Quantum Information Theory

Exercize sheet 2

(Dated: January 25, 2022)

1. (a) Let $\rho_A \in \mathcal{S}(A)$. Recall

$$H(A)_\rho = -D(\rho_A || \mathbb{1}_A)\,, \tag{1}$$

and

$$D(\rho||\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subset \mathrm{supp}(\sigma) \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

Show that

$$0 \leq H(A)_\rho \leq \log \dim(A)\,. \tag{3}$$

Hint: use *Jensen's inequality*: let $\varphi(x)$ be a real concave function, $x_1, \ldots x_n$ be numbers in its domain and $a_i > 0$. Then

$$\frac{\sum_j a_j \varphi(x_j)}{\sum_j a_j} \leq \varphi\left(\frac{\sum_j a_j x_j}{\sum_j a_j}\right)\,, \tag{4}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$ or $\varphi(x)$ is linear on a domain containing $x_1 = x_2 = \ldots = x_n$.

(b) Show that $H(A)_\rho = 0$ iff ρ is a pure state and that $H(A)_\rho = \log\dim(A)$ if and only if ρ is maximally mixed.

(c) Let $\rho_{AB} \in \mathcal{S}(A \otimes B)$. Show that, if $\rho_{AB} = \sigma \otimes \omega$, then

$$H(AB)_\rho = H(A)_\sigma + H(B)_\omega\,. \tag{5}$$

Hint: show that, for σ, ω of maximal rank,

$$\log(\sigma \otimes \omega) = \log(\sigma) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \log(\omega)\,. \tag{6}$$

2. (a) Let $\rho_{AB} \in \mathcal{S}(A \otimes B)$. Recall

$$H(A|B)_\rho = -D(\rho_{AB} || \mathbb{1}_A \otimes \rho_B)\,. \tag{7}$$

Show

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho\,. \tag{8}$$

(b) Let

$$\rho_{AB} = \sum_{x,y} p(x,y) \left|x,y\right\rangle \left\langle x,y\right|\,, \tag{9}$$

be a classical state. Show

$$H(A|B)_\rho \geq 0\,. \tag{10}$$

(c) Show that (10) is not true in general. Hint: take the maximally entangled state $\rho_{AB} = \left|\varphi\right\rangle\left\langle\varphi\right|$, with

$$\left|\varphi\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle_A \left|0\right\rangle_B + \left|1\right\rangle_B \left|1\right\rangle_B)\,. \tag{11}$$

3. (a) Let $\rho_{AB} \in \mathcal{S}(A \otimes B)$. Recall

$$I(A:B) = D(\rho_{AB}||\rho_A \otimes \rho_B)\,. \tag{12}$$

Show that

$$I(A:B) = H(A) + H(B) - H(AB)\,. \tag{13}$$

(b) Show the subaddtivity of entropy

$$H(A) + H(B) \geq H(AB)\,. \tag{14}$$

How large can $I(A:B)$ be for fixed dimensions?

4. Use the data processing inequality for the relative entropy to show *strong subadditivity*, i.e.,

$$H(A|C) + H(B|C) \geq H(AB|C)\,. \tag{15}$$

Hint: It may be helpful to write this inequality in the equivalent form

$$H(A|BC) \leq H(A|C)\,. \tag{16}$$