1. Let $A_q(n, d)$ be the largest $k$ such that a code over alphabet $\{1, \ldots, q\}$ of block length $n$, dimension $k$ and minimum distance $d$ exists (recall that this corresponds to the notation $(n, k; d)_q$). Determine $A_2(3, d)$ for all integers $d \geq 1$.

2. Suppose $C$ is a $(n, k, d)\_2$-code with $d$ odd. Construct using $C$ a code $C'$ that is a $(n + 1, k, d + 1)\_2$-code.

3. By constructing the columns of a parity check matrix in a greedy fashion, show that there exists a binary linear code $[n, k, d]_2$ provided that

$$2^{n-k} > 1 + \binom{n-1}{1} + \cdots + \binom{n-1}{d-2}.$$  

This is a small improvement compared to the general Gilbert-Varshamov bound. In particular, it is tight for the $[7, 4, 3]_2$ Hamming code.

4. The Hadamard code has a nice property that it can be locally decoded. Let $C_{\text{Had}, r} : \{0, 1\}^r \rightarrow \{0, 1\}^{2r}$ be the encoding function of the Hadamard code. Suppose you are interested only in the $i$-th bit $x_i$ of the message $x \in \{0, 1\}^r$. The challenge is that you only have access to $y \in \{0, 1\}^{2r}$ such that $\Delta(C_{\text{Had}, r}(x), y) \leq \frac{2r}{15}$ and you would like to look only at a few bits of $y$. Show that by querying only 2 well-chosen positions (the choice will involve some randomization) of $y$, you can determine $x_i$ correctly with probability $4/5$ (the probability here is over the choice of the queries, in particular $x, y$ and $i$ are fixed).

Hint: You might want to query $y$ at the position labelled by $u \in \{0, 1\}^r$ at random and the position $u + e_i$ where $e_i \in \{0, 1\}^r$ is the binary representation of $i$. 