Quantum homomorphisms

Report - CR 04

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Abstract

In this note, we present some notable results due to Roberson and Mančinska [7] on quantum homomorphisms, precisely, the quantum sandwich theorem, the decision problem, and some results on quantum graph parameters.

Contents

1 Introduction

Unless stated otherwise, all graphs considered in this note are finite and simple. A *homomorphism* from a graph X to a graph Y is a map f from $V(X)$ to $V(Y)$ such that $x \sim x'$ implies $f(x) \sim f(x')$, where '$\sim$' denotes the adjacency relation between two vertices in a graph. If X admits a homomorphism to Y, we will write $X \to Y$, and $X \not\to Y$ otherwise. For deeper discussion of graph homomorphisms, we refer the reader to course note CR-17, lecture 9, and for details of graph definitions and properties throughout this note, the best general reference is textbook *Graph Theory* [2] by Bondy and Murty.

Before defining the quantum version of graph homomorphism, we first describe the homomorphism game, which states as follows. For graphs X and Y, the (X, Y)-*homomorphism game* is played between two players, Alice and Bob, and a referee. The referee sends Alice and Bob vertices $x_A, x_B \in V(X)$ respectively, and then they reply to the referee vertices $y_A, y_B \in V(Y)$ accordingly. Though Alice and Bob are allowed to agree on a strategy beforehand, they are not allowed to communicate during the game. In order to win the (X, Y)-homomorphism game, they must return y_A, y_B satisfying following conditions,

$$\begin{array}{ll} \text{consistency:} & \text{if } x_A = x_B, \text{ then } y_A = y_B; \\ \text{adjacency preserving:} & \text{if } x_A \sim x_B, \text{ then } y_A \sim y_B. \end{array}$$

We say that Alice and Bob can win the game if they have a strategy to win with probability 1. The next proposition yields the equivalence between graph homomorphism and homomorphism game.

Proposition 1. *Alice and Bob can win the* (X, Y)*-homomorphism game if and only if* $X \to Y$.

The *quantum homomorphism game* are completely the same as classical homomorphism game except that Alice an Bob are allowed to share a quantum state beforehand. If quantum players can win the (X, Y)-homomorphism game, we say that there is a *quantum homomorphism* from X to Y, and write $X \xrightarrow{q} Y$, and $X \overset{q}{\not\to} Y$ otherwise. Clearly by the definition, if classical players can win the homomorphism game, then so do quantum players, and hence $X \to Y$ implies $X \xrightarrow{q} Y$. However, the reader may agree that such definition of quantum homomorphism is not very appealing. In the next part, we will develop a mathematical definition of quantum homomorphism.

Before going on, we would like to say briefly about the birth of quantum homomorphism. Quantum homomorphism was introduced very recently [7] by Roberson and Mančinska in the hope of better understanding not only mysterious graph notions as graph homomorphism, chromatic number, clique number, etc. but also the advantage of quantum strategies over classical strategies in quantum information and game theory. The origin of quantum homomorphism can be traced back to 2002 when Galliard and Wolf [5] introduced the quantum version of graph chromatic number. Since then, quantum graph parameters has been a topic of active research in quantum information theory. For a recent account of this topic, we refer the reader to the paper [3] due to Cameron et. al., and [6] due to Pausen et. al.

1.1 An alternative definition

A *projector* E of dimension d is a $d \times d$ matrix which are both idempotent i.e. $E^2 = E$, and Hermitian, i.e. $E = E^\dagger$. A projector is *real* if all its entries are real numbers. Two projectors E, E' are said to be *orthogonal* if $EE' = 0$. A *measurement*, or sometimes called *POVM*, is a collection of positive semi-definite operators $(E_i)_{1 \le i \le k}$ such that $\sum_{i=1}^k E_i = I$. A measurement is *projective* if each E_i is a projector.

Proposition 2. *If* $(F_i)_{1 \le i \le k}$ *are orthogonal projectors, then* $E = \sum_{i=1}^k F_i$ *is a projector.*

Proof. Since F_i are idempotent, Hermitian, and pairwise orthogonal, it follows easily that E is both Hermitian and idempotent by arguments below,

$$E = \sum_i F_i = \sum_i F_i^\dagger = \Big(\sum_i F_i\Big)^\dagger = E^\dagger;$$
$$E^2 = \Big(\sum_i F_i\Big)^2 = \sum_i F_i^2 + \sum_{i \neq j} F_i F_j = \sum_i F_i = E. \qquad \square$$

In the light of definitions above, we now can provide a natural strategy for wining the quantum homomorphism game. Alice and bob first share some quantum state $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, and each possesses a collection of measurements $(\mathcal{E}_x)_{x\in V(X)} = ((E_{xy})_{y\in V(Y)})_{x\in V(X)}$ of dimension d_A and $(\mathcal{F}_x)_{x\in V(X)} = ((F_{xy})_{y\in V(Y)})_{x\in V(X)}$ of dimension d_B, respectively. After receiving x and x' from the referee, Alice performs $\mathcal{E}_x$ while Bob performs $\mathcal{F}_{x'}$ on their part of the share state $|\psi\rangle$, then they return the outputs y and y' to the referee. Clearly,

$$\mathbb{P}\big(\text{output } y, y' | \text{ input } x, x'\big) = \langle\psi|(E_{xy} \otimes F_{x'y'})|\psi\rangle.$$

Thus to win the game, the following conditions need to be met,

consistency:	$\langle\psi|(E_{xy} \otimes F_{xy'})|\psi\rangle = 0$ for $y \neq y'$;
adjacency preserving:	$\langle\psi|(E_{xy} \otimes F_{x'y'})|\psi\rangle = 0$ for $x \sim x'$ and $y \not\sim y'$.

It seems that using the same measurements for both player is feasible, and well chosen measurements could discard $|\psi\rangle$. Precisely, Alice and Bob can win the game if there exists a collection of projectors $\big(E_{xy}\big)_{x\in V(X), y\in V(Y)}$ satisfying

$$\sum_{y\in V(Y)} E_{xy} = I, \ \forall x \in V(X); \tag{1}$$
$$E_{xy} \otimes E_{xy'} = 0 \text{ for } y \neq y'; \tag{2}$$
$$E_{xy} \otimes E_{x'y'} = 0 \text{ for } x \sim x' \text{ and } y \not\sim y', \tag{3}$$

where condition (1) is for ensuring that $\mathcal{E}_x$ are measurements. Interestingly, the inverse direction holds, and when E_{xy} are real projectors.

Theorem 3 ([7, 3])**.** *$X \xrightarrow{q} Y$ if and only if there exists a collection of real projectors $\big(E_{xy}\big)_{x\in V(X), y\in V(Y)}$ satisfying (1), (2), and (3).*

As promised, Theorem 3 yields an alternative definition of quantum homomorphism, which is said in [8] "palatable to mathematicians, and not too offensive to physicists". In fact both definitions are useful in the context of proving mathematical results, which the reader may see throughout the note (or at least, right below).

Proposition 4. *$X \to Y$ implies $X \xrightarrow{q} Y$.*

Proof. Let f be a homomorphism from X to Y. Let $E_{xy} = I$ if $f(x) = y$, and $E_{xy} = 0$ otherwise. It is easily seen that all conditions (1), (2), and (3) hold, and hence $X \xrightarrow{q} Y$. $\square$

For the sake of comprehensiveness, we provide below some properties of quantum homomorphisms. Proofs can be found in [7].

1. *Transitivity:* if $X \xrightarrow{q} Y$ and $Y \xrightarrow{q} Z$, then $X \xrightarrow{q} Z$.
2. $K_m \xrightarrow{q} K_n$ if and only if $m \leq n$.
3. Suppose that $X \xrightarrow{q} Y$ and X is connected, then there is some connected component C of Y such that $X \xrightarrow{q} C$.
4. *Lifting to classical homomorphism:* if $X \xrightarrow{q} Y$, then there exists a graph Y' (not necessarily finite) such that $X \to Y'$.

2 Main results

The order of this section is as follows. We first present the Lovász ϑ number and the quantum version of sandwich theorem. Then we offer the reader some taste about the decision problem of quantum homomorphisms, and finally, provide some other results on quantum graphs parameters.

2.1 Quantum sandwich theorem

The Lovász ϑ number was first introduced by Lovász in 1979, which is often defined as follows. Let A_X range over all symmetric matrices of size $|V(X)|$ such that $a_{ij} = 1$ whenever $i = j$ or $(i,j) \notin E(X)$. Then the *Lovász number* of X, denoted $\vartheta(X)$, is the minimum possible of the largest eigenvalue of A_X. The Lovász theta number of the complement of a graph is often denoted $\vartheta(\overline{X})$; however we here prefer to use $\overline{\vartheta}(X)$ as in [7]. Although Lovász number is defined to deal with the Shannon capacity of graph, it has a natural connection to graph homomorphism.

Proposition 5. *If $X \to Y$, then $\overline{\vartheta}(X) \leq \overline{\vartheta}(Y)$.*

We give below an alternative definition of $\overline{\vartheta}(X)$, which will be more useful in the sense of quantum homomorphism. For $m \in \mathbb{N}$ and $\varepsilon > 0$, let $S(m, \varepsilon)$ be the infinite graph whose vertices are unit vectors in $\mathbb{R}^m$ and two vectors u and v are adjacent if $u^T v = -\varepsilon$. Then

$$\overline{\vartheta}(X) = \min_{X \to S(m,\varepsilon)} (1 + 1/\varepsilon).$$

We here recall definitions of some parameters of graph X in the context of graph homomorphism,

chromatic number:	$\chi(X) := \min\{n : X \to K_n\}$;
clique number:	$\omega(X) := \max\{n : K_n \to X\}$;
independence number:	$\alpha(X) := \omega(\overline{X})$.

Unlike three above parameters, whose decision problems are known to be NP-complete, Lovász number can be computed in polynomial-time. The best known result for Lovász number is the sandwich theorem, which yields a polynomial-time bound for all three parameters (the bound of $\alpha(X)$ can be given via the bound of $\omega(\overline{X})$).

Theorem 6 (Lovász sandwich theorem)**.** $\omega(X) \leq \overline{\vartheta}(X) \leq \chi(X)$.

Thank to the definition of quantum homomorphism, we now can now well define corresponding quantum graph parameters as follows.

$$\begin{array}{ll} \textit{quantum chromatic number:} & \chi_q(X) := \min\{n : X \xrightarrow{q} K_n\}; \\ \textit{quantum clique number:} & \omega_q(X) := \max\{n : K_n \xrightarrow{q} X\}; \\ \textit{quantum independence number:} & \alpha_q(X) := \omega_q(\overline{X}). \end{array}$$

It follows easily that $\omega_q(X) \leq \chi_q(X)$. Indeed suppose $\omega_q(X) = m$ and $\chi_q(X) = n$. Then by transitive property of quantum homomorphism, $K_m \xrightarrow{q} X \xrightarrow{q} K_n$ implies $K_m \xrightarrow{q} K_n$, and hence $m \leq n$, i.e. $\omega_q(X) \leq \chi_q(X)$. The central result in [7] is the quantum version of sandwich theorem, which is much more appealing than the original.

Theorem 7 (Quantum sandwich theorem, [7])**.**

$$\omega(X) \leq \omega_q(X) \leq \overline{\vartheta}(X) \leq \chi_q(X) \leq \chi(X).$$

Proposition 4 immediately establishes the first and last inequalities. To deal with the others, we first prove the quantum version of Proposition 5.

Theorem 8. *If $X \xrightarrow{q} Y$, then $\overline{\vartheta}(X) \leq \overline{\vartheta}(Y)$.*

Proof. Suppose that $\overline{\vartheta}(Y) = 1 + 1/\varepsilon$, then there exists m such that $Y \to S(m, \varepsilon)$. In other words, there are unit vectors $(v_y)_{y \in V(Y)} \in \mathbb{R}^m$ such that $v_y^T v_{y'} = -\varepsilon$ if $y \sim y'$. If there is m' such that $X \to S(m', \varepsilon)$, then $\overline{\vartheta}(X) \leq 1 + 1/\varepsilon$, the theorem is proved. We now find such m'.

Since $X \xrightarrow{q} Y$, there exists real projectors E_{xy} of dimension d satisfying Theorem 3. Let $\mathrm{vec}(E_{xy})$ denote the vector of size d^2 obtained by putting the first column of E_{xy} over the its second column, over the third, etc. For every $x \in V(X)$, we define vectors $u_x \in \mathbb{R}^{md^2}$ as follows,

$$u_x = \frac{1}{\sqrt{d}} \sum_{y \in V(Y)} v_y \otimes \mathrm{vec}(E_{xy}).$$

$$\begin{aligned} \Longrightarrow u_x^T u_{x'} &= \frac{1}{d} \Big(\sum_{y \in V(Y)} v_y \otimes \mathrm{vec}(E_{xy}) \Big)^T \Big(\sum_{y' \in V(Y)} v_{y'} \otimes \mathrm{vec}(E_{x'y'}) \Big) \\ &= \frac{1}{d} \sum_{y, y' \in V(Y)} v_y^T v_{y'} \mathrm{vec}(E_{xy}) \mathrm{vec}(E_{x'y'}) \\ &= \frac{1}{d} \sum_{y, y' \in V(Y)} v_y^T v_{y'} \mathrm{Tr}(E_{xy} E_{x'y'}) \end{aligned}$$

We will prove that the map $f : X \to S(md^2, \varepsilon)$ where $f(x) = u_x$ is a valid homomorphism. To this end, we first prove that $u_x \in V(S(md^2, \varepsilon))$, i.e. u_x are unit vectors.

Condition (2) yields $\mathrm{Tr}(E_{xy}E_{xy'}) = 0$ for all $y \neq y'$. Thank to condition (1) and the facts that E_{xy} are idempotent and v_y are unit vectors, we have

$$\begin{aligned} u_x^T u_x &= \frac{1}{d} \sum_{y \in V(Y)} v_y^T v_y \mathrm{Tr}(E_{xy}E_{xy}) \\ &= \frac{1}{d} \sum_{y \in V(Y)} \mathrm{Tr}(E_{xy}) \\ &= \frac{1}{d} \mathrm{Tr}\Big(\sum_{y \in V(Y)} E_{xy} \Big) \\ &= \frac{1}{d} \mathrm{Tr}(I) = 1. \end{aligned}$$

Hence u_x are unit vectors. We are left with the task of showing that if $x \sim x'$ then $u_x^T u_{x'} = -\varepsilon$. For every $x \sim x'$, condition (3) yields $\mathrm{Tr}(E_{xy}E_{x'y'}) = 0$ for all $y \not\sim y'$. Thus

$$\begin{aligned} u_x^T u_{x'} &= \frac{1}{d} \sum_{y \sim y'} v_y^T v_y \mathrm{Tr}(E_{xy}E_{x'y'}) \\ &= \frac{-\varepsilon}{d} \sum_{y \sim y'} \mathrm{Tr}(E_{xy}E_{x'y'}) \\ &= \frac{-\varepsilon}{d} \sum_{y, y' \in V(Y)} \mathrm{Tr}(E_{xy}E_{x'y'}) \\ &= \frac{-\varepsilon}{d} \mathrm{Tr}\Big(\Big(\sum_{y \in V(Y)} E_{xy} \Big) \Big(\sum_{y' \in V(Y)} E_{x'y'} \Big) \Big) \\ &= \frac{-\varepsilon}{d} \mathrm{Tr}(I^2) = -\varepsilon. \end{aligned}$$

□

Proof of Theorem 7. Let $n = \omega_q(X)$, then $K_n \xrightarrow{q} X$. Theorem 8 yields $n = \overline{\vartheta}(K_n) \leq \overline{\vartheta}(X)$, which establishes the second inequality. The third inequality can be settled by the same manner. It is also valuable to note that $\alpha(X) \leq \alpha_q \leq \vartheta(X)$ can be directly obtained from Theorem 7. □

2.2 Quantum homomorphism decision problem

Though the decision problem if $X \rightarrow Y$ for arbitrary X and Y is known to be NP-complete, until now the complexity of the decision problem for quantum homomorphism is still open (at least, base on our knowledge). In this part, we will discuss about a promising approach to solve such problem. The method is reducing the quantum decision problem into an equivalent problem on deciding the quantum independent number of another graph. In other words, we reduce the problem if $X \xrightarrow{q} Y$ for arbitrary X and Y into if $K_n \xrightarrow{q} Z$ for some graph Z, which is believed to be simpler.

For graphs X and Y, we define their *homomorphic product*, denoted $X \ltimes Y$, to be the graph with vertex set $V(X) \times V(Y)$ and distinct vertices (x, y) and $(x'y')$

being adjacent if either $x = x'$, or $x \sim x'$ and $y \nsim y'$. The homomorphic product has been well studied; the following are their useful properties,

$$K_{|V(Y)|} \to X \ltimes Y; \tag{4}$$

$$\overline{X \ltimes Y} \to K_{|V(X)|}. \tag{5}$$

The next theorem reduces the quantum decision problem to an equivalent one.

Theorem 9. *$X \overset{q}{\to} Y$ if and only if $\alpha_q(X \ltimes Y) = |V(X)|$.*

Proof. Let $|V(X)| = m$ and $|V(Y)| = n$ and $Z = \overline{X \ltimes Y}$. The theorem can be restated as $X \overset{q}{\to} Y$ if and only if $\omega_q(Z) = m$. Property (5) deduces $\chi(Z) \le m$. Combining with quantum sandwich theorem yields $\omega_q(Z) \le m$. Hence

$$\begin{aligned} \omega_q(Z) = m &\iff \omega_q(Z) \ge m \\ &\iff \max\{d : K_d \overset{q}{\to} Z\} \ge m \\ &\iff K_m \overset{q}{\to} Z. \end{aligned}$$

For this reason, it turns out to prove that $X \overset{q}{\to} Y$ if and only if $K_m \overset{q}{\to} Z$.

In one direction, suppose $X \overset{q}{\to} Y$, then Alice and Bob has a quantum strategy for winning the (X, Y)-homomorphism game. We will show $K_m \overset{q}{\to} Z$ by constructing a quantum strategy for wining (K_m, Z)-homomorphism game. Since $|V(K_m)| = |V(X)|$, we can assume that K_m and X share the same vertex set. Now suppose that Alice and Bob receive x_A, x_B for (K_m, Z)-homomorphism game, they will perform measurements for the (X, Y)-homomorphism game and obtain y_A, y_B, then return to the referee $(x_A, y_A), (x_B, y_B) \in V(Z)$. It remains to prove that $(x_A, y_A), (x_B, y_B)$ are successful answers, which can be easily confirmed by examining the two conditions.

- Consistency: if $x_A = x_B$ then $y_A = y_B$, i.e. $(x_A, y_A) = (x_B, y_B)$.
- Adjacency preserving: if $x_A \ne x_B$, then $x_A \sim x_B$ in K_m, there are two cases,
  - $x_A \sim x_B$ in X, then $y_A \sim y_B$ in Y, then $(x_A, y_A) \not\simeq (x_B, y_B)$ in $X \ltimes Y$, where '$\not\simeq$' denotes "neither equal nor adjacent to". Hence $(x_A, y_A) \sim (x_B, y_B)$ in Z.
  - $x_A \not\simeq x_B$ in X, then clearly $(x_A, y_A) \not\simeq (x_B, y_B)$ in $X \ltimes Y$. Hence $(x_A, y_A) \sim (x_B, y_B)$ in Z.

For the other direction, since $K_m \overset{q}{\to} Z$, there are real projectors $F_{i(x,y)}$ of dimension d with $i \in V(K_m)$ and $(x, y) \in V(Z)$ satisfying Theorem 3, which can be expressed as follows,

$$\sum_{\substack{x \in V(X) \\ y \in V(Y)}} F_{i(x,y)} = I, \ \forall i \in V(K_m);$$

$$F_{i(x,y)} \otimes F_{i'(x',y')} = 0 \text{ if } \begin{cases} i = i' \text{ and } (x, y) \ne (x', y'), \text{or} & \text{(a)} \\ i \ne i' \text{ and } x = x', \text{or} & \text{(b)} \\ i \ne i' \text{ and } x \sim x' \text{ and } y \nsim y'. & \text{(c)} \end{cases}$$

For each $x \in V(X)$ and $y \in V(Y)$ we let

$$E_{xy} = \sum_{i \in V(K_m)} F_{i(x,y)}.$$

Proposition 2 ensures that E_{xy} are projectors. The proof is completed by showing that conditions (1), (2) and (3) hold for E_{xy}. Indeed

$$\begin{aligned} E_{xy}E_{x'y'} &= \Big(\sum_{i \in V(K_m)} F_{i(x,y)}\Big)\Big(\sum_{i' \in V(K_m)} F_{i'(x',y')}\Big) \\ &= \sum_{i,i' \in V(K_m)} F_{i(x,y)}F_{i'(x',y')}. \end{aligned}$$

- If $x = x'$ and $y \neq y'$, (a) and (b) imply that all addends of $E_{xy}E_{x'y'}$ are 0, and so $E_{xy}E_{x'y'} = 0$, which confirms (2).
- If $x \sim x'$ and $y \not\sim y'$, (a) and (c) imply that all addends of $E_{xy}E_{x'y'}$ are 0, and so $E_{xy}E_{x'y'} = 0$, which confirms (3).

We are left with the task of verifying (1). To this end, we first show that

$$\operatorname{rank}\Big(\sum_{y \in V(Y)} E_{xy}\Big) = d, \quad \forall x \in V(X). \tag{6}$$

We let $M_x = \sum_{y \in V(Y)} E_{xy}$. Since E_{xy} has are $d \times d$ matrix, clearly $\operatorname{rank}(M_x) \leq d$ for every x. Furthermore,

$$\begin{aligned} \sum_{x \in V(X)} \operatorname{rank}(M_x) &= \sum_{\substack{x \in V(X) \\ y \in V(Y)}} \operatorname{rank}(E_{xy}) \\ &= \sum_{\substack{x \in V(X) \\ y \in V(Y)}} \operatorname{rank}\Big(\sum_{i \in V(K_m)} F_{i(x,y)}\Big) \\ &= \sum_{i \in V(K_m)} \operatorname{rank}\Big(\sum_{\substack{x \in V(X) \\ y \in V(Y)}} F_{i(x,y)}\Big) \\ &= \sum_{i \in V(K_m)} \operatorname{rank}(I) = md. \end{aligned}$$

Hence (6) holds, i.e. M_x has full rank, and so is inversible. By verifying (2), we proved that $(E_{xy})_{Y \in V(Y)}$ are pairwise orthogonal; thus by Proposition 2, M_x is idempotent. Hence

$$M_x = IM_x = M_x^{-1}M_xM_x = M_x^{-1}M_x = I.$$

In other words,

$$\sum_{y \in V(Y)} E_{xy} = I, \quad \forall x \in V(X),$$

which validates (1) and completes the proof. $\square$

We can deduce from Theorem 9 a straightforward corollary as below, which (and also Theorem 8) though fails to provide a certificate for $X \xrightarrow{q} Y$, can still give a certificate for $X \stackrel{q}{\not\to} Y$ in polynomial-time.

Corollary 10. *If $X \xrightarrow{q} Y$, then $\vartheta(X \ltimes Y) = |V(X)|$.*

Proof. By Property (5), quantum sandwich theorem, and Theorem 9, we have

$$|V(X)| = \alpha_q(X \ltimes Y) \leq \vartheta(X \ltimes Y) = \overline{\vartheta}(\overline{X \ltimes Y}) \leq \chi(\overline{X \ltimes Y}) \leq |V(X)|. \quad \square$$

2.3 Graph parameters: classical vs quantum

Another interesting question in quantum homomorphism is characterizing the family of pairs (X, Y) such that $X \xrightarrow{q} Y$ but $X \not\to Y$. The first approach is finding (X, Y) where one of them is a complete graph. Quantum sandwich theorem gives $\chi_q(X) \leq \chi(X)$, so if the inequality turns out to be strict, $X \xrightarrow{q} K_m$ but $X \not\to K_m$ where $m = \chi_q(X)$. The same argument holds for ω and α. It is not hard to find such graphs X. People then started to ask how large the gaps can be. In the next two theorems, Theorems 11 and 12, we will show very impressive results that the gaps between quantum and classical parameters can actually be exponential.

For $n \in \mathbb{N}$, let us denote by Ω_n the graph whose vertices are the ± 1 vectors of length n with orthogonal vectors being adjacent. The *Cartesian product* of X and Y, denoted $X \square Y$, is the graph with vertex set $V(X) \times V(Y)$ with two vertices being adjacent if they are equal in one coordinate and adjacent in the other. Here are some elementary properties of these concepts.

$$\Omega_n \xrightarrow{q} K_n \tag{7}$$

$$X \ltimes K_n = X \square K_n. \tag{8}$$

$$\alpha(X \square Y) \leq \min\{\alpha(X)|V(Y)|, \alpha(Y)|V(X)|\}. \tag{9}$$

Theorem 11 ([1, 4])**.** *For any n divisible by 4, $\chi(\Omega_n) > 2^n$ while $\chi_q(\Omega_n) \leq n$.*

Proof of the former inequality can be found in [4] while proof of the latter is available in [1].

Theorem 12. *There exists an $\varepsilon > 0$ such that for any n divisible by 4,*

$$\frac{\alpha_q(\Omega_n \square K_n)}{\alpha(\Omega_n \square K_n)} \geq \frac{1}{n}\left(\frac{2}{2-\varepsilon}\right)^n.$$

Proof. Properties (7), (8) and Theorem 9 yield

$$\alpha_q(\Omega_n \square K_n) = |V(\Omega_n)| = 2^n.$$

It was proved in [4] that there exists an $\varepsilon > 0$ such that for any n divisible by 4, $\alpha(\Omega_n) \leq (2-\varepsilon)^n$. Property (9) implies $\alpha(\Omega_n \square K_n) \leq n(2-\varepsilon)^n$, which completes the proof. $\square$

3 Perspectives

The quantum sandwich theorem states that

$$\omega(X) \leq \omega_q(X) \leq \overline{\vartheta}(X) \leq \chi_q(X) \leq \chi(X).$$

Theorems 11 and 12 provide exponential gaps for the first and the last inequalities. It is natural to ask can we also obtain exponential gaps for the second and the third ones. Such question is still open, but a strict example for the second and third inequalities can be easily found, for example $\omega_q(C_5) = 2, \overline{\vartheta}(C_5) = \sqrt{2}$, while $\chi_q(C_5) = 3$, where C_5 is the cycle of length 5. Besides, the decision problem is still the central open question in quantum homomorphism. An efficient algorithm for this problem would yield one more example showing the advantage of quantum over classical computing.

Since quantum homomorphism was introduced very recently, there has been little achievement on this topic. The most notable work [6] is due to Paulsen et. al., in which the authors develop not only one but five different definitions of quantum chromatic number. These variances come from the fact that "the set of correlations of quantum experiments may possibly depend on which set of quantum mechanical axioms one chooses to employ". Besides, there are still several interesting aspects of quantum homomorphism discussed in [7] that we do not include into this note, for example quantum version of Shannon capacity and projective rank.

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