HW 3: Typical sets
(due October 9th, before tutorial)

1. (Typical sets) Let $X^n = X_1 \ldots X_n$ be independent and identically distributed bits with $X_1 \sim \text{Ber}(p)$, i.e., $P_{X_1}(0) = 1 - p$ and $P_{X_1}(1) = p$ (assume that $0 < p < 1/2$). Let $\delta > 0$ with $p + \delta \leq 1/2$, show that there exists a set $S_\delta \subseteq \{0, 1\}^n$ with $|S_\delta| \leq 2^n h_2(p + \delta)$ where $h_2(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$ satisfying the property that

$$\lim_{n \to \infty} P\{X^n \in S_\delta\} = 1.$$

You may assume the following inequality for $k \leq n/2$ without proof:

$$1 + \left(\begin{array}{c} n \\ 1 \end{array}\right) + \cdots + \left(\begin{array}{c} n \\ k \end{array}\right) \leq 2^{h_2(k/n)n}.$$  \hspace{1cm} (1)

Remark: We actually proved a more general version of this in class. You are asked here to produce an elementary self-contained proof.

2. (Bonus) Prove inequality (1).

3. Consider a source given by $X^n = X_1 \ldots X_n$ with $X_i$ independent and identically distributed bits with $P\{X_i = 1\} = \frac{1}{4}$. Describe the distribution of the random variable $h_{X^n}(X^n) = - \log_2 P_{X^n}(X^n)$. How many values does it take? What is the probability for each different value? What is the expectation?

4. We showed in class that taking a sequence $X^n = X_1 \ldots X_n$ of independent copies of $X$, we have $h_{X^n}(X^n)/n$ converges weakly to $H(X)$. Now for random variables $X, Y$ we define $i_{X,Y}(X : Y) = \log_2 \frac{P_{XY}(X,Y)}{P_X(X)P_Y(Y)}$. If $X^n$ is $n$ independent copies of $X$ and $Y^n$ is $n$ independent copies of $Y$. What can you say on the random variable $i_{X^n,Y^n}(X^n : Y^n)$ as $n \to \infty$?