Quantum Information Theory

Exercise sheet 8

(Dated: March 15, 2022)

1. We define the Toric-Code Hamiltonian on a $N \times N$ lattice, with qubits placed on each of the $2N^2$ edges. We choose periodic boundary conditions. The Hamiltonian reads

$$\mathcal{H} = -\sum_v A_v - \sum_p B_p \,, \tag{1}$$

where v and p run over all vertices and plaquettes of the lattice, with

$$A_v = \prod_{j \in v} \sigma_j^z \,, \qquad B_p = \prod_{j \in \partial p} \sigma_j^x \,, \tag{2}$$

and where we denoted by ∂p the edges at the boundaries of a plaquette.

   (a) Show that A_v and B_p all commute with the Hamiltonians and that

$$\prod_v A_a = \mathbb{1} \,, \qquad \prod_p B_p = \mathbb{1} \,, \tag{3}$$

   so that there are $2N^2 - 2$ independent stabilizers.

   (b) From the above observation, conclude that the ground-state degeneracy is 4, and that if $|\psi\rangle$ is a ground-state of H, then

$$A_v |\psi\rangle = B_p |\psi\rangle = |\psi\rangle \tag{4}$$

   for all A_v, B_p. What is the ground-state energy?

   (c) Show that

$$|\tilde{0}\tilde{0}\rangle = \frac{1}{\sqrt{2}} \left[\prod_s \left(\frac{\mathbb{1} + A_s}{\sqrt{2}} \right) \right] |+\rangle^{\otimes N} \tag{5}$$

   is a ground-state of the Hamiltonian.

   (d) Show that three additional orthogonal ground-states $|\tilde{0}\tilde{1}\rangle$, $|\tilde{1}\tilde{0}\rangle$, $|\tilde{1}\tilde{1}\rangle$ can be obtained from $|\tilde{0}\tilde{0}\rangle$ applying appropriate string operators.

   (e) Show that the expectation value of any Pauli matrix on any of the ground states is zero.

2. [*Self-duality of the Toric Code on the square lattice*]

   (a) Let H be the Hadamard gate. How do the plaquette and star operators of the Toric Code transform under $H^{\otimes 4}$?

   (b) Show that $H^{\otimes 4}$ can be seen as an operation mapping plaquette and star operators into new plaquette and star operators defined on a dual lattice.

   (c) How does the Toric-Code Hamiltonian look like in the dual lattice?

   (d) Show that

$$\left[\prod_p \left(\frac{\mathbb{1} + B_p}{\sqrt{2}} \right) \right] |0\rangle^{\otimes N} \tag{6}$$

   is also a ground state of $\mathcal{H}$.

3. Let us take $\{|\pm\rangle\}$ as the local basis. For each basis state in the whole Hilbert space,

$$|\alpha_1 \ldots \alpha_{2N^2}\rangle = \bigotimes_{j=1}^{2N^2} |\alpha_j\rangle \tag{7}$$

with $\alpha_j = \pm$, we can associate a graph in the dual lattice: given a qubit, we "color" the corresponding edge in the dual lattice if it is in the state $|-\rangle$, while we do not color it if the qubit is in the state $|+\rangle$. Show that, with this representation, the ground state (5) can be written as an equal superposition of basis states corresponding to colored contractible loops.

4. (a) We want to encode a qubit state $|\phi\rangle_A = \alpha\,|0\rangle_A + \beta\,|1\rangle_A$ into one of the logical qubits of the Toric Code, denoted by $|\tilde{0}\rangle$, $|\tilde{1}\rangle$, via the SWAP$_{\text{in}}$ transformation

$$|\phi\rangle_A \otimes |\tilde{0}\rangle_M \overset{\text{SWAP}_{\text{in}}}{\longrightarrow} |0\rangle_A \otimes |\phi\rangle_M\,. \tag{8}$$

Show how this can be done using controlled string operators of the form

$$\Lambda[S] = |0\rangle_A \langle 0|_A \otimes \mathbb{I} + |\,1\rangle_A \langle 1|_A \otimes S\;, \tag{9}$$

where S is an arbitrary string, and combining them with local unitary operators on the qubit in A.

(b) With the same ingredients, find also a way of obtaining the operator SWAP$_{\text{out}}$, defined as

$$|0\rangle_A \otimes |\tilde{\phi}\rangle_M \overset{\text{SWAP}_{\text{out}}}{\longrightarrow} |\phi\rangle_A \otimes |\tilde{0}\rangle_M \tag{10}$$