Homework due April 3rd, 2022 before 23:59

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Exercise 1: Squashed entanglement

The squashed entanglement is an entanglement measure defined as follows: for a bipartite density operator $\rho_{AB} \in \mathcal{S}(A \otimes B)$, we let

$$E_{\mathrm{sq}}(A:B)_\rho := \frac{1}{2} \inf \left\{ I(A:B|C)_\rho \ : \ \mathrm{tr}_C(\rho_{ABC}) = \rho_{AB} \right\} \ .$$

1. Show that if ρ_{AB} is a pure state, then $E_{\mathrm{sq}}(A:B)_\rho = H(A)_\rho = H(B)_\rho$.

2. Show that if ρ_{AB} is separable, i.e., $\rho_{AB} = \sum_x p(x) \sigma_A^x \otimes \omega_B^x$, where $p(x)$ is a probability distribution and σ^x, ω^x are density operators, we have $E_{\mathrm{sq}}(A:B)_\rho = 0$.

3. Using the chain rule for the conditional mutual information, i.e., $I(X:YW|Z) = I(X:Y|Z) + I(X:W|ZY)$, show that the squashed entanglement satisfies the following "monogamy" relation: for any ρ_{ABC}

$$E_{\mathrm{sq}}(A:B)_\rho + E_{\mathrm{sq}}(A:C)_\rho \leqslant E_{\mathrm{sq}}(A:BC)_\rho \ .$$

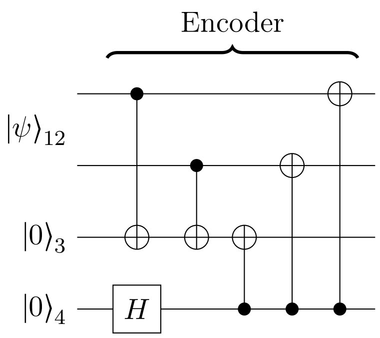
   The word monogamy here refers to the fact that the system A cannot be at the same time maximally entangled with B and with C. This is to be contrasted with classical correlations which can be shared. Taking the mutual information as a measure of correlation, show that there exists a tripartite state ρ_{ABC} (which can be classical) such that $I(A:B)_\rho + I(A:C)_\rho > I(A:BC)_\rho$.

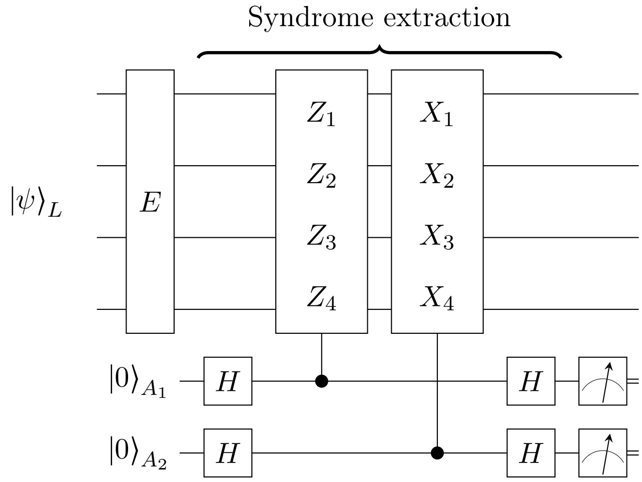
4. Show that the squashed entanglement is additive for tensor product states, i.e., for $\rho_{ABA'B'} = \rho_{AB} \otimes \rho_{A'B'}$, we have

$$E_{\mathrm{sq}}(AA':BB')_\rho = E_{\mathrm{sq}}(A:B)_\rho + E_{\mathrm{sq}}(A':B')_\rho \ .$$

Exercise 2: the $[[4,2,2]]$ detection code

The $[[4,2,2]]$ detection code is the smallest stabilizer code to offer protection against a quantum noise model in which the qubits can suffer both X-errors and Z-errors. An encoder for the $[[4,2,2]]$ detection code is displayed in the left figure on the next page: a two-qubit register $|\psi\rangle_{12}$ is entangled across four qubits to give the logical state $|\psi\rangle_L$.





1. Let us denote by U the encoding circuit above and define the logical operators $|\psi\rangle_L = U(|\psi\rangle_{12}|0\rangle_3|0\rangle_4)$. What are the 4 logical states $|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L$?

2. Check that the operators $X_1X_2X_3X_4$ and $Z_1Z_2Z_3Z_4$ are stabilizers for this code.

3. The syndrome extraction circuit is shown on the right figure above. Assume that the error E is a Pauli error. Prove that measuring the two ancillla qubits A_1 and A_2 allows one to find out whether E commutes or anticommutes with the two stabilizers.

4. Compute the syndrome for each of the 12 single-qubit Pauli errors: $X_1, \cdots, X_4, Z_1$, $\cdots$, $Z_4, Y_1, \cdots Y_4$. Conclude on whether the code can correct such errors or merely detect them.

5. Give the expressions for the 4 logical operators $\bar{X}_1, \bar{Z}_1, \bar{X}_2, \bar{Z}_2$.

6. What is the minimum distance of this code?