Quantum Information Theory

Exercise sheet 7

(Dated: March 5, 2022)

1. Write an expression for a generator matrix encoding k bits using r repetitions for each bit. This is a $[rk, k]$ linear code and should have a $rk \times k$ generator matrix.

2. Let H be a parity check matrix such that any $d-1$ columns are linearly independent, but there exists a set of d linearly dependent columns. Show that the code defined by H has distance d.

3. A *CSS code* $\mathrm{CSS}(\mathcal{C}_1, \mathcal{C}_2)$ is defined from two classical linear codes $\mathcal{C}_1$, $\mathcal{C}_2$ of parameters $[n, k_1]$ and $[n, k_2]$, such that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. The quantum code has parameters $[[n, k_1 - k_2]]$ and is spanned by the vectors

$$|x_j + \mathcal{C}_2\rangle := \frac{1}{2^{k_2/2}} \sum_{y \in \mathcal{C}_2} |x_j + y\rangle,$$

where the elements of $\{x_j\}_{j=1\ldots 2^{k_1-k_2}}$ belong to the quotient $\mathcal{C}_1/\mathcal{C}_2$. In other words, they satisfy $x_i + x_j \notin \mathcal{C}_2$, for any pair $x_i \neq x_j$.

The X-type generators are given by the k_2 rows of $H_2^\perp = G_2^T$ and the Z-type generators are given by the $n - k_1$ rows of H_1.

Let $e \in \mathbb{Z}_2^n$ correspond to a row of $H_2^\perp$ (i.e., $e \in \mathcal{C}_2$), the associated X-generator is

$$X^e = \bigotimes_{i=1}^{n} X_i^{e_i}.$$

Similarly, if f is a row of H_1, then define the corresponding Z-generator as

$$Z^f = \bigotimes_{i=1}^{n} Z_i^{f_i}.$$

Show that such generators commute, and that they stabilize the codewords.

4. Let $\mathcal{C}$ be a linear code $[n, k]$ and define the quantum state

$$|\mathcal{C}\rangle = \frac{1}{\sqrt{|\mathcal{C}|}} \sum_{y \in \mathcal{C}} |y\rangle.$$

Show that

$$H^{\otimes n} |\mathcal{C}\rangle = |\mathcal{C}^\perp\rangle$$

where $\mathcal{C}^\perp$ is the dual code defined as

$$\mathcal{C}^\perp = \{y \in \mathbb{Z}_2^n \ : \ x \cdot y = 0, \forall x \in \mathcal{C}\}. \tag{1}$$

5. Consider a quantum error correcting code that encodes one logical qubit into five physical qubits, with the logical qubit states

$$\begin{aligned}
|\bar{0}\rangle = \frac{1}{4}(&|00000\rangle \\
&+ |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&- |11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\
&- |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle) \\
|\bar{1}\rangle = \frac{1}{4}(&|11111\rangle \\
&+ |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&- |00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\
&- |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle)
\end{aligned}$$

(a) Show that $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are simultaneous eigenstates with eigenvalue $+1$ of the operators

$$XZZXI, IXZZX, XIXZZ, ZXIXZ\,.$$

Note that you don't need to check every case to prove this.

(b) Show that this code can correct an X or Z error acting on any of the five qubits. Hint: explain how the different possible errors would be reflected by a measurement of the syndrome error.

(c) Explain why this implies that the code can correct any single-qubit error.

(d) Find the logical Pauli operators $\bar{X}$ and $\bar{Z}$ satisfying $\bar{X}|\bar{0}\rangle = |\bar{1}\rangle, \bar{X}|\bar{1}\rangle = |\bar{0}\rangle, \bar{Z}|\bar{0}\rangle = |\bar{0}\rangle, \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$.