash t : C_i \to B_i \qquad \Gamma \vdash u : C_i}{\Gamma \vdash t\, u : B_i}\,app \qquad \cdots}{\Gamma \vdash t\, u : \bigcap_{i \in I} B_i}\,\cap \qquad \bigcap_{i \in I} B_i \leq A}{\Gamma \vdash t\, u : A}\,\leq$$

Otherwise, we use:

$$\dfrac{\dfrac{}{\Gamma \vdash t\, u : \Omega}\,\Omega \qquad \Omega \leq A}{\Gamma \vdash t\, u : A}\,\leq$$

□

Statement (Co abstraction inversion with $\cap$ [coInvAbs$_\cap$])
If for all $i \in I$, $\Gamma, x : B_i \vdash t : C_i$ and $\bigcap_{i \in I} B_i \to C_i \leq A$ then $\Gamma \vdash \lambda x.t : A$.

Lemma 48 (Co abstraction inversion with $\cap$)
$(abs) \wedge (\leq) \wedge (\cap) \wedge (\Omega) \Longrightarrow$ coInvAbs$_\cap$.

PROOF: If I is not empty, we have:

$$\frac{}{A \leq A} \qquad \frac{A \leq B \quad B \leq C}{A \leq C} \qquad \frac{}{A \leq \Omega}$$

$$\frac{}{A \cap B \leq A} \qquad \frac{}{A \cap B \leq B} \qquad \frac{}{A \leq A \cap A} \qquad \frac{A \leq C \quad B \leq D}{A \cap B \leq C \cap D}$$

$$\frac{C \leq A \quad B \leq D}{A \to B \leq C \to D} \qquad \frac{}{(A \to B) \cap (A \to C) \leq A \to (B \cap C)} \qquad \frac{}{\Omega \leq \Omega \to \Omega}$$

Table 9: BCD subtyping rules

$$\cfrac{\cfrac{\cdots \quad \cfrac{\Gamma, x : B_i \vdash t : C_i}{\Gamma \vdash \lambda x.t : B_i \to C_i}\,abs \quad \cdots}{\Gamma \vdash \lambda x.t : \bigcap_{i \in I} B_i \to C_i}\,\cap \qquad \bigcap_{i \in I} B_i \to C_i \leq A}{\Gamma \vdash \lambda x.t : A}\,\leq$$

Otherwise, we use:

$$\cfrac{\cfrac{}{\Gamma \vdash \lambda x.t : \Omega}\,\Omega \qquad \Omega \leq A}{\Gamma \vdash \lambda x.t : A}\,\leq$$

□

Statement (Co implicative abstraction inversion with $\cap$ [coInvAbsImp$_\cap$])
If for all $i \in I$, $A \leq A_i$ *and* $\Gamma, x : A_i \vdash t : B_i$, *and* $\bigcap_{i \in I} B_i \leq B$ *then* $\Gamma \vdash \lambda x.t : A \to B$.

Lemma 49 (Co implicative abstraction inversion with $\cap$)
Monot$_\leq$ $\wedge$ (*abs*) $\wedge$ ($\cap$) $\wedge$ (Ω) $\Longrightarrow$ coInvAbsImp$_\cap$.

Proof: For all $i \in I$, by (Monot$_\leq$) we have $\Gamma, x : A \vdash t : B_i$ and then:

$$\cfrac{\cdots \quad \Gamma, x : A \vdash t : B_i \quad \cdots}{\Gamma, x : A \vdash t : \bigcap_{i \in I} B_i}\,\cap$$

If I is empty then:

$$\cfrac{}{\Gamma, x : A \vdash t : \Omega}\,\Omega$$

By (Monot$_\leq$) we deduce $\Gamma, x : A \vdash t : B$ and we conclude with (*abs*). □

4.2 BCD case

The original BCD type system is based on the subtyping rules of Table 9. For this presentation, the transitivity rule cannot be removed: $X \cap Y \leq Y \cap X$ is not provable without transitivity if $X \neq Y$ (if one tries to find a possible last rule, one would need to prove $X \leq Y$), while we have:

$$\cfrac{\cfrac{}{X \cap Y \leq (X \cap Y) \cap (X \cap Y)} \qquad \cfrac{\cfrac{}{X \cap Y \leq Y} \quad \cfrac{}{X \cap Y \leq X}}{(X \cap Y) \cap (X \cap Y) \leq Y \cap X}}{X \cap Y \leq Y \cap X}$$

$$\frac{B \leq A}{A \to C \leq B \to C} \to_l \qquad \frac{C \leq A \to D \qquad D \leq B}{C \leq A \to B} \to_r$$

$$\frac{D \leq C \to A \qquad D \leq C \to B}{D \leq C \to (A \cap B)} \to\cap \qquad \frac{}{B \leq A \to \Omega} \to\Omega$$

Table 10: BCD-like subtyping rules

The system $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$ is the particular case of $\mathsf{IT}_{\leq}$ where the relation $\leq$ is defined *exactly* by the rules of Tables 8 and 10.

Proposition 1 (Equivalence of presentations of BCD)
The subtyping relation generated by the rules of Tables 8 and 10 is the same as the relation generated by the rules of Table 9.

Lemma 50 (Transitivity elimination for $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$)
TRANSELIM *holds for* $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$.

PROOF: Similar to the proof of Lemma 32. □

Lemma 51 (Transitivity-free implication inversion for $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$)
In $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$, TRANSELIM $\Longrightarrow$ IMP$_\cap$.

PROOF: By induction on the derivation of $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B$. We consider each possible last rule from Tables 8 and 10 except (*trans*) (thanks to (TRANSELIM)):

- **(*refl*)** $I = \{1\}$, $A_1 = A$ and $B_1 = B$ thus $A \leq A_1$ and $B_1 \leq B$.
- ($\cap_l^1$) There exists $I' \subseteq I$ such that $\bigcap_{i\in I'}(A_i \to B_i) \leq A \to B$ and, by induction hypothesis, there exists $J \subseteq I' \subseteq I$ such that for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq B$.
- ($\cap_l^2$) Idem.
- ($\to_l$) We have $J = I = \{1\}$, $A \leq A_1$ and $B = B_1$ thus $B_1 \leq B$.
- ($\to_r$) We have $\bigcap_{i\in I}(A_i \to B_i) \leq A \to D$ and $D \leq B$. By induction hypothesis, there exists $J \subseteq I$ such that for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq D$, and we have:

$$\frac{\bigcap_{i\in J} B_i \leq D \qquad D \leq B}{\bigcap_{i\in J} B_i \leq B} \, trans$$

- ($\to\cap$) We have $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B'$ and $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B''$ with $B = B' \cap B''$. By induction hypothesis, there exist $J' \subseteq I$ and $J'' \subseteq I$ such that for all $i \in J'$, $A \leq A_i$ and $\bigcap_{i\in J'} B_i \leq B'$ and for all $i \in J''$, $A \leq A_i$ and $\bigcap_{i\in J''} B_i \leq B''$, we choose $J = J' \cup J'' \subseteq I$ and we get for all $i \in J$, $A \leq A_i$. If both J' and J'' are not empty, we have:

$$\frac{\dfrac{\bigcap_{i\in J'} B_i \leq B'}{\bigcap_{i\in J} B_i \leq B'}\cap_l \qquad \dfrac{\bigcap_{i\in J''} B_i \leq B''}{\bigcap_{i\in J} B_i \leq B''}\cap_l}{\bigcap_{i\in J} B_i \leq B' \cap B''}\cap_r$$

  If J' is empty and J'' is not, we have:

$$\dfrac{\dfrac{\overline{\bigcap_{i\in J''} B_i \leq \Omega}\;\Omega_r \qquad \Omega \leq B'}{\bigcap_{i\in J''} B_i \leq B'}\,trans \qquad \bigcap_{i\in J''} B_i \leq B''}{\bigcap_{i\in J''} B_i \leq B' \cap B''}\,\cap_r$$

with $J'' = J$ (and similarly if J'' is empty but J' is not). Finally if both J' and J'' are empty, then:

$$\dfrac{\Omega \leq B' \qquad \Omega \leq B''}{\Omega \leq B' \cap B''}\,\cap_r$$

($\rightarrow\Omega$) We have $B = \Omega$ and thus $J = \emptyset$ and $\Omega \leq B$. □

Theorem 6 (Subject reduction for $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$)
βSUBJRED holds for $\mathsf{IT}^{\mathsf{BCD}}_{\leq}$.

PROOF: By Lemma 38 we have (GSUBST). By Lemma 36 we have (MONOT$_{\leq}$). By Lemma 37 we have (NFVAR). By Lemma 39 we have (INVVAR$_{\leq}$). By Lemma 40 we have (INVAPP$_{\cap}$). By Lemma 41 we have (INVABS$_{\cap}$). By Lemma 50 we have (TRANSELIM). By Lemma 51 we deduce (IMP$_{\cap}$).

By Lemma 24 we deduce (SUBST). By Lemma 42 we deduce (INVABSIMP$_{\cap}$). By Lemma 43 we deduce (βSUBJRED$_0$). By Lemma 10 we deduce (βSUBJRED). □

4.2.1 Additional results

Statement (Co implication inversion with $\cap$ [COIMP$_{\cap}$])
If $J \subseteq I$ with for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq B$ then $\bigcap_{i\in I}(A_i \rightarrow B_i) \leq A \rightarrow B$.

Lemma 52 (Co implication inversion with $\cap$)
(*Table* 10) $\Longrightarrow$ COIMP$_{\cap}$.

PROOF:

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\dfrac{A \leq A_i}{A_i \rightarrow B_i \leq A \rightarrow B_i}\,\rightarrow_l}{\bigcap_{i\in I} A_i \rightarrow B_i \leq A \rightarrow B_i}\,\cap_l \qquad \cdots}{\bigcap_{i\in I} A_i \rightarrow B_i \leq A \rightarrow \bigcap_{i\in J} B_i}\,\rightarrow\cap \qquad \bigcap_{i\in J} B_i \leq B}{\bigcap_{i\in I}(A_i \rightarrow B_i) \leq A \rightarrow B}\,\rightarrow_r$$

□

5 The η-rule

5.1 General case

Lemma 53 (Subject reduction for η_0 with $\cap$)

$$\text{NFVAR} \wedge \text{INVVAR}_{\leq} \wedge \text{INVAPP}_{\cap} \wedge \text{INVABS}_{\cap} \wedge (\leq) \wedge (\cap) \wedge (\Omega)$$
$$\wedge \left(\dfrac{\Gamma \vdash t : \bigcap_{k\in K} E_k \rightarrow F_k \qquad \cdots\; G \leq E_k \;\cdots \qquad \bigcap_{k\in K} F_k \leq H}{\Gamma \vdash t : G \rightarrow H}\right) \Longrightarrow \eta\text{SUBJRED}_0$$

PROOF: By ($\text{InvAbs}_\cap$), there exists a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ with $\bigcap_{i\in I} B_i \to C_i \leq A$ and, for all $i \in I$, $\Gamma, x : B_i \vdash t\,x : C_i$.

For each $i \in I$, by ($\text{InvApp}_\cap$) applied to $\Gamma, x : B_i \vdash t\,x : C_i$, there exists a set J_i and two families $(D_i)_{i\in I}$ and $(E_i)_{i\in I}$ with $\bigcap_{j\in J_i} E_i^j \leq C_i$ and for all $j \in J_i$, $\Gamma, x : B_i \vdash t : D_i^j \to E_i^j$ and $\Gamma, x : B_i \vdash x : D_i^j$.

For each $j \in J_i$, by (NFVar), $\Gamma \vdash t : D_i^j \to E_i^j$ and, by ($\text{InvVar}_\leq$), $B_i \leq D_i^j$ thus:

$$\dfrac{\dfrac{\cdots \quad \overset{j \in J_i}{\Gamma \vdash t : D_j^i \to E_j^i} \quad \cdots}{\Gamma \vdash t : \bigcap_{j\in J_i} D_i^j \to E_i^j}\,\cap \qquad \cdots\ B_i \leq D_i^j\ \cdots \qquad \bigcap_{j\in J_i} E_i^j \leq C_i}{\Gamma \vdash t : B_i \to C_i}$$

This proves $\Gamma \vdash t : B_i \to C_i$ for each $i \in I$, and we can conclude:

$$\dfrac{\dfrac{\cdots \quad \overset{i \in I}{\Gamma \vdash t : B_i \to C_i} \quad \cdots}{\Gamma \vdash t : \bigcap_{i\in I}(B_i \to C_i)}\,\cap \qquad \bigcap_{i\in I}(B_i \to C_i) \leq A}{\Gamma \vdash t : A}\,\leq$$

We use (Ω) instead of $(\cap)$ if $I = \emptyset$. □

Lemma 54
If $\text{InvVar}_\leq \wedge (var) \wedge (\leq) \wedge$ (*Table 8*) *then:*

$$\left(\dfrac{\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\Gamma \vdash t : G \to H}\right) \Longleftrightarrow (\textit{Table 10})$$

PROOF: We first prove

$$\dfrac{\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\Gamma \vdash t : G \to H} \Longleftrightarrow \dfrac{\cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\bigcap_{k\in K} E_k \to F_k \leq G \to H}$$

For the first implication, we use:

$$\dfrac{\dfrac{}{x : \bigcap_{k\in K} E_k \to F_k \vdash x : \bigcap_{k\in K} E_k \to F_k}\,var \qquad \cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{x : \bigcap_{k\in K} E_k \to F_k \vdash x : G \to H}$$

and by ($\text{InvVar}_\leq$) we have $\bigcap_{k\in K} E_k \to F_k \leq G \to H$. For the second implication, we use:

$$\dfrac{\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \dfrac{\cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\bigcap_{k\in K} E_k \to F_k \leq G \to H}}{\Gamma \vdash t : G \to H}\,\leq$$

Assume now that we have the rules of Table 10. If $K \neq \emptyset$, we can build:

$$\dfrac{\dfrac{\cdots \quad \dfrac{\dfrac{G \leq E_k}{E_k \to F_k \leq G \to F_k}\,{\to_l}}{\bigcap_{k\in K} E_k \to F_k \leq G \to F_k}\,{\cap_l} \quad \cdots}{\bigcap_{k\in K} E_k \to F_k \leq G \to \bigcap_{k\in K} F_k}\,{\to\cap} \qquad \bigcap_{k\in K} F_k \leq H}{\bigcap_{k\in K} E_k \to F_k \leq G \to H}\,{\to_r}$$

Otherwise, if $K = \emptyset$, we have:

$$\dfrac{\dfrac{}{\Omega \leq G \to \Omega}\,{\to\Omega} \qquad \Omega \leq H}{\Omega \leq G \to H}\,{\to_r}$$

Conversely, we consider particular cases of the rule:

$$\dfrac{\cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\bigcap_{k\in K} E_k \to F_k \leq G \to H}$$

With $K = \{1\}$ and $F_1 = H$, we obtain:

$$\dfrac{G \leq E_1 \qquad \dfrac{}{H \leq H}\,\mathit{refl}}{E_1 \to H \leq G \to H}$$

With $K = \{1\}$ and $E_1 = G$, we obtain:

$$\dfrac{C \leq G \to F_1 \qquad \dfrac{\dfrac{}{G \leq G}\,\mathit{refl} \qquad F_1 \leq H}{G \to F_1 \leq G \to H}}{C \leq G \to H}\,\mathit{trans}$$

With $K = \{2\}$, $E_1 = E_2 = G$ and $F_1 \cap F_2 = H$, we obtain:

$$\dfrac{\dfrac{D \leq G \to F_1 \qquad D \leq G \to F_2}{D \leq (G \to F_1) \cap (G \to F_2)}\,{\cap_r} \qquad \dfrac{\dfrac{}{G \leq G}\,\mathit{refl} \quad \dfrac{}{G \leq G}\,\mathit{refl} \quad \dfrac{}{F_1 \cap F_2 \leq F_1 \cap F_2}\,\mathit{refl}}{(G \to F_1) \cap (G \to F_2) \leq G \to (F_1 \cap F_2)}}{D \leq G \to (F_1 \cap F_2)}\,\mathit{trans}$$

With $K = \emptyset$, we obtain:

$$\dfrac{\dfrac{}{B \leq \Omega}\,{\Omega_r} \qquad \dfrac{\dfrac{}{\Omega \leq \Omega}\,{\Omega_r}}{\Omega \leq A \to \Omega}}{B \leq A \to \Omega}\,\mathit{trans}$$

□

Theorem 7 (Subject reduction for η for extensions of $\mathsf{IT}_{\leq}^{\mathsf{BCD}}$)
*η*SubjRed *holds for systems $\mathsf{IT}_{\leq}$ containing the subtyping rules of $\mathsf{IT}_{\leq}^{\mathsf{BCD}}$.*

Proof: By Lemma 38 we have (GSubst). By Lemma 37 we have (NFVar). By Lemma 39 we have (InvVar$_{\leq}$). By Lemma 40 we have (InvApp$_{\cap}$). By Lemma 41 we have (InvAbs$_{\cap}$). (var), ($\leq$), ($\cap$), (Ω) and Tables 8 and 10 hold for $\mathsf{IT}_{\leq}$.

By Lemma 54 we have:

$$\frac{\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \cdots\ G \le E_k\ \cdots \qquad \bigcap_{k\in K} F_k \le H}{\Gamma \vdash t : G \to H}$$

By Lemma 53 we deduce (ηSubjRed$_0$). By Lemma 12 we deduce (ηSubjRed). □

Statement (Implicative types [ImpTyp])
For any type A, there exist a non-empty set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ of types such that $A \le \bigcap_{i\in I} B_i \to C_i$ and $\bigcap_{i\in I} B_i \to C_i \le A$.

Statement (Subject expansion for η_0 [ηSubjExp$_0$])
If $\Gamma \vdash t : A$ and $t \leftarrow_{\eta_0} u$ then $\Gamma \vdash u : A$.

Lemma 55 (Subject expansion for η_0)
$\text{Monot}_{\le} \wedge \text{ImpTyp} \wedge (\textit{Table 7}) \wedge \left(\frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : A}\right) \wedge \left(\frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : B}\right) \Longrightarrow \eta\text{SubjExp}_0$

Proof: By (ImpTyp), we have $A \le \bigcap_{i\in I} B_i \to C_i$ and $\bigcap_{i\in I} B_i \to C_i \le A$. We prove the result by induction on the size of the non-empty set I.

- If I is a singleton $\{1\}$, we have $A \le B_1 \to C_1$ and $B_1 \to C_1 \le A$. By (Monot$_\le$), $\Gamma, x : B_1 \vdash t : A$ thus we can derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x : B_1 \vdash t : A \qquad A \le B_1 \to C_1}{\Gamma, x : B_1 \vdash t : B_1 \to C_1}\le \qquad \dfrac{}{\Gamma, x : B_1 \vdash x : B_1}\mathit{var}}{\Gamma, x : B_1 \vdash t\, x : C_1}\mathit{app}}{\Gamma \vdash \lambda x.(t\, x) : B_1 \to C_1}\mathit{abs} \qquad B_1 \to C_1 \le A}{\Gamma \vdash \lambda x.(t\, x) : A}\le$$

- If I is not a singleton, we have $I = I' \cup I''$ (with both I' and I'' non-empty and disjoint), $A \le \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i$ and $\bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i \le A$. We can derive:

$$\dfrac{\dfrac{\Gamma \vdash t : A \qquad A \le \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i}{\Gamma \vdash t : \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i}\le}{\Gamma \vdash t : \bigcap_{i\in I'} B_i \to C_i}$$

  and

$$\dfrac{\dfrac{\Gamma \vdash t : A \qquad A \le \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i}{\Gamma \vdash t : \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i}\le}{\Gamma \vdash t : \bigcap_{i\in I''} B_i \to C_i}$$

  thus, by induction hypothesis, $\Gamma \vdash \lambda x.(t\, x) : \bigcap_{i\in I'} B_i \to C_i$ and $\Gamma \vdash \lambda x.(t\, x) : \bigcap_{i\in I''} B_i \to C_i$, so that:

$$\dfrac{\dfrac{\Gamma \vdash \lambda x.(t\, x) : \bigcap_{i\in I'} B_i \to C_i \qquad \Gamma \vdash \lambda x.(t\, x) : \bigcap_{i\in I''} B_i \to C_i}{\Gamma \vdash \lambda x.(t\, x) : \bigcap_{i\in I'\cup I''} B_i \to C_i}\cap \qquad \bigcap_{i\in I'\cup I''} B_i \to C_i \le A}{\Gamma \vdash \lambda x.(t\, x) : A}\le$$

□

Statement (Subject expansion for η [ηSubjExp])
If $\Gamma \vdash t : A$ and $t \leftarrow_{\eta} u$ then $\Gamma \vdash u : A$.

Lemma 56 (Subject expansion for η)
$\text{GS}\textsc{ubst} \wedge \eta\text{S}\textsc{ubj}\text{E}\textsc{xp}_0 \Longrightarrow \eta\text{S}\textsc{ubj}\text{E}\textsc{xp}$.

Proof: If $\Gamma \vdash t : A$ and $t \leftarrow_\eta u$ then $t = c\{^{t'}/_x\}$ and $u = c\{^{u'}/_x\}$ with $t' \leftarrow_{\eta_0} u'$. Assume that $\Gamma, \Delta \vdash t' : B$ then by ($\eta\text{S}\textsc{ubj}\text{E}\textsc{xp}_0$) we have $\Gamma, \Delta \vdash u' : B$, thus by ($\text{GS}\textsc{ubst}$) we obtain $\Gamma \vdash u : A$. □

5.1.1 Additional results

Lemma 57

$$\eta\text{S}\textsc{ubj}\text{R}\textsc{ed}_0 \wedge \text{M}\textsc{onot}_{\leq} \wedge (\textit{refl}) \wedge (\cap_l) \wedge (\textit{Table 7}) \Longrightarrow \left(\frac{\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \cdots\ G \leq E_k\ \cdots \qquad \bigcap_{k\in K} F_k \leq H}{\Gamma \vdash t : G \to H} \right)$$

Proof: Assume $\Gamma \vdash t : \bigcap_{k\in K} E_k \to F_k$, if $x \notin t$ we have $\Gamma, x : G \vdash t : \bigcap_{k\in K} E_k \to F_k$ by ($\text{M}\textsc{onot}_{\leq}$) and then for each $k \in K$:

$$\dfrac{\dfrac{\Gamma, x : G \vdash t : \bigcap_{k\in K} E_k \to F_k \qquad \dfrac{\dfrac{}{E_k \to F_k \leq E_k \to F_k}\,\textit{refl}}{\bigcap_{k\in K} E_k \to F_k \leq E_k \to F_k}\,\cap_l}{\Gamma, x : G \vdash t : E_k \to F_k}\,\leq \qquad \dfrac{\dfrac{}{\Gamma, x : G \vdash x : G}\,\textit{var} \qquad G \leq E_k}{\Gamma, x : G \vdash x : E_k}\,\leq}{\Gamma, x : G \vdash t\,x : F_k}\,\textit{app}$$

We then build:

$$\dfrac{\dfrac{\dfrac{\cdots \quad \Gamma, x : G \vdash t\,x : F_k \quad \cdots}{\Gamma, x : G \vdash t\,x : \bigcap_{k\in K} F_k}\,\cap \qquad \bigcap_{k\in K} F_k \leq H}{\Gamma, x : G \vdash t\,x : H}\,\leq}{\Gamma \vdash \lambda x.(t\,x) : G \to H}\,\textit{abs}$$

If $K = \emptyset$, we have:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : G \vdash t\,x : \Omega}\,\Omega \qquad \Omega \leq H}{\Gamma, x : G \vdash t\,x : H}\,\leq}{\Gamma \vdash \lambda x.(t\,x) : G \to H}\,\textit{abs}$$

By ($\eta\text{S}\textsc{ubj}\text{R}\textsc{ed}_0$), we conclude $\Gamma \vdash t : G \to H$. □

5.2 One concrete solution

The system $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$ is the particular case of $\mathsf{IT}_{\leq}$ where the relation $\leq$ is defined *exactly* by the rules of Tables 8, 10 and 11.

Lemma 58 (Transitivity elimination for $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$)
$\text{T}\textsc{rans}\text{E}\textsc{lim}$ *holds for* $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$.

Proof: Similar to the proof of Lemma 32 but using the number of rules plus the numver of (X_l) rules as the size of a derivation. □

$$\frac{}{X \leq A \to X}\ X_l \qquad \frac{A \leq \Omega \to X}{A \leq X}\ X_r$$

Table 11: Extensionality subtyping rules

Lemma 59 (Transitivity-free implication inversion for $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$)
In $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$, TransElim $\Longrightarrow$ Imp$_\cap$.

Proof: By induction on the derivation of $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B$. We consider each possible last rule from Tables 8, 10 and 11 except (*trans*) (thanks to (TransElim)). Since rules from Table 11 are not possible last rules, we can rely on the proof of Lemma 51. □

Theorem 8 (Subject reduction for $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$)
βSubjRed *holds for* $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$.

Proof: By Lemma 38 we have (GSubst). By Lemma 36 we have (Monot$_\leq$). By Lemma 37 we have (NFVar). By Lemma 39 we have (InvVar$_\leq$). By Lemma 40 we have (InvApp$_\cap$). By Lemma 41 we have (InvAbs$_\cap$). By Lemma 58 we have (TransElim). By Lemma 59 we deduce (Imp$_\cap$).

By Lemma 24 we deduce (Subst). By Lemma 42 we deduce (InvAbsImp$_\cap$). By Lemma 43 we deduce (βSubjRed$_0$). By Lemma 10 we deduce (βSubjRed). □

Lemma 60 (Implicative types for extensions of $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$)
ImpTyp *holds for systems containing the subtyping rules of* $\mathsf{IT}_{\leq}^{\mathsf{BCD}\eta}$.

Proof: By induction on the type A:

- If $A = X$, we choose $I = \{1\}$, $B_1 = \Omega$ and $C_1 = X$. We have:

$$\frac{}{X \leq \Omega \to X}\ X_l \qquad \text{and} \qquad \frac{\frac{}{\Omega \to X \leq \Omega \to X}\ \mathit{refl}}{\Omega \to X \leq X}\ X_r$$

- If $A = \Omega$, we choose $I = \{1\}$, $B_1 = \Omega$ and $C_1 = \Omega$. We have:

$$\frac{}{\Omega \leq \Omega \to \Omega}\ {\to}\Omega \qquad \text{and} \qquad \frac{}{\Omega \to \Omega \leq \Omega}\ \Omega_r$$

- If $A = A' \to B'$, we choose $I = \{1\}$, $B_1 = A'$ and $C_1 = B'$.
- If $A = A' \cap A''$, by induction hypothesis, we have I', $(B_i)_{i\in I'}$, $(C_i)_{i\in I'}$, I'', $(B_i)_{i\in I''}$ and $(C_i)_{i\in I''}$ such that $A' \leq \bigcap_{i\in I'} B_i \to C_i$, $\bigcap_{i\in I'} B_i \to C_i \leq A'$, $A'' \leq \bigcap_{i\in I''} B_i \to C_i$ and $\bigcap_{i\in I''} B_i \to C_i \leq A''$. We choose $I = I' \cup I''$ and we have:

$$A' \cap A'' \leq \bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i$$

and

$$\bigcap_{i\in I'} B_i \to C_i \cap \bigcap_{i\in I''} B_i \to C_i \leq A' \cap A''$$

by:

$$\dfrac{\dfrac{D_1 \le E_1}{D_1 \cap D_2 \le E_1}\cap_l^1 \quad \dfrac{D_2 \le E_2}{D_1 \cap D_2 \le E_2}\cap_l^2}{D_1 \cap D_2 \le E_1 \cap E_2}\cap_r$$

□

Theorem 9 (Subject expansion for η for $\mathsf{IT}_{\le}^{\mathsf{BCD}\eta}$)
ηSubjExp *holds for* $\mathsf{IT}_{\le}^{\mathsf{BCD}\eta}$.

Proof: By Lemma 38 we have (GSubst). By Lemma 36 we have (Monot$_{\le}$). By Lemma 60 we have (ImpTyp). Table 7 holds for $\mathsf{IT}_{\le}^{\mathsf{BCD}\eta}$.

$$\dfrac{\Gamma \vdash t : A \cap B \quad \dfrac{\dfrac{}{A \le A}\,\mathit{refl}}{A \cap B \le A}\cap_l^1}{\Gamma \vdash t : A}\le \quad \text{and} \quad \dfrac{\Gamma \vdash t : A \cap B \quad \dfrac{\dfrac{}{B \le B}\,\mathit{refl}}{A \cap B \le B}\cap_l^2}{\Gamma \vdash t : B}\le$$

By Lemma 55 we deduce (ηSubjExp$_0$). By Lemma 56 we deduce (ηSubjExp). □

5.2.1 Additional results

Lemma 61 (Necessity of ImpTyp for ηSubjExp$_0$)
ηSubjExp$_0$ $\wedge$ (*var*) $\wedge$ InvVar$_{\le}$ $\wedge$ InvAbs$_{\cap}$ $\wedge$ InvApp$_{\cap}$ $\wedge$ (*Table 8*) $\wedge$ (*Table 10*) $\Longrightarrow$ ImpTyp.

Proof: By (*var*) and (ηSubjExp$_0$), we have $x : A \vdash \lambda y.x\,y : A$. By (InvAbs$_{\cap}$), there exist I, $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \to C_i \le A$ and for all $i \in I$, $x : A, y : B_i \vdash x\,y : C_i$. For each $i \in I$, by (InvApp$_{\cap}$), there exist J_i, $(D_i^j)_{j\in J_i}$ and $(E_i^j)_{j\in J_i}$ such that $\bigcap_{j\in J_i} E_i^j \le C_i$ and for all $j \in J_i$, $x : A, y : B_i \vdash x : D_i^j \to E_i^j$ and $x : A, y : B_i \vdash y : D_i^j$. By (InvVar$_{\le}$), we obtain $A \le D_i^j \to E_i^j$ and $B_i \le D_i^j$.

We then have:

$$\dfrac{\ldots \quad \dfrac{\dfrac{\ldots \quad \dfrac{A \le D_i^j \to E_i^j \quad \dfrac{B_i \le D_i^j}{D_i^j \to E_i^j \le B_i \to E_i^j}\to_l}{A \le B_i \to E_i^j}\,\mathit{trans} \quad \ldots}{A \le B_i \to \bigcap_{j\in J_i} E_i^j}\to\cap \quad \bigcap_{j\in J_i} E_i^j \le C_i}{A \le B_i \to C_i}\to_r \quad \ldots}{A \le \bigcap_{i\in I} B_i \to C_i}$$

If some J_i is empty, we use:

$$\dfrac{\dfrac{}{A \le B_i \to \Omega}\to\Omega \quad \Omega \le C_i}{A \le B_i \to C_i}\to_r$$

If I is not empty, we are done. Otherwise we have $\Omega \le A$. This entails:

$$\dfrac{}{A \le \Omega \to \Omega}\to\Omega \quad \text{and} \quad \dfrac{\dfrac{}{\Omega \to \Omega \le \Omega}\,\Omega_r \quad \Omega \le A}{\Omega \to \Omega \le A}\,\mathit{trans}$$

□

[Monot] If $\Gamma \vdash t : A$ and $\Delta \supseteq \Gamma$ then $\Delta \vdash t : A$ (where $\Delta \supseteq \Gamma$ means that each typing declaration $x : B$ in Γ appears in Δ).

[Monot$_\leq$] If $\Gamma \vdash t : A$, $\Delta \leq \Gamma$ and $A \leq B$ then $\Delta \vdash t : B$ (where $\Delta \leq \Gamma$ means that for each typing declaration $x : C$ in Γ there is a declaration $x : D$ with $D \leq C$ in Δ).

[NFVar] If $x \notin t$ and $\Gamma, x : B \vdash t : A$ then $\Gamma \vdash t : A$.

[GSubst] Assume that $\Gamma \vdash t\{^v/_x\} : A$ and for all Δ and B, $\Gamma, \Delta \vdash v : B$ implies $\Gamma, \Delta \vdash u : B$, then $\Gamma \vdash t\{^u/_x\} : A$.

[Subst] If $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash u : A$ then $\Gamma \vdash t[^u/_x] : B$.

[coSubst] If $\Gamma \vdash t[^u/_x] : B$ with $x \notin u$ and Γ contains declarations for the free variables of u then there exists a type A such that $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash u : A$.

[InvVar] If $\Gamma \vdash x : A$ then $x : A \in \Gamma$.

[InvVar$_\leq$] If $\Gamma \vdash x : A$ then there exists B such that $B \leq A$ and $x : B \in \Gamma$.

[InvApp] If $\Gamma \vdash t\,u : A$, there exists a type B such that $\Gamma \vdash t : B \to A$ and $\Gamma \vdash u : B$.

[InvApp$_\leq$] If $\Gamma \vdash t\,u : A$, there exist B and C such that $B \leq A$, $\Gamma \vdash t : C \to B$ and $\Gamma \vdash u : C$.

[InvApp$_\cap$] If $\Gamma \vdash t\,u : A$, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq A$ and for all $i \in I$, $\Gamma \vdash t : C_i \to B_i$ and $\Gamma \vdash u : C_i$.

[InvAbs] If $\Gamma \vdash \lambda x.t : A$, there exist B and C such that $A = B \to C$ and $\Gamma, x : B \vdash t : C$.

[InvAbs$_\leq$] If $\Gamma \vdash \lambda x.t : A$, there exist B and C such that $B \to C \leq A$ and $\Gamma, x : B \vdash t : C$.

[InvAbs$_\cap$] If $\Gamma \vdash \lambda x.t : A$, there exist a set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \to C_i \leq A$ and for all $i \in I$, $\Gamma, x : B_i \vdash t : C_i$.

[InvAbsImp] If $\Gamma \vdash \lambda x.t : A \to B$ then $\Gamma, x : A \vdash t : B$.

[InvAbsImp$_\leq$] If $\Gamma \vdash \lambda x.t : A \to B$, there exist A' and B' such that $A \leq A'$, $B' \leq B$ and $\Gamma, x : A' \vdash t : B'$.

[InvAbsImp$_\cap$] If $\Gamma \vdash \lambda x.t : A \to B$, there exist a set I and two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ such that $\bigcap_{i\in I} B_i \leq B$ and for all $i \in I$, $A \leq A_i$ and $\Gamma, x : A_i \vdash t : B_i$.

[TransElim] If $A \leq B$ is derivable then $A \leq B$ is derivable without the (*trans*) rule.

[Imp$_\leq$] If $A \to B \leq C \to D$ then $C \leq A$ and $B \leq D$.

[Imp$_\cap$] If $\bigcap_{i\in I}(A_i \to B_i) \leq A \to B$ then there exists $J \subseteq I$ such that for all $i \in J$, $A \leq A_i$ and $\bigcap_{i\in J} B_i \leq B$.

[ImpTyp] For any type A, there exist a non-empty set I and two families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ of types such that $A \leq \bigcap_{i\in I} B_i \to C_i$ and $\bigcap_{i\in I} B_i \to C_i \leq A$.

[βSubjRed$_0$] If $\Gamma \vdash t : A$ and $t \to_{\beta_0} u$ then $\Gamma \vdash u : A$.

[βSubjRed] If $\Gamma \vdash t : A$ and $t \to_\beta u$ then $\Gamma \vdash u : A$.

[ηSubjRed$_0$] If $\Gamma \vdash t : A$ and $t \to_{\eta_0} u$ then $\Gamma \vdash u : A$.

[ηSubjRed] If $\Gamma \vdash t : A$ and $t \to_\eta u$ then $\Gamma \vdash u : A$.

[βSubjExp$_0$] If $\Gamma \vdash t : A$ with Γ containing declarations for the free variables of u and $t \leftarrow_{\beta_0} u$ then $\Gamma \vdash u : A$.

[βSubjExp] If $\Gamma \vdash t : A$ with Γ containing declarations for the free variables of u and $t \leftarrow_\beta u$ then $\Gamma \vdash u : A$.

[ηSubjExp$_0$] If $\Gamma \vdash t : A$ and $t \leftarrow_{\eta_0} u$ then $\Gamma \vdash u : A$.

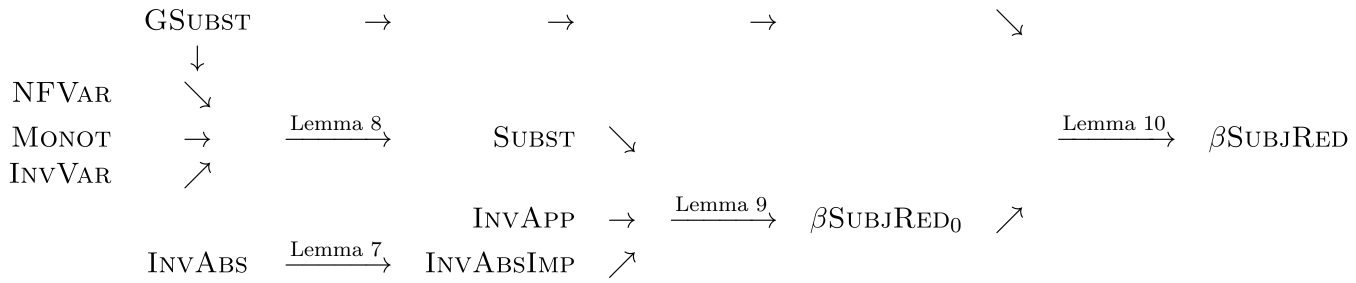
[ηSubjExp] If $\Gamma \vdash t : A$ and $t \leftarrow_\eta u$ then $\Gamma \vdash u : A$.

Table 12: List of the main statements

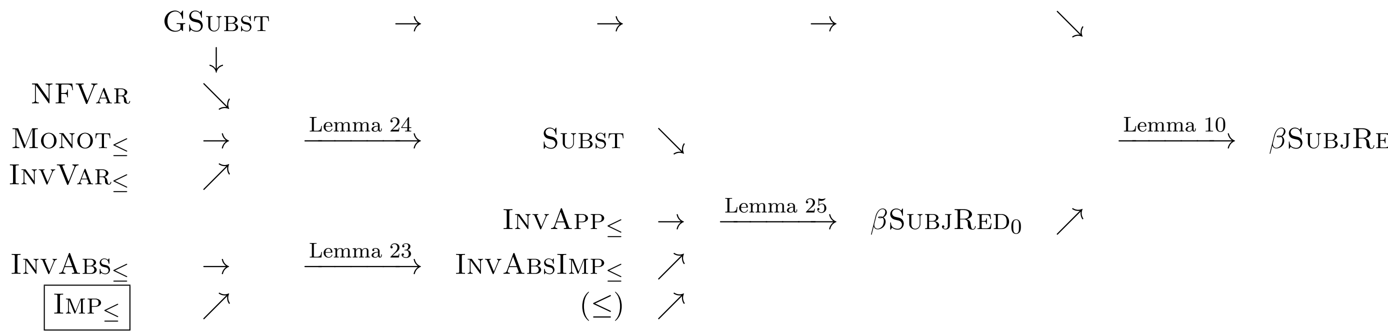
The boxed statements below are those which depend on the subtyping rules in a non monotonic way. A good way to prove them is to rely on TransElim.

A Proofs of βSubjRed

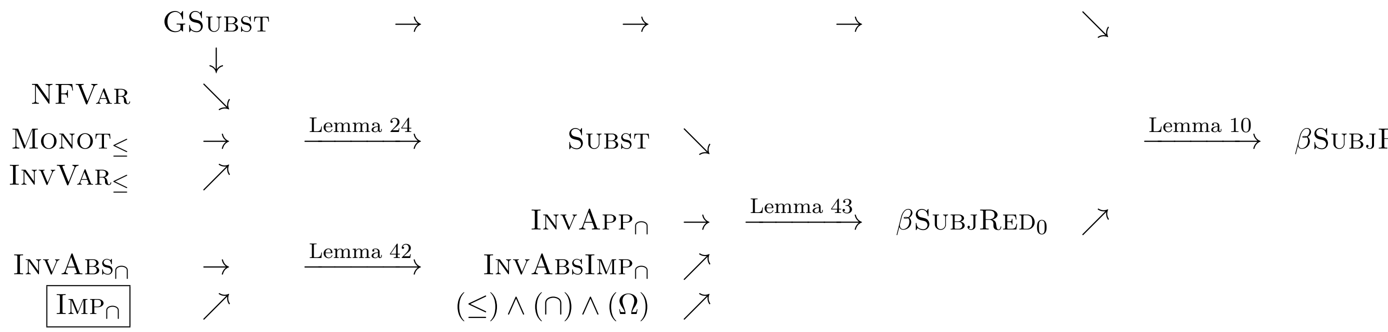
A.1 Simple types



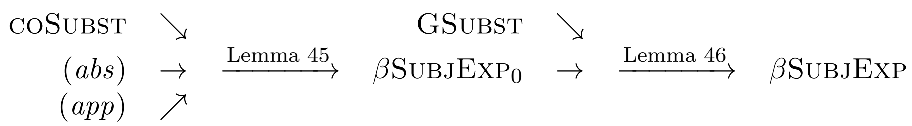
A.2 Simple types with subtyping



A.3 Intersection types



B Proof of βSubjExp with intersection



C Proofs of ηSubjRed

C.1 Simple types

$$\begin{array}{rlcrlcl}
\textsc{NFVar} & \searrow & & & & & \\
\textsc{InvVar} & \searrow & & \textsc{GSubst} & \searrow & & \\
\textsc{InvApp} & \rightarrow & \xrightarrow{\text{Lemma 11}} & \eta\textsc{SubjRed}_0 & \rightarrow & \xrightarrow{\text{Lemma 12}} & \eta\textsc{SubjRed} \\
\textsc{InvAbs} & \nearrow & & & & &
\end{array}$$

C.2 Simple types with subtyping

$$\begin{array}{rlcrlcl}
\textsc{NFVar} & \searrow & & & & & \\
\textsc{InvVar}_{\leq} & \searrow & & \textsc{GSubst} & \searrow & & \\
\textsc{InvApp}_{\leq} & \rightarrow & \xrightarrow{\text{Lemma 34}} & \eta\textsc{SubjRed}_0 & \rightarrow & \xrightarrow{\text{Lemma 12}} & \eta\textsc{SubjRed} \\
\textsc{InvAbs}_{\leq} & \nearrow & & & & & \\
(\rightarrow) & \nearrow & & & & &
\end{array}$$

C.3 Intersection types

$$\begin{array}{rlcrlcl}
\textsc{NFVar} & \searrow & & & & & \\
\textsc{InvVar}_{\leq} & \searrow & & \textsc{GSubst} & \searrow & & \\
\textsc{InvApp}_{\cap} & \rightarrow & \xrightarrow{\text{Lemmas 53 and 54}} & \eta\textsc{SubjRed}_0 & \rightarrow & \xrightarrow{\text{Lemma 12}} & \eta\textsc{SubjRed} \\
\textsc{InvAbs}_{\cap} & \nearrow & & & & & \\
(\mathit{var}) \wedge (\leq) \wedge (\cap) \wedge (\Omega) & \nearrow & & & & & \\
\text{(Tables 8 and 10)} & \nearrow & & & & &
\end{array}$$

D Proof of ηSubjExp with intersection

$$\begin{array}{rlcrlcl}
\textsc{Monot}_{\leq} & \searrow & & \textsc{GSubst} & \searrow & & \\
\textsc{ImpTyp} & \rightarrow & \xrightarrow{\text{Lemma 55}} & \beta\textsc{SubjExp}_0 & \rightarrow & \xrightarrow{\text{Lemma 56}} & \beta\textsc{SubjExp} \\
\text{(Table 7)} & \nearrow & & & & & \\
\left(\dfrac{\Gamma \vdash t : A_1 \cap A_2}{\Gamma \vdash t : A_i}\right) & \nearrow & & & & &
\end{array}$$