Quantum Information Theory

Exercise sheet 5

(Dated: February 15, 2022)

1. Compute the coherent information for the trace channel

$$\mathcal{N}_{\mathrm{D}}(\rho) = \frac{1}{2}\mathrm{tr}[\rho]\mathbb{1}_2\,, \tag{1}$$

and the completley dephasing channel

$$\mathcal{N}_{\varphi}(\rho) = \frac{1}{2}\rho + \frac{1}{2}Z\rho Z\,. \tag{2}$$

2. [*Additivity of coherent information for degradable channels*]
A quantum channel $\mathcal{N}_{A\to B}$ is degradable if there exists a channel $\mathcal{D}_{B\to E}$ such that

$$\mathcal{N}^{c}_{A\to E}\left(\rho_A\right) = \mathcal{D}_{B\to E}\left(\mathcal{N}_{A\to B}\left(\rho_A\right)\right) \tag{3}$$

for any input state ρ_A, where $\mathcal{N}^{c}_{A\to E}$ is the channel complementary to $\mathcal{N}$.
Let $\mathcal{N}$ and $\mathcal{M}$ be any quantum channels that are degradable. Show that the coherent information of the tensor-product channel $\mathcal{N}\otimes\mathcal{M}$ is the sum of their individual coherent informations:

$$Q(\mathcal{N}\otimes\mathcal{M}) = Q(\mathcal{N}) + Q(\mathcal{M})\,. \tag{4}$$

3. Show that the erasure channel

$$\mathcal{N}^{\text{easure},p}_{A\to B}(\rho_A) = (1-p)\rho_B + p|e_B\rangle\langle e_B| \tag{5}$$

is degradable if the erase probability is $p \leq 1/2$.