HW 4: Randomized computation
(due Tuesday, May 2nd before tutorial)

Problem 1. We define the class $\mathbf{BPP}'$ as the set of languages $L$ such that there is a polynomial-time Turing machine $M$ and a polynomial $q$ such that

\[
\begin{align*}
    x \in L & \implies \Pr_{r \in \{0,1\}^{q(n)}} \{ M(x, r) = 1 \} \geq \frac{1}{10} + \frac{1}{n} \\
    x \notin L & \implies \Pr_{r \in \{0,1\}^{q(n)}} \{ M(x, r) = 1 \} < \frac{1}{10},
\end{align*}
\]

where $n = |x|$ is the size of $x$. Prove that $\mathbf{BPP}' = \mathbf{BPP}$.

Problem 2. Show that if $\mathbf{NP} \subseteq \mathbf{BPP}$, then $\mathbf{NP} = \mathbf{RP}$.

Problem 3. The class $\mathbf{PP}$ is defined as the set of languages $L$ such that there exists a polynomial $q(n)$ and a polynomial time machine $M$ such that the following holds.

\[
\begin{align*}
    x \in L & \implies \Pr_{r \in \{0,1\}^{q(n)}} \{ M(x, r) = 1 \} > \frac{1}{2} \\
    x \notin L & \implies \Pr_{r \in \{0,1\}^{q(n)}} \{ M(x, r) = 1 \} \leq \frac{1}{2},
\end{align*}
\]

where $n = |x|$.

1. Show that $\mathbf{NP} \subseteq \mathbf{PP}$.

2. Show that the problem $\#\mathbf{SAT} = \{ (\varphi, k) : \text{the formula } \varphi \text{ has } > k \text{ satisfying assignments} \}$ is $\mathbf{PP}$-complete for Karp reductions.