HW 1: Turing machines
(due Tuesday, February 28th, before tutorial)

For $x \in \Sigma^*$, let us denote by $\bar{x}$ the mirror of $x$ (e.g. \texttt{flow} $= \texttt{wolf}$). The palindrome language over $\Sigma$ is defined as 
$$\text{Pal} = \{ x \in \Sigma^* : x = \bar{x} \}.$$

As we are interested in this problem in deciding a language (i.e., the only possible outputs are “accept” or “reject”), it is convenient not to have an output tape, and instead the outcome of the computation is determined by the halting state. More precisely, instead of having a single halting state $q_{\text{halt}}$, we have two halting states $q_{\text{accept}}$ and $q_{\text{reject}}$.

1. Describe a TM with two tapes that recognizes $\text{Pal}$ in linear time.

2. Now we can imagine a model of Turing machines with only one read and write tape that at the beginning contains the input. Describe a TM with one tape that recognizes $\text{Pal}$ in quadratic time.

3. Solve either one of the two following parts

   (a) (recommended for those who have not worked much with Turing machines) Show how to transform a Turing machine $M$ with $k$ tapes working in time $T(n)$ into a Turing machine $M'$ with a single read-write tape that decides the same language in time $O(T(n)^2)$. Note that it is ok to increase the size of the alphabet $\Gamma$ of the Turing machine.

   (b) (recommended for those who are comfortable with Turing machines) The objective of this part is to show that the simulation argument in (a) is optimal, in particular for the language $\text{Pal}$.

   For this, define the crossing sequence $C_i(x)$ at boundary $i$ of a TM $M$ on input $x$ as the sequence of states (i.e., elements in $Q$) of $M$ when the head crosses between cells $i$ and $i + 1$ (in either directions). We assumed here that we number the cells of the tape from 0 (the starting cell containing $\triangleright$) and the input $x$ is initially written from cell 1 to cell $|x|$.

   Let $M$ be a TM recognizing $\text{Pal}$ running in time $T(n)$. The objective is to show that $T(n) \geq cn^2$ for infinitely many values of $n$. Let $\text{Pal}_n$ the sublanguage of $\text{Pal}$ defined as 
$$\text{Pal}_n = \{ x0^{2n}\bar{x} : x \in \Sigma^n \},$$

assuming that 0 $\in \Sigma$.

   i. Let $y, z \in \text{Pal}_n$ and assume for some $i, j \in \{ n + 1, \ldots, 3n - 1 \}$, $C_i(y) = C_j(z)$. Show that $y = z$.

   ii. Show that for any $y \in \text{Pal}_n$, there is an $i \in \{ n + 1, \ldots, 3n - 1 \}$ such that $|C_i(y)| \leq \frac{T(4n)}{2n-1}$.

   iii. Conclude by observing that for each $y \in \text{Pal}_n$ we can associate a distinct sequence of states of size at most $\frac{T(4n)}{2n-1}$.