Quantum Information Theory

Exercise sheet 4

(Dated: February 8, 2022)

1. [*Holevo bound on accessible information*]

   (a) Suppose Alice prepares a state out of an ensemble $\mathcal{E} = \{p(x), \rho(x)\}$. Bob receives the states prepared by Alice and performs POVM measurements $E = \{E(y)\}_y = \{M^\dagger(y)M(y)\}_y$. The conditional probability that Bob obtains outcome y if Alice sent $\rho(x)$ is therefore

   $$p(y|x) = \mathrm{tr}[E(y)\rho(x)]\,. \tag{1}$$

   The *accessible information* of the ensemble $\mathcal{E}$ is defined as

   $$\mathrm{Acc}(\mathcal{E}) = \max_E [I(X:Y)]\,. \tag{2}$$

   Show the following Holevo bound in terms of the *Holevo information* of the ensemble $\mathcal{E}$

   $$\mathrm{Acc}(\mathcal{E}) \le \chi(\mathcal{E}) := I(X:A)_{\rho_{XA}} \tag{3}$$

   where ρ_{XA} is the classical-quantum state

   $$\rho_{XA} = \sum_x p(x)\,|x\rangle\langle x| \otimes \rho(x)\,. \tag{4}$$

   Hint: Define

   $$\rho'_{XY} = \sum_{x,y} p(x)\mathrm{tr}[E(y)\rho(x)]\,|x\rangle\langle x| \otimes |y\rangle\langle y|\,, \tag{5}$$

   $$\rho'_{XAY} = \sum_{x,y} p(x)\,|x\rangle\langle x| \otimes M(y)\rho(x)M^\dagger(y) \otimes |y\rangle\langle y|\,, \tag{6}$$

   and show

   $$I(X:Y)_{\rho'} \le I(X:AY)_{\rho'} \le I(X:A)_\rho\,. \tag{7}$$

   (b) Take the input ensemble $p_1 = p_2 = p_3 = 1/3$, and $\rho_i = |\varphi_i\rangle\langle\varphi_i|$, with

   $$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\varphi_2\rangle = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \qquad |\varphi_3\rangle = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}\,. \tag{8}$$

   It is possible to prove that the maximum in (2) is achieved by $E = \{E_1, E_2, E_3\}$ with

   $$E_i = \frac{2}{3}\left(\mathbb{1} - |\varphi_i\rangle\langle\varphi_i|\right)\,. \tag{9}$$

   Show explicitly that the Holevo bound is satisfied but not saturated.

2. Given the classical-quantum state (4) and a quantum channel $\mathcal{N}$, define

   $$\rho_{XB} = \sum_x p(x)\,|x\rangle\langle x| \otimes \mathcal{N}_{A\to B}(\rho(x))\,. \tag{10}$$

   Recall that the Holevo information of the quantum channel $\mathcal{N}$ is defined as

   $$\chi(\mathcal{N}) \equiv \max_{\rho_{XA}} I(X;B)_{\rho_{XB}}\,. \tag{11}$$

   Show that the Holevo information for the qubit depolarizing channel

   $$\mathcal{N}_{\mathrm{D}}(\rho) = (1-p)\rho + \frac{p}{2}\mathbb{1}_2 \tag{12}$$

   is

   $$\chi(\mathcal{N}_D) = \log 2 + \left(1 - \frac{p}{2}\right)\log\left(1 - \frac{p}{2}\right) + \frac{p}{2}\log\frac{p}{2}\,. \tag{13}$$

3. Let $|AR_1\rangle$ and $|AR_2\rangle$ be two purifications of a state ρ^A to composite systems AR_1 and AR_2, respectively, with $\dim R_1 \leq \dim R_2$. Prove that there exists an isometry transformation $U : R_1 \to R_2$ s.t. $|AR_2\rangle = (\mathbb{1}_A \otimes U)\,|AR_1\rangle$.

4. Let $\mathcal{E}$ be a (trace-preserving) quantum channel and σ, ρ density operators. Prove

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)\,. \tag{14}$$

   Hint: use Uhlmann's theorem.

5. Find the Choi states, the Kraus operators and the Stinespring representation for the depolarizing channel

$$\mathcal{N}_{\mathrm{D}}(\rho) = (1-p)\rho + \frac{p}{2}\mathrm{tr}[\rho]\mathbb{1}_2\,, \tag{15}$$

   the dephasing channel

$$\mathcal{N}_{\varphi}(\rho) = (1-p)\rho + pZ\rho Z\,, \tag{16}$$

   and the trace channel

$$\mathcal{N}_{\mathrm{tr}}(\rho) = \mathrm{tr}[\rho]\,. \tag{17}$$