HW I: Quantum states (due Sept 27th, before class)

1. Suppose you have a two-qubit system A_1A_2 in the state $|\psi\rangle_{A_1A_2} = |0\rangle_{A_1} \otimes |0\rangle_{A_2}$. I then apply a Hadamard gate on A_1 followed by a controlled-not gate where the control is A_1 and the target is A_2. Recall that a controlled-not gate has the following representation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{1}$$

   What is the state at the outcome? If I perform a measurement, what is the distribution of outcomes?

2. Using only controlled-not gates, build a circuit on two qubits that swaps the two registers A_1 and A_2, i.e., a circuit U such that $U(|\psi\rangle_{A_1} \otimes |\phi\rangle_{A_2}) = |\phi\rangle_{A_1} \otimes |\psi\rangle_{A_2}$ for any $|\phi\rangle, |\psi\rangle \in \mathbb{C}^2$.

3. Suppose I am given a system that is either in the state $|\psi_1\rangle = |0\rangle$ or in in the state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. I would like to determine which is the case by performing a unitary

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{2}$$

   followed by a measurement. What is the maximum probability of correct guessing, and what is the optimal angle θ? You may assume that the probabilities of the two alternatives $|\psi_1\rangle$ and $|\psi_2\rangle$ are equal to $1/2$.