Quantum Information Theory

Exercize sheet 1

(Dated: January 18, 2022)

1. (a) Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, with $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C \simeq \mathbb{C}^2$, and $|\psi\rangle \in \mathcal{H}$ with

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B |0\rangle_C + |1\rangle_A |1\rangle_B |0\rangle_C \right) . \tag{1}$$

Compute the reduced density matrices $\rho_A = \mathrm{Tr}_{BC}(|\psi\rangle \langle\psi|)$, $\rho_C = \mathrm{Tr}_{AB}(|\psi\rangle \langle\psi|)$. Show that ρ_A is a mixed state, while ρ_C is pure.

(b) Let ρ be a density operator. Prove that

$$\mathrm{Tr}[\rho^2] \leq 1 \,, \tag{2}$$

with the equality if and only if ρ is a pure state.

2. Let $\{M_n\}$ with $\sum_m M_m^\dagger M_m = \mathbb{1}$ define a measurement on a system with associated Hilbert space $\mathcal{H}_S$. Show that it is always possible to realize such measurement by means of a unitary transformation in an enlarged Hilbert space, followed by a projective measurement.

3. (a) Show that the transpose map is positive but not completely positive. Hint: consider the state $\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$, with

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \,. \tag{3}$$

(b) [*PPT criterion*] Given $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a density operator $\rho \in \mathcal{H}$ is called separable if there exist $p_k \geq 0$ and density operators $\rho_k^A \in \mathcal{H}_A$, $\rho_k^B \in \mathcal{H}_B$ such that

$$\rho = \sum_k p_k \rho_A^k \otimes \rho_B^k \,. \tag{4}$$

Otherwise ρ is called an entangled state. Show that if ρ is separable, then ρ^{T_B} and ρ^{T_A} are density operators.

4. (a) [*Bloch-sphere interpretation of trace distance*] Let ρ, σ be qubit density operators, with

$$\rho = \frac{1}{2} \left(\mathbb{1} + \vec{r} \cdot \vec{\sigma} \right) , \qquad \sigma = \frac{1}{2} \left(\mathbb{1} + \vec{s} \cdot \vec{\sigma} \right) , \tag{5}$$

and $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$. Prove

$$\Delta(\rho, \sigma) = \frac{|\vec{r} - \vec{s}|}{2} \,. \tag{6}$$

(b) Prove the following formula for the trace distance of two density matrices

$$\Delta(\rho, \sigma) = \max_P \left\{ \mathrm{Tr} \left[P(\rho - \sigma) \right] \right\} , \tag{7}$$

where the maximization is taken either over all positive operators $P \leq \mathbb{1}$, or over all projectors P. Hint: show that for any states ρ, σ there exist positive operators Q, S with support on orthogonal spaces such that

$$\rho - \sigma = Q - S \,. \tag{8}$$

(c) Prove the triangle inequality

$$\Delta(\rho, \sigma) \leq \Delta(\rho, \tau) + \Delta(\tau, \sigma) \,, \tag{9}$$

where ρ, σ, τ are arbitrary density matrices.

(d) Show that $\Delta(\rho, \sigma) = 1$ if and only if ρ and σ have orthogonal support.

(e) [*Operational interpretation of trace distance*] We know that a quantum system is prepared in the state ρ_0 or ρ_1, with classical probability $p = 1/2$. Suppose that we choose a binary POVM $\Lambda = \{\Lambda_0, \Lambda_1\}$ and measure the system. In the case we measure 0 we guess that the state of the system is ρ_0, otherwise we guess it is ρ_1. What is the probability that our guess is correct as a function of Λ? Show that our optimal probability of success is

$$p_{\text{succ}} = \frac{1}{2}\left(1 + \Delta(\rho, \sigma)\right) . \tag{10}$$

What is the optimal probability in the case the system is prepared in three possible density matrices, with probability $p = 1/3$?

5. [*Trace-distance contractivity*]. Prove that for any trace preserving positive linear map $\mathcal{E}$

$$\Delta(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \Delta(\rho, \sigma) . \tag{11}$$

In particular, it follows that the trace distance is contractive for all quantum channels.

6. Show that for any M-code (E, D), there exists an $M/2$-code (E', D') such that

$$p_{\text{err,max}}(E', D') \leq 2 p_{\text{err}}(E, D) . \tag{12}$$