1. The objective here is to generalize Deutsch’s algorithm to a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$. We are guaranteed that $f$ is either constant or balanced, i.e., $|\{x \in \{0, 1\}^n : f(x) = 0\}| = |\{x \in \{0, 1\}^n : f(x) = 1\}| = 2^{n-1}$. The objective is to output 0 if $f$ is constant and 1 if $f$ is balanced with the minimum number of queries. To determine this, we are allowed to perform queries to the oracle $U_f$, where $U_f |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus f(x)\rangle$, $x \in \{0, 1\}^n$ and $y \in \{0, 1\}$. In the classical query model, we use $U_f$ on states of the form $|x\rangle \otimes |y\rangle$ for $x \in \{0, 1\}^n$, $y \in \{0, 1\}$. In the more general quantum model, we can give as input to $U_f$ any state $|\psi\rangle \in \mathbb{C}^{2^n} \otimes \mathbb{C}^2$.

(a) Show that any deterministic classical query algorithm solving the problem has to perform at least $2^n/2 + 1$ queries.

(b) Show how to compute the unitary $S_f$ defined as $S_f |x\rangle = (-1)^{f(x)} |x\rangle$ for all $x \in \{0, 1\}^n$ using only once the unitary $U_f$. You are allowed to use additional qubits that are in the state $|0\rangle$ but they have to be returned to the same state. You are also allowed to use fixed unitaries such as $X$ (NOT gate), CNOT or Hadamard.

(c) Give a quantum query algorithm that solves this problem using 1 query to $U_f$.

(d) Now suppose we consider a classical query algorithm that is allowed to make error with probability $\leq \epsilon$, give an efficient classical query algorithm for this problem.