Quantum Information Theory

Exercise sheet 6

(Dated: February 22, 2022)

1. Let E be an arbitrary 1-qubit unitary. It can be written as

$$E = \alpha_0 \mathbb{1}_2 + \alpha_1 X + \alpha_2 Y + \alpha_3 Z\,, \tag{1}$$

   for some complex coefficients α_i. Show that $\sum_i |\alpha_i|^2 = 1$.

2. (a) Write the 1-qubit Hadamard transform $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the Pauli matrices.

   (b) Suppose that an H-error occurs on the first qubit of $\alpha\,|\bar{0}\rangle + \beta\,|\bar{1}\rangle$ using Shor's 9-qubit code. Give the steps in the error correction procedure that corrects this error.

3. Show that there cannot be a quantum error correcting code that encode one logical qubit into $2n$ physical qubits that can correct arbitrary errors on n qubits. Hint: use the no-cloning theorem.

4. Show that the operations $\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9$ and $\bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$ act as logical Z and X operations on a qubit encoded with Shor's 9-qubit code.

5. Consider encoding a general qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ into the 3-bit repetition code as $\alpha\,|000\rangle + \beta\,|111\rangle$. Recall that a possible choice of generators for the stabilizer group of the 3-bit code is Z_1Z_2, Z_2Z_3. The amplitude damping channel A_γ acts as $A_\gamma(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$ with Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \tag{2}$$

   (a) Consider the effect of A_γ on the first qubit of the encoded state, that is the effect of the 3-qubit channel $A_\gamma \otimes \mathbb{1} \otimes \mathbb{1}$. Find the probabilities for each possible syndrome measurement.

   (b) What is the state of the system after error correction?

   (c) The phase damping channel P_λ acts as $P_\lambda(\rho) = F_0\rho F_0^\dagger + F_1\rho F_1^\dagger + F_2\rho F_2^\dagger$ with Kraus operators

$$F_0 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad F_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}. \tag{3}$$

   Consider encoding the qubit into the 3-bit phase flip code as $\alpha\,|{+}{+}{+}\rangle + \beta\,|{-}{-}{-}\rangle$. Compute the effect of the phase damping channel P_λ on the first qubit, and find the probabilities for each syndrome measurement.

   (d) What is the state of the system after error correction?