HW 2 (due Nov 8th, before class)

1. As any state, the state $H^{\otimes n}|x\rangle$ for $x \in \{0,1\}^n$ can be written as $\sum_{y\in\{0,1\}^n} \alpha_y(x)|y\rangle$. Compute $\alpha_y(x)$.

2. Define the 2-qubit state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$ shared between Alice and Bob. Let U be a 1-qubit unitary. Compute $U \otimes U|\psi\rangle_{AB}$. Conclude that any unitary operation done by Alice on her system A can be undone by Bob only by acting on his system B.

   Note that such a property does not hold for all states, for example a product state $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\theta\rangle_B$ definitely does not satisfy this property.

3. The objectif of this problem is to show that the phase estimation algorithm can still return a good approximation if the angle θ is not of the form $\theta = \frac{j}{2^m}$.

   (a) Show that after applying the phase estimation circuit described in class, we obtain (in the first register) the state

   $$\frac{1}{2^m}\sum_{j=0}^{2^m-1}\left(\sum_{k=0}^{2^m-1} e^{2\pi i k(\theta - j2^{-m})}\right)|j\rangle \ .$$

   (b) We now measure this state in the standard basis. Compute the probability p_j of obtaining outcome j.

   (c) Suppose we are aiming for a precision of $t < m$ bits, i.e., we would like the outcome to be some j such that $|\theta - j2^{-m}| \le 2^{-t-1}$. Compute a lower bound on the probability of obtaining such an outcome. The bound should be such that if t is fixed and m grows, the probability of success should go to 1. For this you may use the following inequality without proof: for $\gamma \in [-\pi, \pi]$, $|1 - e^{i\gamma}| \ge \frac{2}{\pi}|\gamma|$.