Quantum Information Theory

Exercise sheet 3

(Dated: February 1, 2022)

1. [*Binary symmetric channel*] Consider a channel with input space $\{0, 1\}$, where an input 0 is erroneously received as a 1 with probability f, and vice versa. Show that the channel capacity is

$$C(W) = 1 + f \log f + (1 - f) \log(1 - f) =: 1 - H_{\text{bin}}(f) \,. \tag{1}$$

Hint: prove that $I(X : B) \leq 1 - H_{\text{bin}}(f)$, and that equality is achieved by uniform input distribution $p_x = 1/2$.

2. [*Binary erasure channel*] Consider a channel in which some bits are lost, i.e. a fraction α of the bits are erased. Then, we have two inputs, $\{0, 1\}$ and three outputs $\{0, 1, e\}$, where e stands for erased. Show that the channel capacity is

$$C(W) = 1 - \alpha \,. \tag{2}$$

3. Consider a binary symmetric channel with flip probability $f < 0.5$. We consider the simple encoding scheme consisting of repeating each input bit r times, where $r = 2k+1$ and $k \in \mathbb{N}$. The receiver then decodes the string of r bits using a simple majority rule. Show that the error probability ε satisfies

$$\varepsilon \leq e^{-cr} \,, \tag{3}$$

where c is a constant which depends on f. How does the communication rate scale with ε?

Hint: use the Chernoff bound: Suppose $x_1, \ldots, x_n$ are i.i.d. random variables, taking values in $\{0, 1\}$. Let $m = \mathbb{E}[x_1]$ and $s > 0$. Then

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} x_i \geq m + s\right) \leq e^{-D(m+s\|m)n} \,, \qquad \Pr\left(\frac{1}{n}\sum_{i=1}^{n} x_i \leq m - s\right) \leq e^{-D(m-s\|m)n} \,, \tag{4}$$

where

$$D(x\|y) = x \ln \frac{x}{y} + (1 - x) \ln\left(\frac{1 - x}{1 - y}\right) \tag{5}$$

is the Kullback–Leibler divergence between Bernoulli distributed random variables with parameters x and y respectively.

4. (a) Let ρ, σ be density matrices. Prove

$$D(\rho||\sigma) \geq 0 \,, \tag{6}$$

with equality iff $\rho = \sigma$. Hint: prove first the non-negativity of the classical relative entropy

$$\sum_x p(x) \log \frac{p(x)}{q(x)} \geq 0 \,, \tag{7}$$

with equality iff $p(x) = q(x)$ for all x.

(b) The Holevo information of an ensemble $\{p_x, W_x\}$ is defined as

$$H\left(\sum_x p_x W_x\right) - \sum_x p_x H(W_x) \,. \tag{8}$$

Show that this is the same as the mutual information $I(X : B)_{\rho_{XB}}$ of the corresponding classical-quantum state

$$\rho_{XB} = \sum_x p_x \left|x\right\rangle \left\langle x\right| \otimes W_x \,. \tag{9}$$

(c) Prove that the von Neumann entropy is strictly concave, i.e.

$$H\left(\sum_x p_x W_x\right) \geq \sum_x p_x H(W_x)\,. \tag{10}$$

with equality if and only if $W_x = W$ for all x.

(d) Let W be a classical-quantum channel. Using the above results, show that $C(W) = 0$ if and only if the channel is trivial, i.e. $W_x = W$.